CUBO A Mathematical Journal
Vol.16, Nº 01, (21–35). March 2014

Weighted pseudo almost automorphic solutions of fractional functional differential equations

Syed Abbas
School of Basic Sciences,
Indian Institute of Technology Mandi,
Mandi, H.P. - 175001, India.
sabbas.iitk@gmail.com, abbas@iitmandi.ac.in

ABSTRACT

In this paper we discuss the existence of weighted pseudo almost automorphic solution of fractional order functional differential equations. Using the fixed point theorem we establish existence and uniqueness of solution to the problem under consideration. The results obtained extend the theory of *almost automorphic solutions* to a more general class of *weighted pseudo almost automorphic solutions.* These extensions allow to treat infinite dimensional dynamics such as fractional wave and heat equation which are presented in the paper. At the end we give several example to illustrate the analytical findings.

RESUMEN

En este artículo discutimos la existencia de una solución seudo casi automórfica con peso de ecuaciones diferenciales funcionales de orden fraccional. Usando el teorema del punto fijo, establecemos la existencia y unicidad de la solución del problema en estudio. Los resultados obtenidos extienden la teoría de soluciones casi automórficas a clases más generales de soluciones seudo casi automórficas con peso. Estas extensiones permiten estudiar dinámicas infinito-dimensional como la onda fraccionaria y la ecuación del calor, las cuales se presentan en este artículo. Al final, mostramos varios ejemplos para ilustrar los resultados analíticos obtenidos.

Keywords and Phrases: Fractional differential equation, Fixed point theorem, Almost automorphic functions, Abstract differential equations.

2010 AMS Mathematics Subject Classification: 34K40, 34K14.

1 Introduction

In this work we consider the following functional differential equations of fractional order $\alpha \in (1,2)$,

$$\begin{aligned} D_t^{\alpha}u(t) &= Au(t) + D_t^{\alpha-1}f(t,u(t),u_t), \quad t \in \mathbb{R}, \\ u(t) &= \phi(t), \ t \in (-\infty,0], \\ u_t(\theta) &= u(t+\theta), \quad \theta \in (-\infty,0], \end{aligned} \tag{1}$$

where $f : \mathbb{R} \times X \times X \to X$, $\phi \in C^0((-\infty,0],\mathbb{R})$ and $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X. With motivation coming from a wide range of engineering and physical applications, fractional differential equations have recently attracted great attention of mathematicians and scientists. This kind of equations is a generalization of ordinary differential equations to arbitrary non integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy and science. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [36]. Fractional differential equations find numerous applications in the field of visco-elasticity, feedback amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modelling encompassing different branches of physics, chemistry and biological sciences [39]. Many physical processes appear to exhibit fractional order behavior that may vary with time or space. The fractional calculus has allowed the operations of integration and differentiation to any fractional order. The order may take on any real or imaginary value. The existence and uniqueness of solutions to fractional differential equations have been shown by many authors [1, 2, 4, 5, 7, 11, 12, 16, 9, 10, 17, 18, 23, 37, 28, 31, 32, 37, 39]. Agarwal et. al. [6] have shown the existence of weighted pseudo almost periodic solutions of semilinear fractional differential equations.

Since Bohr [15] introduced the concept of almost periodic functions, there have been many important generalizations of this functions in the past few decades. The generalization includes pseudo almost periodic functions [41], where the function can be decomposable in two part. These functions are further generalized to weighted pseudo almost periodic function by Diagana, where the weighted mean of the second component is zero [20]. Another direction of generalization is almost automorphic functions introduced by Bochner [14]. The pseudo almost automorphic functions are natural generalization of almost automorphic functions [14] and introduced by Liang et. al. [33]. These functions are further generalized by Blot et.al. [13] and named weighted pseudo almost automorphic. The authors in [13] have proved very important properties of these functions including composition theorem and completeness property. The study of weighted pseudo almost automorphic solutions of various kind of differential equations are very new and an attractive area of research. For more details on theory and applications of these functions we refer to [13] and references therein. The existence and uniqueness of almost automorphic and pseudo almost automorphic solutions have been established by many authors, for example [3, 13, 26] and references therein.

The problem considered in this work is motivated by the work of Claudio Cuevas and Carlos Lizama [17] work in which they have considered the following fractional differential equations

$$D_t^{\alpha}u(t) = Au(t) + D_t^{\alpha-1}f(t, u(t)), \quad t \in \mathbb{R}, \tag{2}$$

and proved the existence of almost automorphic solutions under certain assumptions. In this paper we discuss existence and uniqueness of weighted pseudo almost automorphic solutions of problem (1). The concept of Stepanov like pseudo almost periodicity is introduced by Diagana [21, 22], which is a generalization of pseudo almost periodicity. Further Stepanov like almost automorphy has been introduced by N'Guerekata and Pankov [29].

2 Preliminaries

Denote $B(X)$ be the Banach space of all linear and bounded operators on X endowed with the norm $\|\cdot\|_{B(X)}$ and $\mathcal{C} = \mathcal{C}(\mathbb{R}, X)$ the set of all continuous functions from $\mathbb{R}$ to X.

Let $\mathbb{U}$ the collection of all positive integrable functions $\rho : \mathbb{R} \to \mathbb{R}$. For each $\rho \in \mathbb{U}$ define

$$m(r, \rho) = \int_{-r}^{r} \rho(s)ds.$$

Denote
$\mathbb{U}_\infty$: The set of all $\rho \in \mathbb{U}$ such that $\lim_{r\to\infty} m(r, \rho) = \infty$
$\mathbb{U}_b$: The set of all bounded $\rho \in \mathbb{U}_\infty$ such that $\inf_{t\in\mathbb{R}} \rho(t) > 0$.

Now we state the definitions of weighted almost automorphic functions.

Definition 2.1. A continuous function $f : \mathbb{R} \to X$ is called almost automorphic if for every real sequence (s_n), there exists a subsequence (s_{n_k}) such that

$$g(t) = \lim_{n\to\infty} f(t + s_{n_k})$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty} g(t - s_{n_k}) = f(t)$$

for each $t \in \mathbb{R}$. The set of all almost automorphic functions from $\mathbb{R}$ to X are denoted by $AA(X)$.

The set of all almost automorphic functions from $\mathbb{R}$ to X are denoted by $AA(X)$ and it is a Banach space equipped with the sup norm

$$\|f\|_\infty = \sup_{t\in\mathbb{R}} \|f(t)\|.$$

Definition 2.2. A continuous function $f : \mathbb{R} \times X \to \mathbb{R}$ is called almost automorphic in t uniformly for x in compact subsets of X if for every compact subset K of X and every real sequence (s_n), there exists a subsequence (s_{n_k}) such that

$$g(t, x) = \lim_{n\to\infty} f(t + s_{n_k}, x)$$

is well defined for each $t \in \mathbb{R}$, $x \in K$ and

$$\lim_{n\to\infty} g(t - s_{n_k}, x) = f(t, x)$$

for each $t \in \mathbb{R}$, $x \in K$. Denote by $AA(\mathbb{R} \times X)$ the set of all such functions.

We denote by

$$AA_0(X) = \left\{ f \in BC(\mathbb{R}, X) : \lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \rho(\xi)\|f(\xi)\| d\xi = 0 \right\},$$

and by $AA_0(\mathbb{R} \times X \times X, X)$ the set of all continuous functions $f : \mathbb{R} \times X \times X \to X$ such that $f(., u, \phi) \in AA_0(X)$ and

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \rho(\xi)\|f(\xi, u, \phi)\| d\xi = 0,$$

uniformly in $(u, \phi) \in X \times X$.

Definition 2.3. *A mapping $f \in BC(\mathbb{R}, X)$ is called weighted pseudo almost automorphic if it can be written as $f = f_1 + f_2$, where $f_1 \in AA(X)$ and $f_2 \in AA_0(X)$.*

The functions f_1 and f_2 are called the almost automorphic and the weighted ergodic perturbation components of f respectively. The set of all such functions will be denoted by $PAA(X)$.

Remark 2.4. A classical example of pseudo almost automorphic function is

$$f(t) = \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t} + \frac{1}{1 + t^2}. \qquad t \in \mathbb{R}.$$

One can easily see that this function is not almost periodic.

Example: Consider the function

$$f(t) = \sin \frac{1}{2 + \cos t + \cos \sqrt{2}t} + e^{\alpha t}$$

It is well known that the function $\sin \frac{1}{2+\cos t+\cos \sqrt{2}t}$ is almost automorphic. Now consider the weight function ρ defined by $\rho(t) = 1 \quad t < 0$ and $\rho(t) = e^{-\beta t} \quad t \geq 0$ for some $\beta > 0$. It is easy to verify that

$$m(r,\rho) = \int_{-r}^{r} \rho(t)dt = \int_{-r}^{0} \rho(t)dt + \int_{0}^{r} \rho(t)dt = r + \frac{1 - e^{-\beta r}}{\beta}.$$

Thus $\lim_{r\to\infty} m(r,\rho) = \infty$ which implies that $\rho \in \mathbb{U}_\infty$. Further

$$\begin{aligned} \int_{-r}^{r} e^{\alpha t}\rho(t)dt &= \int_{-r}^{0} e^{\alpha t}dt + \int_{0}^{r} e^{\alpha t}e^{-\beta t}dt \\ &= \frac{1 - e^{-\alpha r}}{\alpha} + \int_{0}^{r} e^{(\alpha-\beta)t}dt \\ &= \frac{1 - e^{-\alpha r}}{\alpha} + \frac{e^{(\alpha-\beta)r} - 1}{\alpha - \beta}. \end{aligned} \tag{3}$$

Thus for $\alpha \leq \beta$, we have

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} e^{\alpha t}\rho(t)dt = 0.$$

Hence $e^{\alpha t} \in PAA_0(\mathbb{R},\rho)$ and so $f(t) \in WPAA(\mathbb{R})$. It is also interesting to note that

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} e^{\alpha t}dt = \lim_{r\to\infty} \frac{e^{\alpha r} - e^{-\alpha r}}{2\alpha r} = \infty.$$

This implies that $f(t)$ does not belongs to $PAA(\mathbb{R})$, the space of all pseudo almost automorphic functions.

Definition 2.5. *A continuous mapping* $f : \mathbb{R} \times X \times X \to X$ *is called weighted pseudo almost automorphic in* $t \in \mathbb{R}$ *uniformly in* $(x,\phi) \in X \times X$ *if it can be written as* $f = f_1 + f_2$, *where* $f_1 \in AA(\mathbb{R} \times X \times X, X)$ *and* $f_2 \in AA_0(\mathbb{R} \times X \times X, X)$.

We denote the set of all weighted pseudo almost automorphic functions $f : \mathbb{R} \times X \times X \to X$ by $WPAA(\mathbb{R} \times X \times X)$.

The following theorems are from [13].

Theorem 2.6. *The decomposition of a weighted pseudo almost automorphic function is unique for any* $\rho \in U_b$.

Theorem 2.7. *Let* $WPAA(\mathbb{R},\rho) \ni f = g + \phi$ *where* $\rho \in U_\infty$ *and assume that* $f(t,u)$ *is uniformly continuous in any bounded subset* K *of* X *uniformly in* $t \in \mathbb{R}$ *and* $g(t,u)$ *is uniformly continuous in any bounded subset* K *of* X *uniformly in* $t \in \mathbb{R}$. *Then if* $u \in WPAA(\mathbb{R},\rho)$, *implies* $f(\cdot,u(\cdot)) \in WPAA(\mathbb{R},\rho)$.

The above theorem holds if both functions f, g are Lipschitz continuous in u uniformly in $t \in \mathbb{R}$. The weight one functions that is $\rho = 1$, are called pseudo almost automorphic.

3 Weighted pseudo almost automorphic solutions

Assumptions: Let us consider the the following assumptions:

(A1) The function $f : \mathbb{R} \times X \times X \to X$ is weighted pseudo almost automorphic with respect to t uniformly in $(u,\phi) \in X \times X$, and there exists $0 < L < 1$, such that

$$\|f(t,u,\phi) - f(t,v,\psi)\| \leq L(\|u - v\| + \|\phi - \psi\|.$$

(A2) The function f is bounded.

Lemma 3.1. *Let* $\{S(t)\}_{t>0} \subset B(X)$ *be a strongly continuous family of bounded and linear operators such that* $\|S(t)\| \leq \phi(t)$ *for almost all* $t \in \mathbb{R}^+$ *with* $\phi \in L^1(\mathbb{R}^+)$. *If* $f : \mathbb{R} \to X$ *is a weighted pseudo almost automorphic function then* $\int_{-\infty}^{t} S(t-s)f(s)ds \in WPAA(X)$.

A closed and linear operator A is said to be sectorial of type ω and angle θ if there exists $0 < \theta < \frac{\pi}{2}, M > 0$ and $\omega \in \mathbb{R}$ such that its resolvent exists outside the sector

$$\omega + S_\theta := \{\omega + \lambda : \lambda \in \mathbb{C}, \ |arg(-\lambda)| < \theta\},$$

and

$$\|(\lambda - A)^{-1}\| \leq \frac{M}{|\lambda - \omega|}, \ \lambda \notin \omega + S_\theta.$$

Sectorial operators are well studied in the literature. For a recent reference including several examples and properties we refer the reader to [30]. Note that an operator A is sectorial of type ω if and only if $\omega I - A$ is sectorial of type 0.

The equation 1 can be thought as a limiting case of the following equation

$$v'(t) = \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} Av(s)ds + f(t, u(t), u_t), \ t \geq 0, \quad v_t(\theta) = \phi(t), t \in (-\infty, 0), \tag{4}$$

in the sense that the solutions are asymptotic to each other as $t \to \infty$. If we consider that the operator A is sectorial of type ω with $\theta \in [0, \pi(1 - \frac{\alpha}{2}))$, then problem 4 is well posed [19]. Thus we can use variation of parameter formulae to get

$$v(t) = S_\alpha(t)u_0 + \int_0^t S_\alpha(t-s)f(s, u(s), u_s)ds, \quad t \geq 0,$$

where

$$S_\alpha(t) = \frac{1}{2\pi i}\int_\gamma e^{\lambda t}\lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1} d\lambda, \quad t \geq 0,$$

where the path γ lies outside the sector $\omega + S_\theta$. If $S_\alpha(t)$ is integrable then the solution is given by

$$u(t) = \int_{-\infty}^t S_\alpha(t-s)f(s, u(s), u_s)ds.$$

Now one can easily see that

$$v(t) - u(t) = S_\alpha(t)u_0 - \int_t^\infty S_\alpha(s)f(t-s, u(t-s), u_{t-s}).$$

Hence for $f \in L^p(\mathbb{R}^+ \times X \times X, X), \ p \in [1, \infty)$ we have $v(t) - u(t) \to 0$ as $t \to \infty$.

Definition 3.2. *A function* $u : \mathbb{R} \to X$ *is said to be a mild solution to 1 if the function* $S_\alpha(t - s)f(s, u(s), u_s)$ *is integrable on* $(-\infty, t)$ *for each* $t \in \mathbb{R}$ *and*

$$u(t) = \int_{-\infty}^t S_\alpha(t-s)f(s, u(s), u_s)ds,$$

for each $t \in \mathbb{R}$.

Recently, Cuesta in [19], Theorem-1, has proved that if A is a sectorial operator of type $\omega < 0$ for some $M > 0$ and $\theta \in [0, \pi(1 - \frac{\alpha}{2}))$, then there exists $C > 0$ such that

$$\|S_\alpha(t)\| \leq \frac{CM}{1 + |\omega| t^\alpha}$$

for $t \geq 0$. Also the following relation [17], Theorem-3.4 holds,

$$\int_0^\infty \frac{dt}{1 + |\omega| t^\alpha} = \frac{|\omega|^{\frac{-1}{\alpha}} \pi}{\alpha \sin \frac{\pi}{\alpha}}$$

for $\alpha \in (1, 2)$.

Define the operator

$$Fu(t) = \int_{-\infty}^t S_\alpha(t-s) f(s, u(s), u_s) ds, \quad t \in \mathbb{R}.$$

First thing we observe about the operator F is boundedness and continuity. Indeed,

$$\begin{aligned}
\|Fu\| \quad &\leq \int_{-\infty}^t \|S_\alpha(t-s)\| \times \|f(s, u(s), u_s)\| ds \\
&\leq \int_0^\infty \|S_\alpha(s)\| \|f(t-s, u(t-s), u_{t-s})\| ds \\
&\leq CM \int_0^\infty \frac{1}{1 + |\omega| s^\alpha} \|f(t-s, u(t-s), u_{t-s})\| ds \\
&\leq CM \|f\| \int_0^\infty \frac{1}{1 + |\omega| s^\alpha} ds = \frac{CM \|f\| \omega^{-\frac{1}{\alpha}} \pi}{\alpha \sin \frac{\pi}{\alpha}}
\end{aligned} \tag{5}$$

Thus F is bounded. Further, we have

$$\begin{aligned}
&\|Fu(t+h) - Fu(t)\| \\
&= \left\| \int_{-\infty}^{t+h} S_\alpha(t+h-s) f(s, u(s), u_s) ds - \int_{-\infty}^t S_\alpha(t-s) f(s, u(s), u_s) ds \right\| \\
&\leq \int_{-\infty}^t \|S_\alpha(t-s)\| \times \|f(s+h, u(s+h), u_{s+h}) - f(s, u(s), u_s)\| ds \\
&\leq \int_0^\infty \|S_\alpha(s)\| \times \|f(t-s+h, u(t-s+h), u_{t-s+h}) - f(t-s, u(t-s), u_{t-s})\| ds \\
&\leq CM \sup_{t \in \mathbb{R}} \|f(t-s+h, u(t-s+h), u_{t-s+h}) - f(t-s, u(t-s), u_{t-s})\| \\
&\times \int_0^\infty \frac{1}{1 + |\omega| s^\alpha} ds \\
&= \frac{CM \omega^{-\frac{1}{\alpha}} \pi}{\alpha \sin \frac{\pi}{\alpha}} \sup_{t \in \mathbb{R}} \|f(t-s+h, u(t-s+h), u_{t-s+h}) - f(t-s, u(t-s), u_{t-s})\|,
\end{aligned} \tag{6}$$

which goes to zero as $h \to 0$ and hence F is continuous.

It is easy to see that the operator F maps $WPAA(X)$ to $WPAA(X)$, which we represent in the form of a lemma as follows.

Lemma 3.3. *The operator* F *maps* $WPAA(X)$ *to* $WPAA(X)$ *if* $f \in WPAA(X)$.

Proof: As $f \in WPAA(X)$, we can decompose it into two part $f_1 \in AA(X)$ and $f_2 \in AA_0(X)$. Now define the operators

$$F_1u(t) = \int_{-\infty}^{t} S_\alpha(t-s)f_1(s,u(s),u_s)ds, \quad t \in \mathbb{R}$$

and

$$F_2u(t) = \int_{-\infty}^{t} S_\alpha(t-s)f_2(s,u(s),u_s)ds, \quad t \in \mathbb{R}.$$

Also for every sequence t_n there exists a subsequence t_{n_k} such that

$$f_1(t+t_{n_k},u,\psi) \to g_1(t,u,\psi),$$

$$g_1(t-t_{n_k},u,\psi) \to f_1(t,u,\psi), \quad u,\psi \in D,$$

where D is a compact subset of $X \times X$.

$$\begin{aligned} F_1u(t+t_{n_k}) &= \int_{-\infty}^{t+t_{n_k}} S_\alpha(t+t_{n_k}-s)f_1(s,u(s),u_s)ds \\ &= \int_{-\infty}^{t} S_\alpha(t-s)f_1(s+t_{n_k},u(s+t_{n_k}),u_{(s+t_{n_k})})ds \\ &\to \int_{-\infty}^{t} S_\alpha(t-s)g_1(s,u(s),u_s)ds \\ &= (F^*u)(t). \end{aligned} \tag{7}$$

Thus

$$(F_1u)(t+t_{n_k}) \to (F^*u)(t).$$

Similarly one can get

$$(F^*u)(t-t_{n_k}) \to (F_1u)(t).$$

Now we need to show

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \int_{-\infty}^{t} \rho(s)|S_\alpha(t-s)|\|f_2(s,u(s),u_s)\|dsdt = 0.$$

Consider

$$\frac{1}{m(r,\rho)} \int_{-r}^{r} \int_{-\infty}^{t} \rho(s)\|S_\alpha(t-s)\|\|f_2(s,u(s),u_s)\|dsdt \le I_1(r) + I_2(r),$$

where

$$I_1(r) = \frac{1}{m(r,\rho)} \int_{-r}^{r} dt \int_{-r}^{t} \rho(s)\|S_\alpha(t-s)\|\|f_2(s,u(s),u_s)\|ds$$

and

$$I_2(r) = \frac{1}{m(r,\rho)} \int_{-r}^{r} dt \int_{-\infty}^{-r} \rho(s) \|S_\alpha(t-s)\| \|f_2(s, u(s), u_s)\| ds.$$

Thus we have

$$\begin{aligned} I_1(r) \quad \leq \quad & \frac{1}{m(r,\rho)} \int_{-r}^{r} \rho(\xi) \|f_2(\xi, u(\xi), u_\xi)\| d\xi \int_{s}^{r} \|S_\alpha(t-\xi)\| dt \\ & \leq \frac{1}{m(r,\rho)} \int_{-r}^{r} \rho(\xi) \|f_2(\xi, u(\xi), u_\xi)\| d\xi \int_{0}^{r-s} \|S_\alpha(t)\| dt \\ & \leq \frac{1}{m(r,\rho)} \int_{-r}^{r} \rho(\xi) \|f_2(\xi, u(\xi), u_\xi)\| d\xi \int_{0}^{\infty} \|S_\alpha(t)\| dt \\ & \leq \frac{CM}{m(r,\rho)} \int_{-r}^{r} \rho\xi \|f_2(\xi, u(\xi), u_\xi)\| d\xi \frac{|\omega|^{\frac{-1}{\alpha}} \pi}{\alpha \sin \frac{\pi}{\alpha}} \\ & \leq \frac{M_1}{m(r,\rho)} \int_{-r}^{r} \rho(\xi) \|f_2(\xi, u(\xi), u_\xi)\| d\xi, \end{aligned} \tag{8}$$

for some positive constant M_1. The above calculations imply that

$$\lim_{r\to\infty} I_1(r) = 0$$

as $f_2 \in AA_0(\mathbb{R} \times X \times X)$. Now consider

$$\begin{aligned} I_2(r) \quad \leq \quad & \frac{1}{m(r,\rho)} \int_{-r}^{r} dt \int_{t+r}^{\infty} \rho(t-s) \|S_\alpha(s)\| \|f_2(t-s, u(t-s), u_{t-s})\| ds \\ & \leq \frac{1}{m(r,\rho)} \int_{-r}^{r} dt \int_{2r}^{\infty} \rho(t-s) \|S_\alpha(s)\| \|f_2(t-s, u(t-s), u_{t-s})\| ds \\ & \leq \|f_2\|_\infty \int_{2r}^{\infty} \|S_\alpha(s)\| ds. \end{aligned} \tag{9}$$

From the above analysis we get

$$\lim_{r\to\infty} I_2(r) = 0.$$

Thus we have

$$\lim_{r\to\infty} \frac{1}{2r} \int_{-r}^{r} \|F_2(u)(t)\| dt = 0.$$

Hence the result is proved.

Theorem 3.4. *Problem (1) has a unique solution in* $WPAA(X)$ *under the assumption* (A_1) *provided that*

$$\frac{2L|\omega|^{\frac{-1}{\alpha}} \pi}{\alpha \sin \frac{\pi}{\alpha}} < 1.$$

Proof: In order to prove that the operator F has a fixed point, consider

$$\begin{aligned}
&\|Fu_1(t) - Fu_2(t)\| \\
&\leq \int_{-\infty}^{t} \|S_\alpha(t-s)\| \|f(s, u_1(s), u_{1s}) - f(s, u_2(s), u_{2s})\| ds \\
&\leq L \int_{-\infty}^{t} \|S_\alpha(t-s)\| \Big(\|u_1(s) - u_2(s)\| + \|u_{1s} - u_{1s}\|_{B(X)} \Big) ds \\
&\leq 2L \|u_1 - u_2\|_\infty \int_0^\infty \|S_\alpha(t)\| dt. \qquad (10)
\end{aligned}$$

Thus for $2L\int_0^\infty |S_\alpha(t)|dt < 1$, the problem (1) has an unique solution. We have mentioned that

$$\int_0^\infty \frac{1}{1 + |\omega| t^\alpha} = \frac{|\omega|^{\frac{-1}{\alpha}} \pi}{\alpha \sin \frac{\pi}{\alpha}}$$

for $\alpha \in (1, 2)$. Thus the above condition reduces to $\frac{2CML|\omega|^{\frac{-1}{\alpha}}\pi}{\alpha \sin \frac{\pi}{\alpha}} < 1$.

Remark 3.5. *One can easily show that for f Stepanov almost automorphic, the problem (1) has a unique stepanov almost automorphic solutions under the same condition as in both Theorems.*

Remark 3.6. *It is to note that for differential equation*

$$\begin{aligned}
\frac{du(t)}{dt} &= Au(t) + f(t, u(t)), \quad t \in \mathbb{R} \\
&u(0) = u_0, \qquad (11)
\end{aligned}$$

where A generates an exponentially stable C_0 semigroup $\{T(t)\}_{t\geq 0}$, we can conclude that, if f if Lipschitz continuous, bounded and Weyl almost automorphic or Weyl pseudo almost automorphic, then there exists a unique Weyl almost automorphic or Weyl pseudo almost automorphic solution accordingly of the problem provided that $\frac{L_f M_1}{\delta} < 1$, where $\|T(t)\| \leq M_1 e^{-\delta t}$ for some $M \geq 1$ and $\delta > 0$.

4 Examples

Example-1: Consider the following fractional order partial differential equation for $\alpha \in (1, 2)$,

$$\begin{aligned}
\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} &= \frac{\partial^2 u(t,x)}{\partial x^2} + \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}(g(t, u(t,x), u(t-\tau, x))), \ \tau > 0, \\
&\quad t \in \mathbb{R}, \ x \in (0, \pi) \\
u(t, 0) &= u(t, \pi) = 0, \quad t \in \mathbb{R}, \\
u(t, x) &= \phi(t, x) \quad t \in [-\tau, 0], \qquad (12)
\end{aligned}$$

where g is a weighted pseudo almost automorphic function in t. Also assume that g satisfies Lipschitz condition in both variable with Lipschitz constant L_g. Using the transformation $u(t)x =$

$u(t,x)$ and $A = \frac{\partial^2}{\partial x^2}$ with

$$D(A) = \Big\{u \in L^2((0,\pi),\mathbb{R}),\ u^{'} \in L^2((0,\pi),\mathbb{R}),\ u^{''} \in L^2((0,\pi),\mathbb{R}), u(0) = u(\pi) = 0\Big\},$$

the above equation can be transform into

$$\frac{d^\alpha u(t)}{dt^\alpha} = Au(t) + \frac{d^{\alpha-1}}{dt^{\alpha-1}} g(t, u(t), u_t(-\tau)), \tag{13}$$

$t \in \mathbb{R}$ and $u(t) = \phi(t)$ $t \in [-\tau, 0]$. It is to note that A generates an analytic semigroup $\{T(t) : t \geq 0$ on X, where $X = L^2((0,\pi),\mathbb{R})$. Further A has discrete spectrum with eigenvalues of the form $-k^2; k \in N$, and corresponding normalized eigenfunctions given by $z_k(x) = (\frac{2}{\pi})^{\frac{1}{2}} \sin(kx)$. As A is analytic, let us assume that it is sectorial of type ω_1 and let the following relation holds

$$\frac{2L_g|\omega_1|^{\frac{-1}{\alpha}}\pi}{\alpha \sin\frac{\pi}{\alpha}} < 1.$$

Thus under all the required assumption on g, the existence of weighted almost automorphic solutions is ensured accordingly.

Example-2: Consider the following fractional order delay relaxation oscillation equation for $\alpha \in (1,2)$,

$$\begin{aligned}
\frac{\partial^\alpha u(t,x)}{\partial t^\alpha} &= \frac{\partial^2 u(t,x)}{\partial x^2} - pu(t,x) + \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}(f(t, u(t,x), u(t-\tau, x))),\ \tau > 0, \\
&\qquad t \in \mathbb{R},\ x \in (0,\pi) \\
u(t,0) &= u(t,\pi) = 0, \quad t \in \mathbb{R}, \\
u(t,x) &= \phi(t,x) \quad t \in [-\tau, 0],
\end{aligned} \tag{14}$$

where $p > 0$ and f is a weighted pseudo almost automorphic function in t. Also assume that f satisfies Lipschitz condition in both variable with Lipschitz constant L_f. Using the transformation $u(t)x = u(t,x)$ and define $Au = \frac{\partial^2 u}{\partial x^2} - pu,\ u \in D(A)$, where

$$D(A) = \Big\{u \in L^2((0,\pi),\mathbb{C}),\ u^{'} \in L^2((0,\pi),\mathbb{C}),\ u^{''} \in L^2((0,\pi),\mathbb{C}), u(0) = u(\pi) = 0\Big\},$$

the above equation can be transform into

$$\frac{d^\alpha u(t)}{dt^\alpha} = Au(t) + \frac{d^{\alpha-1}}{dt^{\alpha-1}} g(t, u(t), u_t(-\tau)), \tag{15}$$

$t \in \mathbb{R}$ and $u(t) = \phi(t)$ $t \in [-\tau, 0]$. It is to note that A generates an analytic semigroup $\{T(t) : t \geq 0$ on X, where $X = L^2((0,\pi),\mathbb{R})$. Hence $pI - A$ is sectorial of type $\omega = -p < 0$. Further A has discrete spectrum with eigenvalues of the form $-k^2; k \in N$, and corresponding normalized eigenfunctions given by $z_k(x) = (\frac{2}{\pi})^{\frac{1}{2}} \sin(kx)$. As A is analytic. Let us assume that

$$\frac{2L_f|\omega|^{\frac{-1}{\alpha}}\pi}{\alpha \sin\frac{\pi}{\alpha}} < 1.$$

Thus under all the required assumption on f, the existence of weighted almost automorphic solutions is ensured accordingly.

Example-3: Consider the following abstract differential equations of fractional order over a complex Banach space $(X, \|\cdot\|)$,

$$\frac{d^{\alpha}u(t)}{dt^{\alpha}} = Au(t) + \frac{d^{\alpha-1}}{dt^{\alpha-1}}(g(t,u(t)) + Ku(t)), \tag{16}$$

$t \in \mathbb{R}$, where

$$Ku(t) = \int_{-\infty}^{t} k(t-s)u(s)ds$$

and $A : D(A) \subset X \to X$ is a linear densely defined operator of sectorial type on a complex Banach space X. We assume that g is weighted pseudo almost automorphic in t uniformly in u and k satisfy $|k(t)| \le Ce^{-bt}$ for some $C, b > 0$. For u weighted pseudo almost automorphic, it is not difficult to see that $Ku(t)$ is also weighted pseudo almost automorphic. Let us assume that g satisfy Lipschitz condition with Lipschitz constant L_g. Now for $u_1, u_2 \in X$, consider

$$\begin{aligned} \|g(t,u_1(t)) - g(t,u_2(t))\| \quad + \quad & \|Ku_1(t) - Ku_2(t)\| \\ & \le L_g\|u_1 - u_2\| + \int_{-\infty}^{t} |k(t-s)||u_1(s) - u_2(s)|ds \\ & \le L_g\|u_1 - u_2\| + \|u_1 - u_2\| \int_0^{\infty} |k(s)|ds \\ & \le L_g\|u_1 - u_2\| + \frac{C}{b}\|u_1 - u_2\| \\ & \le \left(L_g + \frac{C}{b}\right)\|u_1 - u_2\|. \end{aligned} \tag{17}$$

Thus we have

$$\|g(t,u_1) - g(t,u_2)\| + \|Ku_1 - Ku_2\| \le \left(L_g + \frac{C}{b}\right)\|u_1 - u_2\|.$$

Considering $t - s = s_1$ we have

$$Ku(t) = \int_0^{\infty} k(s_1)u(t+s_1) = \int_0^{\infty} k(s)u_t(s).$$

Thus if we take $g_1(t, u(t), u_t) = g(t, u(t)) + Ku(t)$, the above equation is similar to (1). From the above analysis, one can easily see that g_1 satisfies Lipschitz condition with Lipschitz constant $L_g + \frac{C}{b}$. Further assume that A is sectorial of type ω_2 and the following condition hold

$$\frac{2(L_g + \frac{C}{b})|\omega_2|^{\frac{-1}{\alpha}}\pi}{\alpha \sin\frac{\pi}{\alpha}} < 1.$$

One can easily see that for $u \in WPAA(X), Ku(t) \in WPAA(X)$. Thus we can apply our result to ensure the existence of weighted almost automorphic solution for g weighted almost automorphic.

Received: November 2012. Accepted: May 2013.

References

[1] Abbas, S., *Existence of solutions to fractional order ordinary and delay differential equations and applications*, Electron. J. Diff. Equ., Vol. 2011 (2011), No. 09, pp. 1-11.

[2] Abbas, S., Banerjee, M., Momani, S.,*Dynamical analysis of fractional-order modified logistic model*, Comp. Math.Appl., 62 (3), 1098-1104.

[3] Abbas, S., *Pseudo almost automorphic solutions of some nonlinear integro-differential equations*, Comp. Math.Appl., 62 (5), 2259-2272.

[4] Agarwal, R.P., Zhou, Yong, He, Yunyun, *Existence of fractional neutral functional differential equations*, Comp. Math. Appl., 59 (2010) 1095-1100.

[5] Agarwal, R. P.; Benchohra, M.; Hamani, S.; *Boundary value problems for fractional differential equations*, Georgian Math. J., 16, 3 (2009), 401-411.

[6] Agarwal, R. P., Andradec, B., Cuevas, C., *Weighted pseudo almost periodic solutions of a class of semilinear fractional differential equations*, Nonlinear Analysis: RWA, 11 (2010), 3532-3554.

[7] Ahmad, Bashir, Juan J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions* Comp. Math. Appl. 58 (2009) 1838-1843.

[8] Andres, J., Bersant, A. M., Lesnika, K., *On Some Almost-Periodicity Problems in Various Metrics*, Acta Applicandae Mathematicae 65 (2001), 35-57.

[9] Cuevas, Claudio; N'Guérékata, G. M.; Sepulveda, *A. Pseudo almost automorphic solutions to fractional differential and integro-differential equations.* Commun. Appl. Anal. 16 (2012), no. 1, 131-152.

[10] Cuevas, Claudio; Sepulveda, Alex; Soto, Herme, *Almost periodic and pseudo-almost periodic solutions to fractional differential and integro-differential equations.* Appl. Math. Comput. 218 (2011), no. 5, 1735-1745.

[11] El-Sayed, A. M. A.; *On the fractional differential equations.* Appl. Math. Comput., Vol. 49 (1992), no. 2-3, 205-213,

[12] El-Sayed, Ahmed M. A.; *Nonlinear functional-differential equations of arbitrary orders.* Nonlinear Anal. 33 (1998), no. 2, 181-186.

[13] Blot, J., Mophu, G.M., N'Guérékata, G. M., Pennequin, D., *Weightedd pseudo almost automorphic functions and applications to abstarct differential equations*, Nonlinear Anal., Vol.71, (2009), 903-909.

[14] Bochner, S., Neumann, Von, J., *On compact solutions of operational-differential equations. I*, Annals of Mathematics, vol. 36, no. 1 (1935), 255-291.

[15] Bohr, H., *Zur Theorie der fastperiodischen Funktionen I*, Acta Math., 45 (1925), 29-127.

[16] Cao, J., Yang, Q., Huang, Z., *Existence of anti-periodic mild solutions for a class of semilinear fractional differential equations*, Comm. Nonl. Sci. Num. Siml., In press, 2011.

[17] Cuevas, Claudio, Lizama, Carlos, *Almost automorphic solutions to a class of semilinear fractional differential equations*, Appl. Maths. Letters 21 (2008) 1315-1319.

[18] Chen, Anping, Chen, Fulai, Deng, Siqing, *On almost automorphic mild solutions for fractional semilinear initial value problems*, Comp. Math. Appl. 59 (2010) 1318-1325.

[19] Cuesta, E., *Asymptotic behaviour of the solutions of fractional integro-differential equations and some time discretizations*, Discrete Contin. Dyn. Syst. (Supplement) (2007) 277285.

[20] Diagana, T., *Weighted pseudo alomst periodic functions and applications*, Compt. Rendus Math., Vol. 343 (2006), No. 103, 643-646.

[21] Diagana, T., *Stepanov like pseudo alomst periodic functions and their applications to differential equations*, Comm. Math. Anal. Vol. 3 (2007), No. 1, 9-18.

[22] Diagana, T., *Stepanov like pseudo alomst periodic functions and their applications to nonautonomous differential equations*, Nonlinear Anal, TMA, Vol. 69 (2008), No. 12, 4227-4285.

[23] Diethelm, K., *The analysis of fractional differential equations*, Lecture Notes in Mathematics, 2004, Springer Verlag Berlin Heidelberg, 2010.

[24] Mophou, Gisele M.; N'Guérékata, G. M., *On a class of fractional differential equations in a Sobolev space*. Appl. Anal. 91 (2012), no. 1, 1534.

[25] N'Guérékata, G. M., *Existence and uniqueness of almost automorphic mild solutions of some semilinear abstract differential equations*, Semigroup Forum, 69 (2004) 8086.

[26] N'Guérékata, G. M., *Topics in Almost Automorphy*, Springer-Verlag, New York, 2005.

[27] N'Guérékata, G. M., *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Academic, New York, Boston, Moscow, London, 2001.

[28] N'Guérékata, G. M., *A cauchy problem for some abstract differential equation with nonlocal conditions*, Nonlinear Anal. TMA, 70 (2009), 18731876.

[29] N'Guérékata, G. M., Pankov, A., *Stepanov like almost automorphic functions and monotone evolution equations*, Nonlinear Analysis, TMA, 68, 9 (2008), 2658-2667.

[30] Haase, M., *The functional calculus for sectorial operators, in: Operator Theory: Advances and Applications*, vol. 169, Birkhuser Verlag, Basel, 2006.

[31] Ibrahim, Rabha W., Momani, S., *On the existence and uniqueness of solutions of a class of fractional differential equations*, J. Math. Anal. Appl. 334 (2007), 1-10.

[32] Kilbas, A., Srivastava, H., Trujillo, J., *Theory and applications of fractional differential equations*, North Holland Math. Studies, 204, Elsevier Science, 2006.

[33] Liang, Jin, Zhang, Jun, Xiao, Ti-Jun., *Composition of pseudo almost automorphic and asymptotically almost automorphic functions*, J. Math. Anal. Appl., 340, 2(2008) 1493-1499.

[34] Liu, J., N'Guérékata, G. M., N. Van Minh, *Almost automorphic solutions of second order evolution equations*, Appl. Anal., 84 (11) (2005) 11731184.

[35] Long, W., Ding, H. S., *Composition theorem of Stepanov almost periodic functions and Stepanov like pseudo almost periodic functions*, Advances in difference equations, Vol. 2011, Article ID, 654695, 12 pages.

[36] Mainardi, F., *Fractional Calculus: Some basic problems in continuum and statistical mechanics, in Fractals and Fractional Calculus in Continuum Mechanics*, Carpinteri, A. and Mainardi, F. (eds), Springer, New York, 1997.

[37] Mophou, G., N'Guérékata, G. M., *Existence of mild solution for some fractional differential equations with nonlocal conditions*, Semigroup Forum, 79 (2009), 315-322.

[38] Heymans N, Podlubny I., *Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives*, Rheol Acta, 45 (2006), 765-771.

[39] Podlubny, I., *Fractional Differential Equations*, Academic Press, London, 1999.

[40] Weyl, H., *Integralgleichungen und fastperiodische funktionen*, Math Ann., 97 (1927), 338-356.

[41] Zhang, C. Y., *Pseudo almost periodic solutions of some differential equations*, J. Math. Anal. Appl., 181(1), 90 (1994), 62-76.