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On a result of Q. Han, S. Mori and K. Tohge concerning uniquesness of meromorphic functions.

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ABSTRACT

In the paper we prove a result on the uniqueness of meromorphic functions that is related to a result of Q. Han, S. Mori and K. Tohge and is originated from a result of H.Ueda and two subsequent results of G. Brosch.

RESUMEN

En este artículo probamos un resultado de unicidad de funciones meromórficas que se relaciona a un resultado de Q. Han, S. Mori y K. Tohge, y se origina de un resultado de H. Ueda y dos resultados derivados de G. Brosch.

Keywords and Phrases: Meromorphic function, Uniqueness, Weighted sharing.

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1 Introduction, Definitions and Results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$ we say that f and g share the value a CM (counting multiplicities) if f, g have the same a-points with the same multiplicities. If we do not take the multiplicities into account then f, g are said to share the value a IM (ignoring multiplicities). For the standard notations and definitions of the value distribution theory we refer to [5] and [15] . However we require following notations.

Definition 1. *Let k be a positive integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_{k)}(a; f)$ and $\overline{E}_{k)}(a; f)$ the collection of those a-points of f whose multiplicities does not exceed k, with counting multiplicities and with ignoring multiplicities respectively.*

Definition 2. *Let k be a positive integer and $a \in \mathbb{C} \cup \{\infty\}$. Then by $N(r, a; f| \leq k)$ we denote the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than k. By $\overline{N}(r, a; f| \leq k)$ we denote the corresponding reduced counting funcion.*

In an analogous manner we define $N(r, a; f| \geq k)$ and $\overline{N}(r, a; f| \geq k)$.

Also by $N(r, a; f| = k)$ and $\overline{N}(r, a; f| = k)$ we denote respectively the counting function and reduced counting function of those a-points of f whose multiplicities are exactly k.

In 1980 H.Ueda[14]{see also p. 327 [15]}prove the following result.

Theorem A. *[14] Let f and g be nonconstant entire functions sharing $0, 1$ CM, and $a(\neq 0, 1, \infty)$ be a complex number. If $E_{\infty)}(a; f) \subset E_{\infty)}(a; g)$, then f is a bilinear transformation of g.*

Improving Theorem A in 1989 G.Brosch[2] proved the following result.

Theorem B. *[2] Let f and g be two nonconstant meromorphic functions sharing $0, 1, \infty$ CM, and $a(\neq 0, 1, \infty)$ be a complex number. If $\overline{E}_{\infty)}(a; f) \subset \overline{E}_{\infty)}(a; g)$, then f is a bilinear transformation of g.*

Following example shows that in Theorem B the condition $\overline{E}_{\infty)}(a; f) \subset \overline{E}_{\infty)}(a; g)$ cannot be replaced by $\overline{E}_{\infty)}(a; f) \subset \overline{E}_{\infty)}(b; g)$ for $b \neq a, 0, 1, \infty$.

Example 1. *Let $f = e^{2z} + e^{z} + 1$, $g = e^{-2z} + e^{-z} + 1$, $a = \frac{3}{4}$ and $b = 3$. Then f, g share $0, 1, \infty$ CM and $f - a = \frac{1}{4}(2e^{z} + 1)^2$, $g - b = e^{-2z}(1 + 2e^{z})(1 - e^{z})$. So $\bar{E}_{\infty)}(a; f) \subset \bar{E}_{\infty)}(b; g)$ but f is not a bilinear transformation of g.*

Considering the possibility $a \neq b$, G.Brosch[2] proved the following theorem.

Theorem C. *[2] Let f and g be two nonconstant meromorphic functions sharing $0, 1, \infty$ CM, and a, b be two complex numbers such that $a, b \notin \{0, 1, \infty\}$. If $\overline{E}_{\infty)}(a; f) = \overline{E}_{\infty)}(b; g)$, then f is a bilinear transformation of g.*

In 2001 the idea of weighted sharing of values was introduced {cf.[6], [7]} which provides a scaling between IM sharing and CM sharing of values. We now explain this notion in the following definition.

Definition 3. *[11] Let* k *be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a, f)$ *the set of all* a*-points of* f*, where an* a*-point with multiplicity* m *is counted* m *times if* $m \leq k$ *and* $k+1$ *times if* $m > k$*. If* $E_k(a, f) = E_k(a, g)$*, we say that* f, g *share the value* a *with weight* k*.*

The definition means that z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if z_0 is a zero of g with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity $m(> k)$ if and only if z_0 is a zero of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

In 2004 using the idea of weighted value sharing T.C. Alzahari and H.X.Yi [1] improved Theorem C in the following manner .

Theorem D. *[1] Let* f*,* g *be two nonconstant meromorphic functions sharing* $(a_1, 1), (a_2, \infty)$*,* (a_3, ∞)*, where* $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$*, and let* a*,* b *be two finite complex numbers such that* $a, b \notin \{0, 1\}$ *. If* $\bar{E}_{\infty)}(a; f) = \bar{E}_{\infty)}(b; g)$*, then* f *is a bilinear transformation of* g*. Moreover* f *and* g *satisfy exactly one of the following relations:*

(i) $f \equiv g$;

(ii) $fg \equiv 1$;

(iii) $bf \equiv ag$;

(iv) $f + g \equiv 1$;

(v) $f \equiv ag$;

(vi) $f \equiv (1 - a)g + a$;

(vii) $(1 - b)f \equiv (1 - a)g + (a - b)$;

(viii) $(1 - a + g)f \equiv ag$;

(ix) $f\{(b - a)g + (a - 1)b\} \equiv a(b - 1)g$;

(x) $f(g-1) \equiv g$;

The cases (ii) and (v) may occur if $ab = 1$, *cases (iv) and (viii) may occur if* $a + b = 1$, *cases (vi) and (x) may occur if* $ab = a + b$.

Improving Theorem D recently I.Lahiri and P.Sahoo [12] proved the following theorem.

Theorem E. *[12] Let* f, g *be two distinct nonconstant meromorphic functions sharing* $(a_1, 1)$, (a_2, m), (a_3, k), *where* $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$ *and* $(m-1)(mk-1) > (1+m)^2$.*If for two values* $a, b \notin \{0, 1, \infty\}$ *the functions* $f - a$ *and* $g - b$ *share* $(0, 0)$ *then* f, g *share* $(0, \infty)$, $(1, \infty)$, (∞, ∞) *and* $f - a$, $g - b$ *share* $(0, \infty)$. *Also there exists a non-constant entire function* λ *such that* f *and* g *are one of the following forms:*

(i) $f = ae^{\lambda}$ *and* $g = be^{-\lambda}$, *where* $ab = 1$;

(ii) $f = 1 + ae^{\lambda}$ *and* $g = 1 + (1 - \frac{1}{b})e^{-\lambda}$, *where* $ab = a + b$;

(iii) $f = \frac{a}{a+e^{\lambda}}$ *and* $g = \frac{e^{\lambda}}{1-b+e^{\lambda}}$, *where* $a + b = 1$:

(iv) $f = \frac{e^{\lambda}-a}{e^{\lambda}-1}$ *and* $g = \frac{be^{\lambda}-1}{e^{\lambda}-1}$, *where* $ab = 1$;

(v) $f = \frac{be^{\lambda}-a}{be^{\lambda}-b}$ *and* $g = \frac{be^{\lambda}-a}{ae^{\lambda}-a}$, *where* $a \neq b$;

(vi) $f = \frac{a}{1-e^{\lambda}}$ *and* $g = \frac{be^{\lambda}}{e^{\lambda}-1}$, *where* $ab = a + b$;

(vii) $f = \frac{b-a}{(b-1)(1-e^{\lambda})}$ *and* $g = \frac{(b-a)e^{\lambda}}{(a-1)(1-e^{\lambda})}$, *where* $a \neq b$;

(viii) $f = a + e^{\lambda}$ *and* $g = b(1 + \frac{1-b}{e^{\lambda}})$, *where* $a + b = 1$;

(ix) $f = e^{\lambda} - \frac{a(b-1)}{a-b}$ *and* $g = \frac{b(a-1)}{a-b}\{1 - \frac{a(b-1)}{(b-a)e^{\lambda}}\}$, *where* $a \neq b$;

Q.Han, S.Mori and K.Tohge [4] further improved Theorem C, Theorem D, Theorem E and proved the following.

Theorem F. *[4] Let* f *and* g *be two distinct nonconstant meromorphic functions sharing* (a_1, k_1), (a_2, k_2) *and* (a_3, k_3) *for three distinct values* $a_1, a_2, a_3 \in \mathbb{C}\cup\{\infty\}$, *where* $k_1k_2k_3 > k_1 + k_2 + k_3 + 2$. *Furthermore if* $\overline{E}_{k)}(a_4; f) = \overline{E}_{k)}(a_5; g)$ *for values* a_4, a_5 *in* $\mathbb{C}\cup\{\infty\}\backslash\{a_1, a_2, a_3\}$ *and for some positive integer* $k(\geq 2)$, *then* f *is a bilinear transformation of* g.

Example 1 with $a = b = \frac{3}{4}$ shows that the conclusion of Theorem F does not hold for $k = 1$. This suggests that some further investigation is necessary for the case $k = 1$. In the paper we take up this problem and prove the following result.

Theorem 1.1. *Let f, g be two distinct nonconstant meromorphic functions sharing (a_1, k_1), $(a_2, k_2), (a_3, k_3)$ where $a_1, a_2, a_3 \in \mathbb{C} \cup \{\infty\}$ are distinct and $k_1k_2k_3 > k_1 + k_2 + k_3 + 2$. Further let $E_{1)}(a; f) \subset \overline{E}_{\infty)}(b; g)$ for two complex numbers $a, b \notin \{a_1, a_2, a_3\}$ and $E_{1)}(0; f') \subset \overline{E}_{\infty)}(0; g')$. Then f is a bilinear transformation of g.*

If, in particular, $\{a_1, a_2, a_3\} = \{0, 1, \infty\}$, then there exists a non-constant entire function λ such that f and g assume exactly one of the following forms:

(i) $f = ae^{\lambda}$ *and* $g = be^{-\lambda}$ *where* $ab = 1$;

(ii) $f = 1 + ae^{\lambda}$ *and* $g = 1 + (1 - \frac{1}{b})e^{-\lambda}$ *where* $ab = a + b$;

(iii) $f = \frac{a}{a+e^{\lambda}}$ *and* $g = \frac{e^{\lambda}}{1-b+e^{\lambda}}$ *where* $a + b = 1$:

(iv) $f = \frac{e^{\lambda}-a}{e^{\lambda}-1}$ *and* $g = \frac{e^{\lambda}-a}{ae^{\lambda}-a}$ *where* $\overline{E}_{\infty)}(a; f) = \phi$;

(v) $f = \frac{be^{\lambda}-a}{be^{\lambda}-b}$ *and* $g = \frac{be^{\lambda}-a}{ae^{\lambda}-a}$ *where* $a \neq b$;

(vi) $f = \frac{a}{1-e^{\lambda}}$ *and* $g = \frac{ae^{\lambda}}{(1-a)(1-e^{\lambda})}$ *where* $\overline{E}_{\infty)}(a; f) = \phi$;

(vii) $f = \frac{b-a}{(b-1)(1-e^{\lambda})}$ *and* $g = \frac{(b-a)e^{\lambda}}{(a-1)(1-e^{\lambda})}$ *where* $a \neq b$;

(viii) $f = a + e^{\lambda}$ *and* $g = (1 - a)(1 + \frac{a}{e^{\lambda}})$ *where* $\overline{E}(a; f) = \phi$;

(ix) $f = e^{\lambda} - \frac{a(b-1)}{a-b}$ *and* $g = \frac{b(a-1)}{a-b}\{1 - \frac{a(b-1)}{(b-a)e^{\lambda}}\}$ *where* $a \neq b$;

Considering Example 1 we see that the condition $E_{1)}(0; f') \subset \overline{E}_{\infty)}(0, g')$ is essential for Theorem 1.1.

2 Lemmas

In the section we present some necessary lemmas.

Lemma 2.1. *[3] Let f and g share $(0, 0), (1, 0), (\infty, 0)$. Then $T(r, f) \leq 3T(r, g) + S(r, f)$ and $T(r, g) \leq 3T(r, f) + S(r, f)$.*

From this we conclude that $S(r, f) = S(r, g)$. Henceforth we denote either of them by $S(r)$.

Lemma 2.2. *[16] Let* f *and* g *share* $(0, k_1), (1, k_2), (\infty, k_3)$ *and* $f \not\equiv g$*, where* $k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2$*. Then*

$$\overline{N}(r, 0; f \mid\geq 2) + \overline{N}(r, 1; f \mid\geq 2) + \overline{N}(r, \infty; f \mid\geq 2) = S(r).$$

Following can be proved in the line of Theorem 3.2 of [11].

Lemma 2.3. *Let f and g be two distinct nonconstant meromorphic functions sharing* $(0, k_1), (1, k_2)$, (∞, k_3)*, where* $k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2$*. If* $N_0(r) + N_1(r) \geq \lambda T(r, f) + S(r)$ *for some* $\lambda > \frac{1}{2}$*, then f is a bilinear transformation of g and*

$$N_0(r) + N_1(r) = T(r, f) + S(r) = T(r, g) + S(r),$$

where $N_0(r)(N_1(r))$ *denotes the counting function of those simple(multiple) zeros of* $f - g$ *which are not the zeros of* $f(f - 1)$ *and* $\frac{1}{f}$*.*

Lemma 2.4. *[13] Let* f *and* g *be two distinct noncostant meromorphic functions sharing* $(0, 0)$, $(1, 0), (\infty, 0)$*. Further suppose that f is a bilinear transformation of g and* $E_{1)}(a; f) \subset \overline{E}_{\infty)}(b; g)$*, where* $a, b \notin \{0, 1, \infty\}$*. Then there exists a nonconstant entire function* λ *such that f and g assume exactly one of the forms given in Theorem1.1.*

Following can be proved in the line of Lemma 2.4 [13].

Lemma 2.5. *Let* f *and* g *share* $(0, k_1), (1, k_2), (\infty, k_3)$ *and* $f \not\equiv g$*, where* $k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2$*. If f is not a bilinear transformation of g, then for a complex number* $a \notin \{0, 1, \infty\}$ *each of the following holds:*

(i) $N(r, a; f \mid\geq 3) + N(r, a; g \mid\geq 3) = S(r)$*;*

(ii) $T(r, f) = N(r, a; f \leq 2) + S(r)$*;*

(iii) $T(r, g) = N(r, a; g \leq 2) + S(r)$*.*

In the line of Lemma 5 [9] we can prove the following.

Lemma 2.6. *Let* f, g *share* $(0, k_1), (1, k_2), (\infty, k_3)$ *and* $f \not\equiv g$*, where* $k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2$*. If* $\alpha = \frac{f-1}{g-1}$ *and* $\beta = \frac{g}{f}$*, then* $\overline{N}(r, a; \alpha) = S(r)$ *and* $\overline{N}(r, a; \beta) = S(r)$ *for* $a = 0, \infty$*.*

Following is an analogue of Lemma 2.6 [13].

Lemma 2.7. *Let f and g be two distinct meromorphic functions sharing* $(0, k_1), (1, k_2), (\infty, k_3)$*, where* $k_1 k_2 k_3 > k_1 + k_2 + k_3 + 2$*. Then* $T(r, \frac{\alpha^{(p)}}{\alpha}) + T(r, \frac{\beta^{(p)}}{\beta}) = S(r)$*, where p is a positive integer and* α, β *are defined as in Lemma 2.6.*

Using the techniques of [8] and [10] we can prove the following.

Lemma 2.8. *Let* f, g *share* $(0,k_1)$, $(1,k_2)$, (∞,k_3) *and* $f \not\equiv g$, *where* $k_1k_2k_3 > k_1+k_2+k_3+2$. *If* f *is not a bilinear transformation of* g, *then each of the following holds :*

(i) $T(r,f)+T(r,g) = N(r,0;f\mid\leq 1)+N(r,1;f\mid\leq 1)+N(r,\infty;f\mid\leq 1)+N_0(r)+S(r)$,

(ii) $T(r,f) = N(r,0;g'\mid\leq 1)+N_0(r)+S(r)$,

(iii) $T(r,g) = N(r,0;f'\mid\leq 1)+N_0(r)+S(r)$,

(iv) $N_1(r) = S(r)$,

(v) $N_0(r,0;g'\mid\geq 2) = S(r)$,

(vi) $N_0(r,0;f'\mid\geq 2) = S(r)$,

(vii) $\overline{N}(r,0;g'\mid\geq 2) = S(r)$,

(viii) $\overline{N}(r,0;f'\mid\geq 2) = S(r)$,

(ix) $N(r,0;f-g\mid\geq 2) = S(r)$,

(x) $N(r,0;f-g\mid f=\infty) = S(r)$,

where $N_0(r,0;g'\mid\geq 2)(N_0(r,0;f'\mid\geq 2))$ *is the counting function of those multiple zeros of* $g'(f')$ *which are not the zeros of* $f(f-1)$ *and* $N(r,0;f-g\mid f=\infty)$ *is the counting function of those zeros of* $f-g$ *which are poles of f.*

3 Proof of Theorem 1.1

Proof. If necessary considering a bilinear transformation we may choose $\{a_1,a_2,a_3\}=\{0,1,\infty\}$. We now consider the following cases

CASE 1. Let $a=b$. If possible, we suppose that f is not a blinear transformation of g. We put

$$\Phi = \frac{f'(f-a)}{f(f-1)} - \frac{g'(g-a)}{g(g-1)}.$$

Let $\Phi \not\equiv 0$. Since $\Phi = a\frac{\beta'}{\beta}+(1-a)\frac{\alpha'}{\alpha}$, by Lemma 2.7 we get $T(r,\Phi)=S(r)$. Since $E_{1)}(a;f)\subset \overline{E}_{\infty)}(a;g)$ and $E_{1)}(0;f')\subset \overline{E}_{\infty)}(0;g')$, it follows that

$$N(r,a;f\mid\leq 2) \leq 2N(r,0;\Phi) = S(r),$$

which contradicts (ii) of Lemma 2.5. Therefore $\Phi\equiv 0$ and so

$$\frac{f'(f-a)}{f(f-1)} = \frac{g'(g-a)}{g(g-1)} \tag{3.1}$$

If z_0 is a double zero of $g - a$, then from (3.1) we see that z_0 is a common zero of f' and g'.Hence z_0 is a zero of $\frac{\alpha'}{\alpha} = \frac{f'}{f-1} - \frac{g'}{g-1}$. So by (i) of Lemma 2.5 and Lemma 2.7 we get

$$\begin{aligned} N(r, a; g \mid\geq 2) &= 2N(r, 0; \frac{\alpha'}{\alpha}) + S(r) \\ &= S(r). \end{aligned} \tag{3.2}$$

Again if z_1 is a zero of g' which is not a zero of $g(g-1)(g-a)$, then from (3.1) and the hypotheses of the theorem it follows that z_1 is a zero of f' and so of $\frac{\alpha'}{\alpha}$. Hence from Lemma 2.2, Lemma 2.7 and (3.2) we get

$$\begin{aligned} N(r, 0; g' \mid\leq 1) &\leq N(r, a; g \mid\geq 2) + \overline{N}(r, 0; f \mid\geq 2) + \overline{N}(r, 1; f \mid\geq 2) + N(r, 0; \frac{\alpha'}{\alpha}) \\ &= S(r). \end{aligned} \tag{3.3}$$

Now from (ii) and (iv) of Lemma 2.8 and (3.3) we obtain

$$N_0(r) + N_1(r) = T(r, f) + S(r),$$

which is impossible by Lemma 2.3. Therefore f is a bilinear transformation of g and so by Lemma 2.4 f and g take one of the forms (i)-(iv),(vi) and (viii).

CASE 2. Let $a \neq b$. If f is a bilinear transformation of g, then by Lemma 2.4 f and g assume one of the forms $(i) - (ix)$. So we suppose that f is not a bilinear transformation of g. Following two subcases come up for consideration.

Subcase (i) Let $N(r, a; f \mid\geq 2) \neq S(r)$.

We put $\Psi = \frac{f'(f-b)}{f(f-1)} - \frac{g'(g-b)}{g(g-1)}$. Since a double zero of $f - a$ is a zero of f' and so a zero of g', if $\Psi \not\equiv 0$, then we get by Lemma 2.5(i) and Lemma 2.7,

$$N(r, a; f \mid\geq 2) \leq 2N(r, 0; \Psi) + S(r) = S(r)$$

which is a contradiction. Hence $\Psi \equiv 0$ and so

$$\frac{f'(f-b)}{f(f-1)} = \frac{g'(g-b)}{g(g-1)}.$$

This shows that $f - a$ has no simple zero because $E_{1)}(a; f) \subseteq \overline{E}_{\infty)}(b; g)$.

Since $\frac{\alpha'}{\alpha} = \frac{f'}{f-1} - \frac{g'}{g-1}$. and $E_{1)}(0; f') \subseteq \overline{E}_{\infty)}(0; g')$, it follows that a double zero of $f - a$ is a zero of $\frac{\alpha'}{\alpha}$. So by Lemma 2.7 we get $N(r, a; f \mid= 2) \leq 2N(r, 0; \frac{\alpha'}{\alpha}) = S(r)$, which contradicts (ii) of Lemma 2.5.

Subcase (ii) Let $N(r, a; f \mid\geq 2) = S(r)$. Since f is not a bilinear transformation of g, we see that α, β and $\alpha\beta$ are non-constant. Also we note that $f = \dfrac{1-\alpha}{1-\alpha\beta}$ and $g = \dfrac{(1-\alpha)\beta}{1-\alpha\beta}$.

We put $F = (f-a)(1-\alpha\beta) = a\alpha\beta - \alpha + 1 - a$ and $w = \frac{F'}{F}$. Also we note that $F = (f-a)\frac{g-f}{f(g-1)}$. Since by Lemma 2.6 $\overline{N}(r,\infty;F) = S(r)$ and w has only simple poles (if there is any), we get

$$T(r,w) = m(r,w) + N(r,w) = \overline{N}(r,0;F) + S(r). \tag{3.4}$$

Now by Lemma 2.2 and (ix), (x) of Lemma 2.8 we obtain

$$\begin{aligned}\overline{N}(r,0;F \mid\geq 2) &\leq N(r,a;f \mid\geq 2) + N(r,0;f-g \mid\geq 2) + \overline{N}(r,\infty;f \mid\geq 2)\\ &\quad +N(r,0;f-g \mid f=\infty)\\ &= S(r).\end{aligned} \tag{3.5}$$

Hence from (3.4) and (3.5) we get

$$\begin{aligned}T(r,w) &= N(r,0;F \mid\leq 1) + S(r)\\ &= N(r,a;f \mid\leq 1) + N_0(r) + N_2(r) + S(r),\end{aligned} \tag{3.6}$$

where $N_2(r)$ is the counting function of those simple poles of f which are non-zero regular points of $f-g$.

From the definitions of α and β we get

$$\left\{g - \frac{\alpha'\beta}{(\alpha\beta)'}\right\}\left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\right) \equiv \frac{f'(g-f)}{f(f-1)}. \tag{3.7}$$

From (3.7) we see that a simple pole of f which is a non-zero regular point of $f-g$ is a regular point of $\left\{g - \frac{\alpha'\beta}{(\alpha\beta)'}\right\}\left(\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\right)$. Hence it is either a pole of $\frac{\alpha'\beta}{(\alpha\beta)'}$ or a zero of $\frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}$. Therefore by Lemma 2.7 and the first fundamental theorem we get

$$\begin{aligned}N_2(r) &\leq T\left(r, \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\right) + T\left(r, \frac{\alpha'\beta}{(\alpha\beta)'}\right)\\ &\leq T\left(r, \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}\right) + T\left(r, \frac{1}{1+\frac{\alpha\beta'}{\alpha'\beta}}\right)\\ &\leq 2T\left(r, \frac{\alpha'}{\alpha}\right) + 2T\left(r, \frac{\beta'}{\beta}\right) + O(1)\\ &= S(r).\end{aligned}$$

So from (3.6) we get

$$T(r,w) = N(r,a;f \mid\leq 1) + N_0(r) + S(r). \tag{3.8}$$

By (ii) of Lemma 2.5 we get from (3.8)

$$T(r,w) = T(r,f) + N_0(r) + S(r). \tag{3.9}$$

Let

$$\begin{aligned}\tau_1 &= \frac{a-1}{b-1}(\xi - b\delta),\\ \tau_2 &= \frac{1}{2}\cdot\frac{a-1}{b-1}\{\xi' + \xi^2 - b(\delta' + \delta^2)\}\\ \text{and}\quad \tau_3 &= \frac{1}{6}\cdot\frac{a-1}{b-1}\{\xi'' + 3\xi\xi' + \xi^3 - b(\delta'' + 3\delta\delta' + \delta^3)\},\end{aligned}$$

where $\xi = \frac{\alpha'}{\alpha}$ and $\delta = \frac{\alpha'}{\alpha} + \frac{\beta'}{\beta}$. By Lemma 2.7 we see that $T(r, \xi) = S(r)$ and $T(r, \delta) = S(r)$.

If $\tau_1 \equiv 0$, from (3.7) we get

$$(g - b)\delta \equiv \frac{f'(g - f)}{f(f-1)}. \tag{3.10}$$

Since $E_{1)}(a; f) \subset \overline{E}(b; g)$, it follows from (3.10) that a simple zero of $f - a$, which is neither a zero nor a pole of δ, is a zero of $g - b$ and so is a zero of f'. Hence $N(r, a; f \mid\leq 1) = S(r)$, which contradicts (ii) of Lemma 2.5. Therefore $\tau_1 \not\equiv 0$.

Let z_0 be a simple zero of $f - a$ and $\tau_1(z_0) \neq 0$. Then $g(z_0) = b$ and so $\alpha(z_0) = \frac{a-1}{b-1}$ and $\beta(z_0) = \frac{b}{a}$. Expanding F around z_0 in Taylor's series we get

$$-F(z) = \tau_1(z_0)(z - z_0) + \tau_2(z_0)(z - z_0)^2 + \tau_3(z_0)(z - z_0)^3 + O((z - z_0)^4).$$

Hence in some neighbourhood of z_0 we obtain

$$w(z) = \frac{1}{z - z_0} + \frac{B(z_0)}{2} + C(z_0)(z - z_0) + O((z - z_0)^2),$$

where $B = \frac{2\tau_2}{\tau_1}$ and $C = \frac{2\tau_3}{\tau_1} - \left(\frac{\tau_2}{\tau_1}\right)^2$.

We put

$$H = w' + w^2 - Bw - A, \tag{3.11}$$

where $A = 3C - \frac{B^2}{4} - B'$.

Clearly $T(r, A) + T(r, B) + T(r, C) = S(r)$ and since $w = \frac{F'}{F}$ and $F = (f - a)\frac{g - f}{f(g-1)}$, we get by Lemma 2.1 and (3.9) that $S(r, w) = S(r)$.

Let $H \not\equiv 0$. Then it is easy to see that z_0 is a zero of H. So

$$\begin{aligned}N(r, a; f \mid\leq 1) &\leq N(r, 0; H) + S(r)\\ &\leq T(r, H) + S(r)\\ &= N(r, H) + S(r).\end{aligned} \tag{3.12}$$

From (ii) of Lemma 2.5 and (3.12) we get

$$T(r,f) \leq N(r,H) + S(r). \tag{3.13}$$

Let z_1 be a pole of F. Then z_1 is a simple pole of w. So if z_1 is not a pole of A and B, then z_1 is at most a double pole of H. Hence by Lemma 2.6 we get

$$N(r,\infty;H \mid F=\infty) \leq 2\overline{N}(r,\infty;F) + S(r) = S(r), \tag{3.14}$$

where $N(r,\infty;H \mid F=\infty)$ denotes the counting function of those poles of H which are also poles of F.

Let z_2 be a multiple zero of F. Then z_2 is a simple pole of w. So if z_2 is not a pole of A and B, then z_2 is a pole of H of multiplicity at most two. Hence by (3.5) we get

$$N(r,\infty;H \mid F=0,\geq 2) \leq 2\overline{N}(r,0;F \mid \geq 2) + S(r) = S(r), \tag{3.15}$$

where $N(r,\infty;H \mid F=0,\geq 2)$ denotes the counting function of those poles of H which are multiple zeros of F.

Let z_3 be a simple zero of F which is not a pole of A and B. Then in some neighbourhood of z_3 we get $F(z) = (z-z_3)h(z)$, where h is analytic at z_3 and $h(z_3) \neq 0$. Hence in some neightbourhood of z_3 we obtain

$$H(z) = \left(\frac{2h'}{h} - B\right)\frac{1}{z-z_3} + h_1,$$

where $h_1 = \left(\frac{h'}{h}\right)' + \left(\frac{h'}{h}\right)^2 - \frac{Bh'}{h} - A$.

This shows that z_3 is at most a simple pole of H. Since a simple zero of $f-a$ is a zero of H and $N(r,0;F \mid f=t) \leq N(r,0;f-g \mid \geq 2)$ for $t=0,1$ and $F = (f-a)\frac{g-f}{f(g-1)}$, we get from (3.14) and (3.15) in view of (ix) of Lemma 2.8

$$\begin{aligned} N(r,H) &= N(r,\infty;H \mid F=\infty) + N(r,\infty;H \mid F=0) + S(r) \\ &\leq N(r,0;F \mid \leq 1) - N(r,a;f \mid \leq 1) + S(r) \\ &= N_0(r) + N_2(r) + S(r) \\ &= N_0(r) + S(r), \end{aligned} \tag{3.16}$$

where $N(r,0;F \mid f=t)$ denotes the counting function of those zeros of F which are zeros of $f-t$ and $N(r,\infty;H \mid F=0)$ denotes the counting function of those poles of H which are zeros of F

From (3.13) and (3.16) we obtain $T(r,f) \leq N_0(r) + S(r)$, which by (iv) of Lemma 2.8 and

Lemma 2.3 implies a contradiction. Therefore $H \equiv 0$ and so

$$w' + w^2 - Bw - A \equiv 0$$
$$\text{i.e.,} \quad \frac{w'}{w} \equiv \frac{A}{w} - w + B$$
$$\text{i.e.,} \quad F'' \equiv AF + BF'.$$

Since $F' = a(\alpha\beta)' - \alpha'$ and $F'' = a(\alpha\beta)'' - \alpha''$, we get from above

$$K\alpha\beta + L\alpha \equiv A(f - a)(1 - \alpha\beta), \tag{3.17}$$

where $K = a\{\frac{(\alpha\beta)''}{\alpha\beta} - B\frac{(\alpha\beta)'}{\alpha\beta}\}$ and $L = B\frac{\alpha'}{\alpha} - \frac{\alpha''}{\alpha}$.

By Lemma 2.7 we see that $T(r, K) = S(r)$ and $T(r, L) = S(r)$. Since $\alpha\beta = \frac{g(f-1)}{f(g-1)}$ and $\alpha = \frac{f-1}{g-1}$, we get from (3.17)

$$Kg + Lf \equiv \frac{A(f - a)(g - f)}{(f - 1)} \tag{3.18}$$

Let z_0 be a simple zero of $f - a$ which is not a pole of A. Since $E_{1)}(a; f) \subset \overline{E}_{\infty)}(b; g)$, it follows from 3.18 that z_0 is a zero of $bK + aL$. Hence

$$N(r, a; f \mid\leq 1) \leq N(r, 0; bK + aL) + N(r, \infty; A) \equiv S(r),$$

which contradicts (ii) of Lemma 2.5.This proves the theorem. □

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References

[1] T. C. Alzahary and H.X.Yi, *Weighted sharing three values and uniqueness of meromorphic functions*, J. Math.Anal. Appl., Vol. 295 (2004), pp.247-257.

[2] G. Brosch, *Eindeutigkeissätze für Meromorphe Funktionen (Thesis)*, Technical University of Aachen, (1989).

[3] G.G.Gundersen, *Meromorphic functions that share three or four values*, J.London Math. Soc.,Vol.20, No.2 (1979),pp.457-466.

[4] Q.Han, S. Mori and K. Tohge, *On results of H. Ueda and G. Brosch concerning the unicity of meromorphic functions*, J. Math.Anal. Appl., 335 (2007), pp.915-934.

[5] W. K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[6] I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math.J.161 (2001),pp.193-206

[7] I. Lahiri, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl., Vol. 46, No. 3 (2001), pp. 241-253.

[8] I. Lahiri, *Characteristic functions of a meromorphic functions sharing three values with finite weights*, Complex Var.Theory Appl.,50(1) (2005),pp.69-78.

[9] I. Lahiri, and A.Sarkar, *On a uniqueness theorem of Tohge*, Arch. Math. (Basel),84 (2005),pp.461-469.

[10] I. Lahiri, *Sharing three values with small weights*, Hokkaido Math. J., Vol. 36 (2007),pp. 129-142.

[11] I. Lahiri, *Weighted sharing of three values by meromorphic functions*, Hokkaido Math. J., Vol.37 (2008), No.1,pp.41-58.

[12] I. Lahiri, and P.Sahoo, *On a result of G. Brosch*, J. Math. Anal. Appl.,331(1) (2007),pp.532-546.

[13] I. Lahiri, and R.Pal, *On a result of G. Brosch and T. C. Alzahari*, J. Math. Anal. Appl., Vol. 341, Issue 1 (2008), pp. 91-102.

[14] H. Ueda, Unicity theorems for meromorphic or entire functions, Kodai Math. J.,3 (1980),pp. 457-471.

[15] C. C. Yang and H. X. Yi, *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers (2003).

[16] H.X.Yi, *Meromorphic functions with weighted sharing of three values*, Complex Var. Theory Appl., Vol.50 (2005), pp.923-934.