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Stationary Boltzmann equation and the nonlinear alternative of Leray-Schauder type

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ABSTRACT

By applying a nonlinear alternative of Leray-Schauder type, a fixed point of an operator is found, which, in turn, comes to be a solution of stationary Boltzmann equation with boundary conditions of Maxwellian type.

RESUMEN

Aplicando una alternativa no lineal del tipo Leray-Schauder, se encontró un punto fijo de un operador, el cual corresponde a la solución de una ecuación de Boltzmann estacionaria con condiciones de frontera del tipo Maxwelliano.

Keywords and Phrases: Nonlinear alternative of Leray-Schauder type, Fixed point, Solution of stationary Boltzmann equation.

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1 Introduction

Let us consider the Banach space $E = \left\{u \in L^1(\Omega \times B_{3R}(0)) : \nu_i \dfrac{\partial u}{\partial x_i} \in L^1(\Omega \times B_{3R}(0))\right\}$ with norm $\|u\|_E := \|u\|_{L^1(\Omega \times B_{3R}(0))} + \left\|\nu_i \dfrac{\partial u}{\partial x_i}\right\|_{L^1(\Omega \times B_{3R}(0))}$ and we expect to find $u(x,\nu) \geq 0$, such that

$$\begin{cases} \nu.\nabla_x u = Q(u,u), \quad u \in B_E(0,R) \\ u(x,\nu) = M(\nu) = e^{-|\nu|^2} \ (Maxwellian), \quad u \in \partial B_E(0,R), \ (R > 0) \end{cases} \tag{1.1}$$

Here $\quad Q(u,u)(\nu) = \displaystyle\int_{B_{3R}(0)} \int_{|p|=1} [p \cdot (\nu - z)] p [u(x,z')u(x,\nu') - u(x,z)u(x,\nu)] dp dz,$

is the collision operator, Ω bounded and regular and the speeds related by the following relations:

$$\begin{cases} \nu' = \nu - [p \cdot (\nu - z)]p \\ z' = z + [p \cdot (\nu - z)]p \end{cases} \tag{1.2}$$

$(z,\ \nu)$ and $(z',\ \nu')$ son the pre-collision and post-collision speeds, respectively. It can be noted that if $z,\ \nu \in B_R(0)$ in $\mathbb{R}^n$, then $z',\ \nu' \in B_{3R}(0)$.
Here, the following problem will be proved.

Theorem 1.1. *Let us suppose that* $Q(u,u) \in B_E(u, R/2)$; $\nu_i \dfrac{\partial u}{\partial x_i} \in B_E(0, R/2n)$ *y* $\nu_i \dfrac{\partial u_n}{\partial x_i} \in B_E\left(\nu_i \dfrac{\partial u}{\partial x_i}, R^{**}\right)$, $R^{**} \geq 0$, *for* $n = 1, 2, \ldots,$ *moreover* $0 < \displaystyle\int_{B_{3R}(0)} \int_{|p|=1} \|\nu - z\| dp dz < \infty$ *and* $0 < \displaystyle\int_{B_{3R}(0)} \int_{|p|=1} \|\nu - z\| dp d\nu < \infty$, *then there exits a solution for* $u \in B_E(0,R)$ *of problem* (1.1).

In these stationary problems, the flows of quantities as entropy control and compactness properties are under control, but they do not imply, per se, the desired results. Anyway, energy control and similar properties are available from momentum flows and mass control that can be imposed on the problem to replace entropy-bounding non-availability. There are controls based on involution of entropy dissipation..
Using such tools, in the last years Arkeryd L. and Noury A., [2] -[3] -[4]- [5], have made a development focused on the results of solutions existence in the L^1 context for nonlinear equations Boltzmann type and also for those presenting maxwellian equilibrium. The case of perturbation on the global maxwellian equilibrium has been typically studied since the 60's. Methods of general type as Hilbert spaces and contraction mapping have been used, being the pioneers [6] -[7] -[9] -[10]; in [11] exposed, generally discussed the problem of boundary value for the stationary equation, in [12] and [14] is proved the theorem for the stationary equation Povzner with certain spatial boundary conditions of type *Maxwellian* and in [13] there are applications to dynamics of fluids, we present the main result five lemmas.

2 Development

The problem (1.1), $v.\nabla_x u = Q(u,u)$ is equivalent to $u + v.\nabla_x u = u + Q(u,u)$, this implies that $u = u + v.\nabla_x u - Q(u,u)$ which suggests the following operator:

$$J(u) := u + v.\nabla_x u - Q(u,u)$$

Defined on

$$E = \left\{ u \in L^1(\Omega \times B_{3R}(0)) : v_i \frac{\partial u}{\partial x_i} \in L^1(\Omega \times B_{3R}(0)) \right\},$$

E is a Banach space with the norm

$$\|u\|_E := \|u\|_{L^1(\Omega \times B_{3R}(0))} + \left\| v_i \frac{\partial u}{\partial x_i} \right\|_{L^1(\Omega \times B_{3R}(0))}$$

Finding fixed points of J, coincides with finding solutions of (1.1). So we will work to find fixed points, via alternative Leray-Schauder type, in effect:
Let $C = \overline{B_E(0,R)}$, this is a convex and closed set in E and $U := B_E(0,R)$, open ball centered in 0 and radius R.

Lemma 2.1. *Let us suppose that* $Q(u,u) \in B_E(u, R/2)$ *and* $v_i \frac{\partial u}{\partial x_i} \in B_E(0, R/2n)$, $u = 1, 2, \ldots,$ *then* J *sends* C *in* C.

Proof. Let $u \in C = \overline{B_E(0,R)}$, as $J(u) := u + v.\nabla_x u - Q(u,u)$, then

$$|J(u)| \le |u - Q(u,u)| + |v.\nabla_x u| = |u - Q(u,u)| + \sum_{i=1}^{n} \left| v_i \frac{\partial u}{\partial x_i} \right|, \quad \text{i.e,}$$

$$\|J(u)\|_{L^1(\Omega \times B_{3R}(0))} \le \|u - Q(u,u)\|_{L^1(\Omega \times B_{3R}(0))} + \sum_{i=1}^{n} \left\| v_i \frac{\partial u}{\partial x_i} \right\|_{L^1(\Omega \times B_{3R}(0))} \tag{2.1}$$

now:

$$\frac{\partial J(u)}{\partial x_i} = \frac{\partial u}{\partial x_i} + \frac{\partial}{\partial x_i} \left[\sum_{i=1}^{n} v_i \frac{\partial u}{\partial x_i} \right] + \frac{\partial Q(u,u)}{\partial x_i}, \quad \text{then}$$

$$\left| \frac{\partial J(u)}{\partial x_i} \right| \le \left| \frac{\partial}{\partial x_i} \big(u - Q(u,u) \big) \right| + \left| \frac{\partial}{\partial x_i} \sum_{i=1}^{n} v_i \frac{\partial u}{\partial x_i} \right|, \quad \text{i.e.,}$$

$$\left\| \frac{\partial J(u)}{\partial x_i} \right\|_{L^1(\Omega \times B_{3R}(0))} \le \left\| \frac{\partial}{\partial x_i} \big(u - Q(u,u) \big) \right\|_{L^1(\Omega \times B_{3R}(0))} + \sum_{i=1}^{n} \left\| \frac{\partial}{\partial x_i} \left(v_i \frac{\partial u}{\partial x_i} \right) \right\|_{L^1(\Omega \times B_{3R}(0))}. \tag{2.2}$$

from (2.1) and (2.2) we conclude that:

$$\|J(u)\|_E \le \|u - Q(u,u)\|_E + \sum_{i=1}^{n} \left\| v_i \frac{\partial u}{\partial x_i} \right\|_E \le \frac{R}{2} + \frac{R}{2} = R,$$

that is to say, $J(u) \in C = B_E(0,R)$. ☐

Lemma 2.2. *If* $u_n \in B_E(u,R)$, *such that* $|u_n(x,v)| \leq R$, $|u(x,v)| \leq R$, *for every* n, $x \in \Omega$, $v \in B_{3R}(0)$, *moreover*

$$0 < \int_{B_{3R}} \int_{|p|=1} \|(v-z)\| dp dz < \infty \quad y \quad 0 < \int_{B_{3R}} \int_{|p|=1} \|(v-z)\| dp dv < \infty,$$

then $Q(u_n,u_n) \in B_E(Q(u,u), r^*R)$ *with*

$$r^* = 4R \max\left\{ \int_{B_{3R}} \int_{|p|=1} \|(v-z)\| dp dz, \int_{B_{3R}} \int_{|p|=1} \|(v-z)\| dp dv \right\}. \qquad (r^* \geq 0)$$

Proof. By definition of $Q(u,u)$, leads to

$$\begin{aligned}
\Big| Q(u_n,u_n)(v) - Q(u,u)(v) \Big| \\
&= \Big| \int_{B_{3R}(0)} \int_{|p|=1} [p\cdot(v-z)]p[u_n(x,z')u_n(x,v') - u_n(x,z)u_n(x,v)] dp dz \\
&\quad - \int_{B_{3R}(0)} \int_{|p|=1} [p\cdot(v-z)]p[u(x,z')u(x,v') - u(x,z)u(x,v)] dp dz \Big|,
\end{aligned}$$

That is to say:

$$\begin{aligned}
\Big| Q(u_n,u_n)(v) - Q(u,u)(v) \Big| \\
&= \Big| \int_{B_{3R}(0)} \int_{|p|=1} [p\cdot(v-z)]p\Big[u_n(x,z')u_n(x,v') - u(x,z')u(x,v') \\
&\quad - u_n(x,z)u_n(x,v) + u(x,z)u(x,v)\Big] dp dz \Big| \\
&\leq \int_{B_{3R}(0)} \int_{|p|=1} |p\cdot(v-z)| \Big| u_n(x,z')u_n(x,v') - u_n(x,z')u(x,v') \\
&\quad + u_n(x,z')u(x,v') - u(x,z')u(x,v') + u(x,z)u(x,v) \\
&\quad - u(x,z)u_n(x,v) + u(x,z)u_n(x,v) - u_n(x,z)u_n(x,v) \Big| dp dz
\end{aligned}$$

then

$$\begin{aligned}
\Big| Q(u_n,u_n)(v) - Q(u,u)(v) \Big| \\
&\leq \int_{B_{3R}(0)} \int_{|p|=1} |p\cdot(v-z)| \Big[\big|u_n(x,z')\big| \, \big|u_n(x,v') - u(x,v')\big| \\
&\quad + \big|u(x,v')\big| \, \big|u_n(x,z') - u(x,z')\big| + \big|u(x,z)\big| \, \big|u_n(x,v) - u(x,v)\big| \\
&\quad + \big|u_n(x,v)\big| \, \big|u_n(x,z) - u(x,z)\big| \Big] dp dz
\end{aligned}$$

so,

$$\begin{aligned}
\Big\|Q(u_n,u_n)-Q(u,u)\Big\|_{L^1(\Omega\times B_{3R}(0))} &\le \int_{\overline{\Omega}}\int_{B_{3R}(0)}\Big|Q(u_n,u_n)(v)-Q(u,u)(v)\Big|dxdv\\
&\le \int_{\overline{\Omega}}\int_{B_{3R}(0)}\int_{B_{3R}(0)}\int_{|p|=1}|p\cdot(v-z)|\,|u_n(x,z')|\,|u_n(x,v')-u(x,v')|dpdzdvdx\\
&\quad+\int_{\overline{\Omega}}\int_{B_{3R}(0)}\int_{B_{3R}(0)}\int_{|p|=1}\|v-z\|\,|u(x,v')|\,|u_n(x,z')-u(x,z')|dpdzdvdx\\
&\quad+\int_{\overline{\Omega}}\int_{B_{3R}(0)}\int_{B_{3R}(0)}\int_{|p|=1}\|v-z\|\,|u(x,z)|\,|u_n(x,v)-u(x,v)|dpdzdvdx\\
&\quad+\int_{\overline{\Omega}}\int_{B_{3R}(0)}\int_{B_{3R}(0)}\int_{|p|=1}\|v-z\|\,|u_n(x,z)-u(x,z)|dpdzdvdx\\
&\le \int_{B_{3R}(0)}\int_{|p|=1}R\|v-z\|dpdz\int_{\overline{\Omega}}\int_{B_{3R}(0)}|u_n(x,v')-u(x,v')|dvdx\\
&\quad+\int_{B_{3R}(0)}\int_{|p|=1}R\|v-z\|dpdv\int_{\overline{\Omega}}\int_{B_{3R}(0)}|u_n(x,z')-u(x,z')|dzdx\\
&\quad+\int_{B_{3R}(0)}\int_{|p|=1}R\|v-z\|dpdz\int_{\overline{\Omega}}\int_{B_{3R}(0)}|u_n(x,v)-u(x,v)|dvdx\\
&\quad+\int_{B_{3R}(0)}\int_{|p|=1}R\|v-z\|dpdv\int_{\overline{\Omega}}\int_{B_{3R}(0)}|u_n(x,z)-u(x,z)|dzdx
\end{aligned}$$

Making the change of variables $v'\to v$ y $z'\to z$, whose Jacobians are 1, then:

$$\begin{aligned}
&\Big\|Q(u_n,u_n)-Q(u,u)\Big\|_{L^1(\Omega\times B_{3R}(0))}\\
&\quad\le 2R\left[\int_{B_{3R}(0)}\int_{|p|=1}\|v-z\|dpdz+\int_{B_{3R}(0)}\int_{|p|=1}\|v-z\|dpdv\right]\|u_n-u\|_{L^1(\Omega\times B_{3R}(0))}\\
&\quad\le 2R\left[\int_{B_{3R}(0)}\int_{|p|=1}\|v-z\|dpdz+\int_{B_{3R}(0)}\int_{|p|=1}\|v-z\|dpdv\right]\|u_n-u\|_{E}\\
&\quad\le 4R\max\left\{\int_{B_{3R}}\int_{|p|=1}\|(v-z)\|dpdz,\ \int_{B_{3R}}\int_{|p|=1}\|(v-z)\|dpdv\right\}\|u_n-u\|_{E}.
\end{aligned}$$

Hence $Q(u_n,u_n)\in B_E(Q(u,u),r^*R)$. ☐

Lemma 2.3. *If* $u_n\in B_E(0,R)\cap B_E(u,R)$, $u\in B_E(0,R)$, *and* $v_i\dfrac{\partial u_n}{\partial x_i}\in B_E\left(v_i\dfrac{\partial u}{\partial x_i},\ R^{**}\right)$, *with* $r^{**}=r*R+R+nR^{**}\ge 0$ *such that*

$$0<\int_{B_{3R}}\int_{|p|=1}\|(v-z)\|dpdz<\infty \quad \textit{and} \quad 0<\int_{B_{3R}}\int_{|p|=1}\|(v-z)\|dpdv<\infty,$$

then $J(u_n)\in B_E(J(u),r^{**})$

Proof.
$$|J(u_n)-J(u)| = |u_n+v.\nabla_x u_n - Q(u_n,u_n) - u - v.\nabla_x u + Q(u,u)|$$
$$\leq |u_n-u| + |Q(u_n,u_n)-Q(u,u)| + |v.\nabla_x u_n - v.\nabla_x u|, \quad \text{luego:}$$

$$\begin{aligned} \left\|J(u_n)-J(u)\right\|_{L^1(\Omega\times B_{3R}(0))} &\leq \left\|u_n-u\right\|_{L^1(\Omega\times B_{3R}(0))} + \left\|Q(u_n,u_n)-Q(u,u)\right\|_{L^1(\Omega\times B_{3R}(0))} \\ &+ \left\|\sum_{i=1}^{n} v_i \frac{\partial}{\partial x_i}(u_n-u)\right\|_{L^1(\Omega\times B_{3R}(0))} \end{aligned} \tag{2.3}$$

now calculating $\dfrac{\partial J(u)}{\partial x_i}$, we obtain that:

$$\begin{aligned} \|J(u_n)-J(u)\|_E &\leq \|u_n-u\|_E + \|Q(u_n,u_n)-Q(u,u)\|_E + \sum_{i=1}^{n}\left\|v_i\frac{\partial}{\partial x_i}(u_n-u)\right\|_E \\ &\leq r^*R + R + nR^{**} = r^{**} \end{aligned}$$

therefore $J(u_n) \in B_E(J(u), r^{**})$. □

Lemma 2.4. *The operator $J : \overline{U} \longrightarrow C$ is compact. Dunford-Pettis Criterion will be applied, see [8], in fact:*

i) $\displaystyle\int_\Omega \Big|J(u)\Big| du \leq \int_\Omega 2R\, du = 2R\, m(\Omega) \leq 2R\delta$, *defining* $\varepsilon = 2R\delta$, *then the existence of* δ, *such that if* $m(\Omega) \leq \delta$, *then* $\displaystyle\int_\Omega \Big|J(u)\Big| du \leq \varepsilon$.

ii) Given $\varepsilon^* > 0$, *exists a closed,* $F \subset \Omega$ *such that if* $m(F) < \infty$, *then* $\displaystyle\int_{\Omega-F} \Big|J(u)\Big| du \leq 2R\, m(\Omega - F) \leq 2R\varepsilon^*$, *if we defined* $2R\varepsilon^* \leq \varepsilon$, *then* $\displaystyle\int_{\Omega-F} \Big|J(u)\Big| du \leq \varepsilon$.

Lemma 2.5. *For every $u \in \partial U$ you have $u = J(u)$. In fact, if $u \in \partial U$, then $u = e^{-|v|^2}$, and:*

$$J(u) = u + v.\nabla_x u - Q(u,u) = u = e^{-|v|^2}$$

then for every $u \in \partial U$ *y* $\lambda \in (0,1)$ *must be* $u \neq \lambda J(u)$.

Therefore the nonlinear alternative Leray-Schauder type, see [1], page 48, we conclude that there is a fixed point of the operator J a solution resulting from (1.1).

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