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On centralizers of standard operator algebras with involution

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ABSTRACT

The purpose of this paper is to prove the following result. Let X be a complex Hilbert space, let $\mathcal{L}(X)$ be the algebra of all bounded linear operators on X and let $\mathcal{A}(X) \subset \mathcal{L}(X)$ be a standard operator algebra, which is closed under the adjoint operation. Let $T : \mathcal{A}(X) \to \mathcal{L}(X)$ be a linear mapping satisfying the relation $2T(AA^*A) = T(A)A^*A + AA^*T(A)$ for all $A \in \mathcal{A}(X)$. In this case T is of the form $T(A) = \lambda A$ for all $A \in \mathcal{A}(X)$, where λ is some fixed complex number.

RESUMEN

El propósito de este artículo es probar el siguiente resultado. Sea X un espacio de Hilbert complejo, sea $\mathcal{L}(X)$ el álgebra de todos los operadores lineales acotados sobre X y sea $\mathcal{A}(X) \subset \mathcal{L}(X)$ la álgebra de operadores clásica, la cual es cerrada bajo la operación adjunto. Sea $T : \mathcal{A}(X) \to \mathcal{L}(X)$ una aplicación lineal satisfaciendo la relación $2T(AA^*A) = T(A)A^*A + AA^*T(A)$ para todo $A \in \mathcal{A}(X)$. En este caso, T es de la forma $T(A) = \lambda A$ para todo $A \in \mathcal{A}(X)$, donde λ es un número complejo fijo.

Keywords and Phrases: ring, ring with involution, prime ring, semiprime ring, Banach space, Hilbert space, standard operator algebra, H^*-algebra, left (right) centralizer, two-sided centralizer.

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This research has been motivated by the work of Vukman, Kosi-Ulbl [5] and Zalar [13]. Throughout, R will represent an associative ring with center $Z(R)$. Given an integer $n \geq 2$, a ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $x \mapsto x^*$ on a ring R is called involution if $(xy)^* = y^*x^*$ and $x^{**} = x$ hold for all pairs $x, y \in R$. A ring equipped with an involution is called a ring with involution or *-ring. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. We denote by Q_r and C the Martindale right ring of quotients and the extended centroid of a semiprime ring R, respectively. For the explanation of Q_r and C we refer the reader to [2].

An additive mapping $T : R \to R$ is called a left centralizer in case $T(xy) = T(x)y$ holds for all pairs $x, y \in R$. In case R has the identity element, $T : R \to R$ is a left centralizer iff T is of the form $T(x) = ax$ for all $x \in R$, where a is some fixed element of R. For a semiprime ring R all left centralizers are of the form $T(x) = qx$ for all $x \in R$, where $q \in Q_r$ is some fixed element (see Chapter 2 in [2]). An additive mapping $T : R \to R$ is called a left Jordan centralizer in case $T(x^2) = T(x)x$ holds for all $x \in R$. The definition of right centralizer and right Jordan centralizer should be self-explanatory. We call $T : R \to R$ a two-sided centralizer in case T is both a left and a right centralizer. In case $T : R \to R$ is a two-sided centralizer, where R is a semiprime ring with extended centroid C, then T is of the form $T(x) = \lambda x$ for all $x \in R$, where $\lambda \in C$ is some fixed element (see Theorem 2.3.2 in [2]). Zalar [13] has proved that any left (right) Jordan centralizer on a semiprime ring is a left (right) centralizer.

Let us recall that a semisimple H^*-algebra is a complex semisimple Banach*-algebra whose norm is a Hilbert space norm such that $(x, yz^*) = (xz, y) = (z, x^*y)$ is fulfilled for all $x, y, z \in A$. For basic facts concerning H^*-algebras we refer to [1]. Vukman [10] has proved that in case there exists an additive mapping $T : R \to R$, where R is a 2-torsion free semiprime ring satisfying the relation $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$, then T is a two-sided centralizer. Kosi-Ulbl and Vukman [9] have proved the following result. Let A be a semisimple H^*-algebra and let $T : A \to A$ be an additive mapping such that $2T(x^{n+1}) = T(x)x^n + x^nT(x)$ holds for all $x \in R$ and some fixed integer $n \geq 1$. In this case T is a two-sided centralizer. Recently, Benkovič, Eremita and Vukman [3] have considered the relation we have just mentioned above in prime rings with suitable characteristic restrictions. Kosi-Ulbl and Vukman [9] have proved that in case there exists an additive mapping $T : R \to R$, where R is a 2-torsion free semiprime *-ring, satisfying the relation $T(xx^*) = T(x)x^*$ $(T(xx^*) = xT(x^*))$ for all $x \in R$, then T is a left (right) centralizer. For results concerning centralizers on rings and algebras we refer to [4–13], where further references can be found.

Let X be a real or complex Banach space and let $\mathcal{L}(X)$ and $\mathcal{F}(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $\mathcal{L}(X)$, respectively. An algebra $\mathcal{A}(X) \subset \mathcal{L}(X)$ is said to be standard in case $\mathcal{F}(X) \subset \mathcal{A}(X)$. Let us point out that any standard operator algebra is prime, which is a consequence of a Hahn-Banach theorem. In case X is a real or complex Hilbert space, we denote by A^* the adjoint operator of $A \in \mathcal{L}(X)$. We denote by X^* the dual space of a real or complex Banach space X.

Vukman and Kosi-Ulbl [5] have proved the following result.

Theorem 0.1. *Let R be a 2-torsion free semiprime ring and let $T : R \to R$ be an additive mapping. Suppose that*

$$2T(xyx) = T(x)yx + xyT(x) \tag{1}$$

holds for all $x, y \in R$. In this case T is a two-sided centralizer.

In case we have a *-ring, we obtain, after putting $y = x^*$ in the relation (1), the relation

$$2T(xx^*x) = T(x)x^*x + xx^*T(x).$$

It is our aim in this paper to prove the following result, which is related to the above relation.

Theorem 0.2. *Let X be a complex Hilbert space and let $\mathcal{A}(X)$ be a standard operator algebra, which is closed under the adjoint operation. Suppose $T : \mathcal{A}(X) \to \mathcal{L}(X)$ is a linear mapping satisfying the relation*

$$2T(AA^*A) = T(A)A^*A + AA^*T(A) \tag{2}$$

for all $A \in \mathcal{A}(X)$. In this case T is of the form $T(A) = \lambda A$, where λ is a fixed complex number.

Proof. Let us first consider the restriction of T on $\mathcal{F}(X)$. Let A be from $\mathcal{F}(X)$ (in this case we have $A^* \in \mathcal{F}(X)$). Let $P \in \mathcal{F}(X)$ be a self-adjoint projection with the property $AP = PA = A$ (we also have $A^*P = PA^* = A^*$). Putting P for A in (2) we obtain

$$2T(P) = T(P)P + PT(P).$$

Left multiplication by P in the above relation gives $PT(P) = PT(P)P$. Similarly, right multiplication by P in the above relation leads to $T(P)P = PT(P)P$. Therefore

$$T(P) = T(P)P = PT(P) = PT(P)P. \tag{3}$$

Putting $A + P$ for A in the relation (2) we obtain

$$\begin{aligned} &2T(A^2) + 2T(AA^* + A^*A) + 4T(A) + 2T(A^*) = \\ &\quad = T(A)(A + A^*) + T(A)P + T(P)A^*A + T(P)(A + A^*) + \\ &\quad + (A + A^*)T(A) + PT(A) + AA^*T(P) + (A + A^*)T(P). \end{aligned}$$

Putting $-A$ for A in the above relation and comparing the relation so obtained with the above relation gives

$$\begin{aligned} &2T(A^2) + 2T(AA^* + A^*A) = \\ &\quad = T(A)(A + A^*) + T(P)A^*A + (A + A^*)T(A) + AA^*T(P) \end{aligned} \tag{4}$$

and

$$4T(A) + 2T(A^*) = \\ = T(A)P + PT(A) + T(P)(A + A^*) + (A + A^*)T(P). \tag{5}$$

So far we have not used the assumption of the theorem that X is a complex Hilbert space. Putting iA for A in the relations (4) and (5) and comparing the relations so obtained with the above relations, respectively, we obtain

$$2T(A^2) = T(A)A + AT(A), \tag{6}$$

$$4T(A) = T(A)P + PT(A) + T(P)A + AT(P). \tag{7}$$

Putting A^* for A in the relation (5) gives

$$4T(A^*) + 2T(A) = \\ = T(A^*)P + PT(A^*) + T(P)(A + A^*) + (A + A^*)T(P).$$

Putting iA for A in the above relation and comparing the relation so obtained with the above relation leads to

$$2T(A) = T(P)A + AT(P).$$

Comparing the above relation and (7), we obtain

$$2T(A) = T(A)P + PT(A). \tag{8}$$

Right (left) multiplication by P in the above relation gives $T(A)P = PT(A)P$ and $PT(A) = PT(A)P$, respectively. Hence, $PT(A) = T(A)P$, which reduces the relation (8) to

$$T(A) = T(A)P.$$

From the above relation one can conclude that T maps $\mathcal{F}(X)$ into itself. We therefore have a linear mapping $T : \mathcal{F}(X) \to \mathcal{F}(X)$ satisfying the relation (6) for all $A \in \mathcal{F}(X)$. Since $\mathcal{F}(X)$ is prime, one can conclude, according to Theorem 1 in [10] that T is a two-sided centralizer on $\mathcal{F}(X)$. We intend to prove that there exists an operator $C \in \mathcal{L}(X)$, such that

$$T(A) = CA \tag{9}$$

for all $A \in \mathcal{F}(X)$. For any fixed $x \in X$ and $f \in X^*$ we denote by $x \otimes f$ an operator from $\mathcal{F}(X)$ defined by $(x \otimes f)y = f(y)x, y \in X$. For any $A \in \mathcal{L}(X)$ we have $A(x \otimes f) = (Ax) \otimes f$. Now let us choose such f and y that $f(y) = 1$ and define $Cx = T(x \otimes f)y$. Obviously, C is linear and applying the fact that T is a left centralizer on $\mathcal{F}(X)$, we obtain

$$(CA)x = C(Ax) = T((Ax) \otimes f)y = T(A(x \otimes f))y = T(A)(x \otimes f)y = T(A)x$$

for any $x \in X$. We therefore have $T(A) = CA$ for any $A \in \mathcal{F}(X)$. As T is a right centralizer on $\mathcal{F}(X)$, we obtain $C(AB) = T(AB) = AT(B) = ACB$. We therefore have $[A, C]B = 0$ for any

$A, B \in \mathcal{F}(X)$, whence it follows that $[A, C] = 0$ for any $A \in \mathcal{F}(X)$. Using closed graph theorem one can easily prove that C is continuous. Since C commutes with all operators from $\mathcal{F}(X)$, we can conclude that $Cx = \lambda x$ holds for any $x \in X$ and some fixed complex number λ, which gives together with the relation (9) that T is of the form

$$T(A) = \lambda A \tag{10}$$

for any $A \in \mathcal{F}(X)$ and some fixed complex number λ. It remains to prove that the relation (10) holds on $\mathcal{A}(X)$ as well. Let us introduce $T_1 : \mathcal{A}(X) \to \mathcal{L}(X)$ by $T_1(A) = \lambda A$ and consider $T_0 = T - T_1$. The mapping T_0 is, obviously, additive and satisfies the relation (2). Besides, T_0 vanishes on $\mathcal{F}(X)$. It is our aim to show that T_0 vanishes on $\mathcal{A}(X)$ as well. Let $A \in \mathcal{A}(X)$, let $P \in \mathcal{F}(X)$ be a one-dimensional self-adjoint projection and $S = A + PAP - (AP + PA)$. Such S can also be written in the form $S = (I - P)A(I - P)$, where I denotes the identity operator on X. Since $S - A \in \mathcal{F}(X)$, we have $T_0(S) = T_0(A)$. It is easy to see that $SP = PS = 0$. By the relation (2) we have

$$\begin{aligned}
&T_0(S)S^*S + SS^*T_0(S) = \\
&= 2T_0(SS^*S) = \\
&= 2T_0((S + P)(S + P)^*(S + P)) = \\
&= T_0(S + P)(S + P)^*(S + P) + (S + P)(S + P)^*T_0(S + P) \\
&= T_0(S)S^*S + T_0(S)P + SS^*T_0(S) + PT_0(S).
\end{aligned}$$

We therefore have

$$T_0(S)P + PT_0(S) = 0.$$

Considering $T_0(S) = T_0(A)$ in the above relation, we obtain

$$T_0(A)P + PT_0(A) = 0. \tag{11}$$

Multiplication from both sides by P in the above relation leads to

$$PT_0(A)P = 0.$$

Right multiplication by P in the relation (11) and considering the above relation gives

$$T_0(A)P = 0.$$

Since P is an arbitrary one-dimensional self-adjoint projection, it follows from the above relation that $T_0(A) = 0$ for all $A \in \mathcal{A}(X)$, which completes the proof of the theorem. □

We conclude the paper with the following conjecture.

Conjecture 0.3. *Let R be a semiprime *-ring with suitable torsion restrictions and let $T : R \to R$ be an additive mapping satisfying the relation*

$$2T(xx^*x) = T(x)x^*x + xx^*T(x)$$

for all $x \in R$. In this case T is a two-sided centralizer.

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References

[1] W. Ambrose: *Structure theorems for a special class of Banach algebras*, Trans. Amer. Math. Soc. 57 (1945), 364-386.

[2] K. I. Beidar, W. S. Martindale 3rd, A. V. Mikhalev: *Rings with generalized identities*, Marcel Dekker, Inc., New York, (1996).

[3] D. Benkovič, D. Eremita, J. Vukman: *A characterization of the centroid of a prime ring*, Studia Sci. Math. Hungar. 45 (3) (2008), 379-394.

[4] I. Kosi-Ulbl, J. Vukman: *An equation related to centralizers in semiprime rings*, Glas. Mat. 38 (58) (2003), 253-261.

[5] I. Kosi-Ulbl, J. Vukman: *On centralizers of semiprime rings*, Aequationes Math. 66 (2003), 277-283.

[6] I. Kosi-Ulbl, J. Vukman: *On certain equations satisfied by centralizers in rings*, Internat. Math. J. 5 (2004), 437-456.

[7] I. Kosi-Ulbl, J. Vukman: *Centralizers on rings and algebras*, Bull. Austral. Math. Soc. 71 (2005), 225-234.

[8] I. Kosi-Ulbl, J. Vukman: *A remark on a paper of L. Molnár*, Publ. Math. Debrecen. 67 (2005), 419-421.

[9] I. Kosi-Ulbl, J. Vukman: *On centralizers of standard operator algebras and semisimple* H^*-*algebras*, Acta Math. Hungar. 110 (3) (2006), 217-223.

[10] J. Vukman: *An identity related to centralizers in semiprime rings*, Comment. Math. Univ. Carol. 40 (1999), 447-456.

[11] J. Vukman: *Centralizers of semiprime rings*, Comment. Math. Univ. Carol. 42 (2001), 237-245.

[12] J. Vukman: *Identities related to derivations and centralizers on standard operator algebras*, Taiwan. J. Math. Vol. 11 (2007), 255-265.

[13] B. Zalar: *On centralizers of semiprime rings*, Comment. Math. Univ. Carol. 32 (1991), 609-614.