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Fundamentals of scattering theory and resonances in quantum mechanics

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ABSTRACT

We present the basics of two-body quantum-mechanical scattering theory and the theory of quantum resonances. The wave operators and S-matrix are constructed for smooth, compactly-supported potential perturbations of the Laplacian. The meromorphic continuation of the cut-off resolvent is proved for the same family of Schrödinger operators. Quantum resonances are defined as the poles of the meromorphic continuation of the cut-off resolvent. These are shown to be the same as the poles of the meromorphically continued S-matrix. The basic problems of the existence of resonances and estimates on the resonance counting function are described and recent results are presented.

RESUMEN

Presentamos los conceptos básicos de la teoría de dispersión cuanto-mecánica de dos cuerpos y la teoría de resonancias cuánticas. El operador de ondas y la matriz S se construyen para perturbaciones del potencial suaves y de soporte compacto del Laplaciano. La continuación meromórfica de la resolvente truncada se prueba para la misma familia de operadores de Schrdinger. Las resonancias cuánticas se definen como los polos de la continuación meromórifca de la resolvente truncada. Se muestra que ellas son las mismas que los polos de la matriz S continuada meromórficamente. Los problemas básicos de la existencia de resonancias y las estimaciones de la función de conteo de la resonancia se describen y resultados recientes se presentan.

Keywords and Phrases: Scattering theory, resonances, Schrödinger equation, wave operators, quantum mechanics

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1 Introduction: Schrödinger operators

The purpose of these notes is to present the necessary background and the current state-of-the-art concerning quantum resonances for Schrödinger operators in a simple, but nontrivial, setting. The unperturbed Hamiltonian $H_0 = -\Delta$ is the Laplacian on $L^2(\mathbb{R}^d)$. In quantum mechanics, the Schrödinger operator or Hamiltonian H_0 represents the kinetic energy operator of a free quantum particle. Many interactions are represented by a potential V that is a real-valued function with $V \in L^\infty_0(\mathbb{R}^d)$, the essentially bounded functions of compact support. Occasionally, we need the potential to have some derivatives and this will be indicated. If, for example, the potential $V \in C^\infty_0(\mathbb{R}^d)$, then all the results mentioned here hold true. The perturbed Hamiltonian is $H_V = -\Delta + V$.

A fundamental property shared by both Hamiltonians is self-adjointness. The unperturbed Hamiltonian H_0 is self-adjoint on its natural domain $H^2(\mathbb{R}^d)$, the Sobolev space of order two, which is dense in $L^2(\mathbb{R}^d)$. The self-adjoint operator H_0 is the generator of a one-parameter strongly-continuous unitary group $t \in \mathbb{R} \to U_0(t) = e^{-iH_0 t}$.

The potential V is relatively H_0-bounded with relative bound zero. By the Kato-Rellich Theorem [14, Theorem 13.5], the perturbed operator H_V is self-adjoint on the same domain $H^2(\mathbb{R}^d)$. This self-adjoint operator generates a one-parameter strongly-continuous unitary group $t \in \mathbb{R} \to U_V(t) = e^{-iH_V t}$.

The unitary groups $U_0(t)$ and $U_V(t)$ provide solutions to the initial value problem for the Schrödinger operator in $L^2(\mathbb{R}^d)$. For example, the solution to

$$i\frac{\partial \psi(t)}{\partial t} = H_V \psi(t), \quad \psi(0) = \psi_0 \in H^2(\mathbb{R}^d), \tag{1.1}$$

is formally given by $\psi(t) = U_V(t)\psi_0$. In this way, the unitary group $U_V(t)$ provides the time-evolution of the initial state ψ_0.

Scattering theory seeks to provide a description of the perturbed time-evolution $U_V(t)$ in terms of the simpler (as we will show below) time-evolution $U_0(t)$. Although we will work on the Hilbert space $L^2(\mathbb{R}^d)$, much of scattering theory can be formulated in a more abstract setting. Consequently, we will often write $\mathcal{H}$ for a general Hilbert space.

Suppose we take a state $f \in \mathcal{H}$ and consider the interacting time-evolution $U_V(t)f$. What is the behavior of $U_V(t)f$ as $t \to \pm\infty$? There is one exactly solvable case, although, as we will see, it is not too interesting. Suppose that f is an eigenfunction of H_V with eigenvalue E so that f satisfies the eigenvalue equation $H_V f = Ef$. Then, the time evolution is rather simple since $U_V(t)f = e^{-itE}f$, as is easily verified by differentiation. We do not expect this simple oscillating state to be approximated by the free dynamics so we should eliminate these states from our consideration. Let $\mathcal{H}_{\rm cont}(H_V)$ be the closed subspace of $\mathcal{H}$ orthogonal to the span of all the eigenfunctions of H_V. We will call these states the *scattering states* of H_V. Given $f \in \mathcal{H}_{\rm cont}(H_V)$, can we find a state $f_+ \in \mathcal{H}$ so that as time runs to plus infinity, the state $U_V(t)f$ looks approximately like the free time-evolved state $U_0(t)f_+$? In particular, we ask if given $f \in \mathcal{H}_{\rm cont}(H_V)$, does there

exist a state $f_+ \in \mathcal{H}$ so that

$$U_V(t)f - U_0(t)f_+ \to 0, \quad \text{as} \quad t \to +\infty. \tag{1.2}$$

When it is possible to find such a vector f_+, we have a simpler description of the dynamics $U_V(t)$ generated by H_V in terms of the free dynamics $U_0(t)$ generated by H_0. We can also pose the question concerning the existence of a state f_- so that (1.2) holds for $t \to -\infty$ with f_- replacing f_+.

We understand (1.2) to mean convergence as a vector in $\mathcal{H}$, that is

$$\lim_{t\to+\infty} \|U_V(t)f - U_0(t)f_+\|_{\mathcal{H}} = 0. \tag{1.3}$$

Note that if f_+ is an eigenfunction of H_0 with eigenvalue E, that is $H_0 f_+ = Ef_+$, then $U_0(t)f_+ = e^{-itE}f_+$, we would not expect the limit (1.3) to exist. Hence, we want f_+ to be a state with nontrivial free time evolution. This means that we want f_+ to be a scattering state for H_0, that is, $f_+ \in \mathcal{H}_{\text{cont}}(H_0)$. For our specific example, $H_0 = -\Delta$, there are no eigenfunctions so $\mathcal{H}_{\text{cont}}(H_0) = \mathcal{H}$.

Because the operators $U_0(t)$ and $U_V(t)$ are unitary, the limit in (1.3) is equivalent to

$$\lim_{t\to+\infty} \|f - U_V(t)^* U_0(t)f_+\|_{\mathcal{H}} = 0. \tag{1.4}$$

Since $H_0 = -\Delta$ has no eigenvalues and only continuous spectrum, we expect that the limit

$$\lim_{t\to+\infty} U_V(t)^* U_0(t)f_+ = f, \tag{1.5}$$

if it exists, should exist for all states $f_+ \in \mathcal{H}$. Similarly, we might expect that the limit

$$\lim_{t\to-\infty} U_V(t)^* U_0(t)f_- = f, \tag{1.6}$$

exists for all $f \in \mathcal{H}$. We will prove in section 2 that these limits do exist and define bounded operators $\Omega_\pm(H_V, H_0)$ on $\mathcal{H}$ called the *wave operators* for the pair (H_0, H_V).

If we consider the original problem: Given $f \in \mathcal{H}_{\text{cont}}(H_V)$, find $f_\pm$ so that the limit in (1.2), and the similar limit for $t \to -\infty$, it might seem strange that we consider $\Omega_\pm(H_V, H_0)$ rather than the limit of the operators in the other order, namely, $U_0(t)^* U_V(t)$ on the scattering states of H_V. As we will see, it is much more difficult to prove the existence of the latter limit. Let us consider, however, the inner product $(g, \Omega_\pm(H_V, H_0)f)$ for g in the range of the wave operator $\Omega_\pm(H_V, H_0)$. Using the definition and unitarity of the time evolution groups, we have

$$\begin{aligned}
(g, \Omega_\pm(H_V, H_0)f) &= \lim_{t\to\pm\infty} (g, U_V(t)^* U_0(t)f) \\
&= \lim_{t\to\pm\infty} (U_0(t)^* U_V(t)g, f) \\
&= (\Omega_\pm(H_V, H_0)^* g, f).
\end{aligned} \tag{1.7}$$

Since this holds for all $f \in \mathcal{H}$, it follows that for $g \in$ Ran $\Omega_\pm(H_V, H_0)$,

$$\lim_{t\to\pm\infty} U_0(t)^* U_V(t)g = \Omega_\pm(H_V, H_0)^* g. \tag{1.8}$$

Comparing this to (1.3), it is clear that we obtain the desired states by $f_\pm = \Omega_\pm(H_V, H_0)^* f$. As we will see in Proposition 4, the existence of the strong limits of $U_0(t)^* U_V(t)$ on the scattering states of H_V as $t \to \pm\infty$ is related to asymptotic completeness.

The existence of the wave operators $\Omega_\pm(H_V, H_0)$ allow us to define states $f_\pm$ for any scattering state $f \in \mathcal{H}_{\rm cont}(H_V)$. The map $S : f_- \to f_+$ plays an important role in scattering theory. This map is called the S-operator for the pair (H_0, H_V).

Two technical remarks. 1) The subspace of scattering states $\mathcal{H}_{\rm cont}(H_V)$ is technically the absolutely continuous spectral subspace of H_V (see section 8.1). The unperturbed operator $H_0 = -\Delta$ has spectrum equal to the half-line $[0, \infty)$ and is purely absolutely continuous. In our setting, the perturbed operator H_V has only absolutely continuous spectrum and possibly eigenvalues. In general, it is a difficult task to prove the absence of singular continuous spectrum. There is an orthogonal spectral projector $E_{\rm cont}(H_V)$ so that $\mathcal{H}_{\rm cont}(H_V) = E_{\rm cont}(H_V)\mathcal{H}$. We will use either notation interchangeably. 2) The type of convergence described in (1.5) and (1.6) is called *strong convergence of operators*. We say that a sequence of bounded operators A_n on $\mathcal{H}$ converges *strongly* to $A \in B(\mathcal{H})$ if for all $f \in \mathcal{H}$, we have $\lim_{n\to\infty} A_n f = Af$.

2 Fundamentals of two-body scattering theory

The basic objects of scattering theory are the wave operators and the scattering operator. The crucial property of the wave operators $\Omega_\pm(H_V, H_0)$ is called *asymptotic completeness*. This condition guarantees the unitarity of the scattering operator. On the level of spectral theory, asymptotic completeness means that the restrictions of the operators H_0 and H_V to their absolutely continuous subspaces are unitarily equivalent. From this viewpoint, scattering theory is a tool for studying the absolutely continuous spectral components of the pair (H_0, H_V) of self-adjoint operators. The theory has been developed to a very abstract level and the reader is referred to the references for further details (for example, [32, 45]).

2.1 Wave operators

Another way to write (1.4) is

$$\lim_{t\to\infty} U_V(t)^* U_0(t) f_+ = f, \tag{2.1}$$

so one of our first tasks is to ask whether the limit on the left side of (2.1) exists.

Proposition 1. Suppose that the real-valued potential $V \in L^\infty_0(\mathbb{R}^d)$ and that $d \geq 3$. For any $f \in \mathcal{H}$, the limit

$$\lim_{t\to\infty} U_V(t)^* U_0(t) f \tag{2.2}$$

exists. This limit defines a bounded linear transformation $\Omega_+(H_V, H_0)$ with $\|\Omega_+(H_V, H_0)\| = 1$.

The linear operator $\Omega_+(H_V, H_0)$ is called a *wave operator*. We can also consider the limit in (2.1) as time runs to minus infinity. We introduce another wave operator $\Omega_-(H_V, H_0)$ defined by

$$s - \lim_{t \to -\infty} U_V(t)^* U_0(t) \equiv \Omega_-(H_V, H_0), \tag{2.3}$$

when the strong limit exists. Of course, we can introduce another pair of wave operators by interchanging the order of H_V and H_0. We will consider these wave operators $\Omega_\pm(H_0, H_V)$ in section 2.3 when we discuss asymptotic completeness. We will see that it is much more difficult to prove the existence of these wave operators. We prove Proposition 1 using the classic Cook-Hack method (see, for example, [31, section XI.4]). In the following proof, we drop the Hamiltonians from the notation for the wave operators and simply write $\Omega_\pm$ for the wave operators $\Omega_\pm(H_V, H_0)$.

Proof. 1. The proof of Proposition 1 relies on an explicit estimate for the free propagation given by $U_0(t)$. For any $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, and for $t \neq 0$, we have

$$\|U_0(t)f\|_\infty \leq \frac{C_d}{t^{d/2}} \|f\|_1. \tag{2.4}$$

This estimate is proved (see [1, Lemma 3.12]) using an explicit formula for $U_0(t)f$, $t \neq 0$. For any $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, we have

$$(U_0(t)f)(x) = \left(\frac{1}{4\pi i t}\right)^{d/2} \int_{\mathbb{R}^d} e^{i|x-y|^2/(4t)} \, f(y) \, d^d y. \tag{2.5}$$

This representation is based on the fact that the Fourier transform (see (3.4) and (3.4)) of the action of the free propagation group is

$$(F(U_0(t)f))(k) = e^{-i|k|^2 t} (Ff)(k). \tag{2.6}$$

Formally, formula (2.5) is obtained by computing the inverse Fourier transform. This involves a singular integral:

$$\int_{\mathbb{R}^d} e^{ik\cdot(x-y)} e^{-i|k|^2 t} \, d^d k. \tag{2.7}$$

This integral can be done by first regularizing the integrand by replacing t by $t - i\epsilon$, for $\epsilon > 0$. This results in a Gaussian function of k, and the Fourier transform is explicitly computable. It is also a Gaussian function. One can then take $\epsilon \to 0$ and recover the formula (2.5) since $f \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ guarantees convergence of the integral.

2. Given this result (2.4), we proceed as follows. Let us define $\Omega(t)$ by

$$\Omega(t) \equiv U_V(t)^* U_0(t). \tag{2.8}$$

From this definition, we compute for any $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$

$$\begin{aligned} (\Omega(t) - 1)f &= \int_0^t \frac{d}{ds} U_V(s)^* U_0(s) f \, ds \\ &= i \int_0^t U_V(s)^* V U_0(s) f \, ds. \end{aligned} \tag{2.9}$$

Since $U_0(t)$ maps $L^2(\mathbb{R}^d)$ to itself and $V \in L^\infty_0(\mathbb{R}^d)$, the integral on the right is well-defined. To prove the existence of the limit, consider $0 << t_1 < t_2$ and note that from (2.9) and the estimate (2.4), we have

$$\begin{aligned} \left\| \int_{t_1}^{t_2} U_V(s)^* V U_0(s) f \, ds \right\| &\leq \|V\|_{L^2(\mathbb{R}^d)} \int_{t_1}^{t_2} \|U_0(s) f\|_{L^\infty(\mathbb{R}^d)} \, ds \\ &\leq C_d \|V\|_{L^2(\mathbb{R}^d)} \, \|f\|_1 \int_{t_1}^{t_2} s^{-d/2} \, ds \\ &\leq \tilde{C}_d \|V\|_{L^2(\mathbb{R}^d)} \|f\|_1 (t_1^{1-d/2} - t_2^{1-d/2}). \end{aligned} \tag{2.10}$$

It follows that for $d \geq 3$, we have the bound

$$\|(\Omega(t_2) - \Omega(t_1) f\| \leq \tilde{C}_d \|V\| \|f\|_1 (t_1^{1-d/2} - t_2^{1-d/2}). \tag{2.11}$$

Consequently, for any sequence $t_n \to \infty$, the sequence of vectors $\Omega(t_n)f$ is a norm-convergent Cauchy sequence so $\lim_{t\to\infty} \Omega(t) f \equiv \tilde{f}_+$ exists. We must show that the map $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d) \to \tilde{f}_+$ defines a linear bounded operator. Since $\|\Omega(t_n)f\| \leq \|f\|_{L^2(\mathbb{R}^d)}$, for any t_n, it follows that $\|\tilde{f}_+\| \leq \|f\|$. This defines $\Omega_+ : f \to \tilde{f}_+$ on a dense domain $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. A densely-defined bounded linear operator can be extended to $\mathcal{H}$ without increasing the norm. Finally, one verifies that $s - \lim_{t\to\infty} \Omega(t) = \Omega_+$ by approximating any $g \in \mathcal{H}$ by a sequence in $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and using a triangle inequality argument. □

The simplicity of this proof relies on the estimate (2.4) for the group $U_0(t)$. It is more difficult to consider the strong limit of $U_0(t)^* U_V(t)$ since no general formula is available for $U_V(t)f$.

2.2 Properties of wave operators

The wave operators $\Omega_\pm$ are bounded operators on $\mathcal{H}$ with $\|\Omega_\pm\| = 1$. They satisfy a number of important properties.

First, they are *partial isometries* in the sense that $E_\pm \equiv \Omega_\pm^* \Omega_\pm$ are orthogonal projections. In our case, $E_\pm = I$, the identity operator on $\mathcal{H}$. In the general case, the operator $E_\pm$ is the projection onto the continuous subspace of H_0. For any $f, g \in \mathcal{H}$, we have

$$(\Omega_\pm f, \Omega_\pm g) = (f, E_\pm g) = (E_\pm f, E_\pm g), \tag{2.12}$$

so that

$$\|\Omega_\pm f\| = \|E_\pm f\|. \tag{2.13}$$

It follows that $\Omega_\pm$ are isometries on $E_\pm \mathcal{H}$ and that the kernel of $\Omega_\pm$ is $(1 - E_\pm)\mathcal{H}$. We have that $\Omega_\pm E_\pm = \Omega_\pm$. The subspaces of $\mathcal{H}$ given by $E_\pm \mathcal{H}$ are called the *initial spaces* of the partial isometries $\Omega_\pm$.

Second, the adjoints $\Omega_\pm^*$ are partial isometries. Since $(\Omega_\pm^*)^* \Omega_\pm^* = \Omega_\pm \Omega_\pm^*$, the operator $F_\pm \equiv \Omega_\pm \Omega_\pm^*$ satisfies $F_\pm^2 = \Omega_\pm (\Omega_\pm^* \Omega_\pm) \Omega_\pm^* = \Omega_\pm E_\pm \Omega_\pm^* = F_\pm$, and in a similar manner $F_\pm^* = F_\pm$,

so $F_\pm$ are orthogonal projections. It follows that $F_\pm\Omega_\pm^* = \Omega_\pm^*$ and that $\|\Omega_\pm^* f\| = \|F_\pm f\|$. One can show that $F_\pm$ are the orthogonal projections onto the closed ranges of the wave operators $\mathrm{Ran}\,\Omega_\pm = F_\pm\mathcal{H}$. The subspaces $F_\pm\mathcal{H}$ are called the *final subspaces* of the partial isometries $\Omega_\pm$.

Proposition 2. The wave operators satisfy the following intertwining relations:

$$\begin{aligned} \Omega_\pm U_0(t) &= U_V(t)\Omega_\pm \\ U_0(t)\Omega_\pm^* &= \Omega_\pm^* U_V(t). \end{aligned} \tag{2.14}$$

Proof. These relations follow from the existence of the wave operators and the simple properties of the unitary evolution groups. For any $f \in \mathcal{H}$, we have

$$\begin{aligned} U_V(t)\Omega_+ f &= \lim_{s\to\infty} U_V(t)U_V(s)^* U_0(s) f \\ &= \lim_{s\to\infty} [U_V(s-t)^* U_0(s-t)]U_0(t) f \\ &= \lim_{u\to\infty} [U_V(u)^* U_0(u)]U_0(t) f \\ &= \Omega_+ U_0(t) f, \end{aligned} \tag{2.15}$$

proving the first intertwining relation. The second is proven in the same manner. □

2.3 Asymptotic completeness

The existence of the wave operators $\Omega_\pm(H_V, H_0)$ means the existence of a orthogonal projectors onto the initial space $E_\pm \equiv \Omega_\pm(H_V, H_0)^*\Omega_\pm(H_V, H_0) = I$ and final subspaces $F_\pm \equiv \Omega_\pm(H_V, H_0)\Omega_\pm(H_V, H_0)^*$ that are the ranges of the wave operators $\Omega_\pm(H_V, H_0)$. The range of the wave operators must be contained in the continuous spectral subspace of H_V.

Definition 3. The pair of self-adjoint operators (H_0, H_V) is said to be asymptotically complete if $F_-\mathcal{H} = F_+\mathcal{H} = E_{\mathrm{cont}}(H_V)\mathcal{H}$, that is, if $\mathrm{Ran}\,\Omega_- = \mathrm{Ran}\,\Omega_+ = E_{\mathrm{cont}}(H_V)\mathcal{H}$.

In our situation, with $H_0 = -\Delta$, the spectrum of H_0 is purely absolutely continuous and $E_{\mathrm{cont}}(H_0)\mathcal{H} = \mathcal{H}$. In particular, $E_\pm = 1_{\mathcal{H}}$. Also, neither operator H_0 nor H_V has singular continuous spectrum. In more general situations, one needs to prove that the perturbed operator H_V has no singular continuous spectrum. In these more general cases, the subspace $\mathcal{H}_{\mathrm{cont}}(H_V)$ must be taken as the absolutely continuous spectral subspace.

One can also consider wave operators $\Omega_\pm(H_0, H_V)$ defined by switching the order of the unitary operators in (2.2):

$$\Omega_\pm(H_0, H_V) \equiv s - \lim_{t\to\pm\infty} U_0(-t)U_V(t)E_{\mathrm{cont}}(H_V). \tag{2.16}$$

At first sight, it would seem that the existence of these wave operators would be equivalent to the existence of $\Omega_\pm(H_V, H_0)$. However, we have no explicit control over the dynamics generated by H_V such as formula (2.5). Consequently, it is difficult to use the Cook-Hack method to prove the existence of the wave operators $\Omega_\pm(H_0, H_V)$. In fact, the existence of the wave operators $\Omega_\pm(H_0, H_V)$ is equivalent to asymptotic completeness.

Proposition 4. Suppose that the wave operators $\Omega_\pm(H_V, H_0)$ exist. Then the pair of operators (H_0, H_V) are asymptotically complete if and only if the wave operators $\Omega_\pm(H_0, H_V)$ exist.

Proof. 1. Suppose that both sets of wave operators exist. Then, we know that the projection $E_{\rm cont}(H_V) = \Omega_\pm(H_V, H_V)$. But, we have

$$U_V(-t)U_V(t) = U_V(-t)U_0(t) \cdot U_0(-t)U_V(t), \tag{2.17}$$

from which it follows that

$$\Omega_\pm(H_V, H_V) = \Omega_\pm(H_V, H_0)\Omega_\mp(H_0, H_V). \tag{2.18}$$

This implies that $\mathcal{H}_{\rm cont}(H_V) \subset$ Ran $\Omega_\pm(H_V, H_0)$. Since the existence of $\Omega_\pm(H_V, H_0)$ means that Ran $\Omega_\pm(H_V, H_0) \subset \mathcal{H}_{\rm cont}(H_V)$, these two inclusions mean that Ran $\Omega_+(H_V, H_0) = \Omega_-(H_V, H_0) = \mathcal{H}_{\rm cont}(H_V)$.

2. To prove the other implication, we assume that the wave operators $\Omega_\pm(H_V, H_0)$ exist and are asymptotically complete. Then, for any $\phi \in \mathcal{H}_{\rm cont}(H_V)$, there exists a $\psi \in \mathcal{H}$ so that $\phi = \Omega_+(H_V, H_0)\psi$. This means that $U_0(t)\psi - U_V(t)\phi$ converges to zero as $t \to +\infty$. By unitarity of the operator $U_0(t)$, this means that $\lim_{t\to+\infty} U_0(-t)U_V(t)\phi = \psi$ for all $\phi \in \mathcal{H}_{\rm cont}(H_V)$. This implies the existence of $\Omega_+(H_0, H_V)$. The proof of the existence of the other wave operator is analogous. □

We now turn to proving the existence of the wave operators $\Omega_\pm(H_0, H_V)$. Many methods have been developed over the years in order to do this. The classic result of Birman [31, Theorem XI.10] is perhaps the simplest to apply to our simple two-body situation. There are more elegant and far-reaching methods. The Enss method, in particular, is based on a beautiful phase-space analysis of the scattering process. A thorough account of the Enss method may be found in Perry's book [27]. Perry combined the Enss method with the Melin transform in [26] to present a new, clear, and short proof of asymptotic completeness for two-body systems more general than those considered here. Finally, the problem of asymptotic completeness for N-body Schrödinger operators with short-range, two-body potentials, was solved by Sigal and Soffer [38]. They developed a very useful technique of local decay estimates.

In preparation, we recall that a bounded operator K is in the *trace class* if the following condition is satisfied. The singular values of a compact operator A are given by $\mu_j(A) = \sqrt{\lambda_j(A^*A)}$, where $\{\lambda_j(B)\}$ are the eigenvalues of B. We say that K *is in the trace class* if $\sum_j \mu_j(K) < \infty$. We say that K *is in the Hilbert-Schmidt class* if $\sum_j \mu_j(K)^2 < \infty$. We refer to [29] or [39] for details concerning the von Neumann-Schatten trace ideals of bounded of operators.

Theorem 5. Let $V \in L^\infty_0(\mathbb{R}^d)$ be a real-valued potential and $d \geq 3$. Then the pair $(H_0, H_V)$is asymptotically complete.

Proof. 1. By Proposition 4, it suffices to prove that $\Omega_\pm(H_0, H_V)$ exist since we know from Proposition 1 that the wave operators $\Omega_\pm(H_V, H_0)$ exist. For any interval $I \subset \mathbb{R}$ and self-adjoint operator

A, let $E_I(A)$ denote the spectral projection for A and the interval I. In the first step, we note that

$$E_I(H_0)VE_I(H_V), E_I(H_V)VE_I(H_0) \in \mathcal{I}_1. \tag{2.19}$$

The trace class property of these operators is easily demonstrated by proving that $|V|^{1/2}R_0(i)^k$ is a Hilbert-Schmidt operator for $k > d/2$ and noting that $E_I(H_0)R_0(i)^{-k}$ is a bounded operator.

2. Next, we need the following result called Pearson's Theorem in [31, Theorem XI.7]. Let $a > 0$ and define the bounded operator $J_a \equiv E_{(-a,a)}(H_0)E_{(-a,a)}(H_V)$. The trace class property (2.19) means that $H_0J_a - J_aH_V \in \mathcal{I}_1$. The main result of [31, Theorem XI.7] is that

$$s - \lim_{t\to\pm\infty} U_0(t)^*J_aU_V(t)E_{\text{cont}}(H_V) \tag{2.20}$$

exists. Let $0 < a_0 < a$ and choose $\phi \in E_{(-a_0,a_0)}(H_V)E_{\text{cont}}(H_V)\mathcal{H}$. We then have

$$U_0(t)^*E_{(-a,a)}(H_0)U_V(t)\phi = U_0(t)^*J_aU_V(t)\phi, \tag{2.21}$$

so by (2.20), the strong limit of the term on the left in (2.21) exists.

3. We can now write the expression that gives the wave operator acting on any $\phi \in E_{(-a_0,a_0)}(H_V)E_{\text{cont}}(H_V)\mathcal{H}$:

$$U_0(t)^*U_V(t)\phi = U_0(t)^*[E_{(-a,a)}(H_0) + E_{\mathbb{R}\setminus(-a,a)}(H_0)]U_V(t)\phi. \tag{2.22}$$

Since the strong limit of the first term on the right in (2.22) exists by (2.21), it suffices to prove that

$$\lim_{a\to\infty}\left\{\sup_{t\in\mathbb{R}}\|U_0(t)^*E_{\mathbb{R}\setminus(-a,a)}(H_0)U_V(t)\phi\|\right\} = 0. \tag{2.23}$$

Once this is proven, we can first take $a \to \infty$ and then $a_0 \to \infty$ so that the limit in (2.22) holds for any $\phi \in E_{\text{cont}}(H_V)\mathcal{H}$.

4. To prove (2.23), we need some estimates. Let $f(s) = s^2 + 1 \geq 1$. The fact that V is relatively H_0-bounded means that

$$\|f(H_V)f(H_0)^{-1}\| < C_1 < \infty. \tag{2.24}$$

Next, recall that $\phi \in E_{(-a_0,a_0)}(H_V)\mathcal{H}$, for $0 < a_0 < a$, so that

$$\|f(H_V)U_V(t)\phi\| \leq \sup_{|s|\leq a_0} f(s) = a_0^2 + 1 < \infty. \tag{2.25}$$

Finally, since f is invertible, we have

$$\|f(H_0)^{-1}E_{\mathbb{R}\setminus(-a,a)}(H_0) \leq \left[\inf_{|s|\geq a_0} f(s)\right]^{-1} = (a^2+1)^{-1}. \tag{2.26}$$

Note that this vanishes as $a \to \infty$.

5. Returning to (2.23), we write the norm as

$$\begin{aligned} &\|U_0(t)^* E_{\mathbb{R}\setminus(-a,a)}(H_0) U_V(t)\phi\| \\ &\quad \leq \|U_0(t)^* \cdot f(H_0)^{-1} E_{\mathbb{R}\setminus(-a,a)}(H_0) \cdot f(H_0) f(H_V)^{-1} \cdot f(H_V) U_V(t)\phi\| \\ &\quad \leq \|f(H_0)^{-1} E_{\mathbb{R}\setminus(-a,a)}(H_0)\| \, \|f(H_0) f(H_V)^{-1}\| \, \|f(H_V) U_V(t)\phi\| \\ &\quad \leq C_1 (a_0^2+1)(a^2+1)^{-1}, \end{aligned} \tag{2.27}$$

independently of t. Taking $a \to \infty$ proves (2.23). □

The asymptotic completeness of (H_0, H_V) means that the absolutely continuous parts of each operator are unitarily equivalent. Recall that our condition on the real-valued potential $V \in L^\infty_0(\mathbb{R}^d)$ means that $V(H_0+i)^{-1}$ is compact. By Weyl's Theorem (see, for example, [14, Theorem 14.6]), the essential spectrum of H_V is the same as the essential spectrum of H_0 that is $[0,\infty)$. Hence, the perturbation can add at most a discrete set of isolated eigenvalues with finite multiplicities. The property of asymptotic completeness goes beyond this and establishes the unitary equivalence of the absolutely continuous components.

3 The scattering operator

The existence of the wave operators $\Omega_\pm(H_V, H_0)$ guarantees the existence of the asymptotic states $f_\pm$. For any $f \in \text{ Ran } \Omega_\pm(H_V, H_0) \subset E_{\text{cont}}(H_V)\mathcal{H}$, we have $f_\pm = \Omega_\pm(H_V, H_0)^* f$. The S-operator maps f_- to f_+. It is a bounded operator on $L^2(\mathbb{R}^d)$. Furthermore, the S-operator commutes with the free time evolution $U_0(t)$. This allows for a reduction of the S-operator to a family of operators $S(\lambda)$ defined on $L^2(\mathbb{S}^{d-1})$ called the S-matrix.

3.1 Basic properties of the S-operator

An important use of the wave operators is the construction of the S-operator on $\mathcal{H}$. For any $f \in \text{ Ran } \Omega_\pm(H_V, H_0)$, we have from section 2.1 that $f_\pm = \Omega_\pm(H_V, H_0)^* f$, or, for example, $f = \Omega_- f_-$. As a result, we can compute a formula for the map $f_- \to f_+$ in terms of the wave operators: $Sf_- = f_+ = \Omega_+^* f = \Omega_+^* \Omega_- f_-$. Consequently, the S-operator is defined as the bounded operator

$$S \equiv \Omega_+^* \Omega_- : \mathcal{H} \to \mathcal{H}. \tag{3.1}$$

Proposition 6. Suppose $\text{Ran } \Omega_- \subset \text{ Ran } \Omega_+$. Then, the scattering operator is a partial isometry on $L^2(\mathbb{R}^d)$.

Proof. To prove this, we need to show that S^*S is an orthogonal projection. This follows from the properties of the wave operators:

$$S^*S = (\Omega_+^*\Omega_-)^*(\Omega_+^*\Omega_-) = \Omega_-^*[\Omega_+\Omega_+^*]\Omega_- = \Omega_-^* P_+ \Omega_-. \tag{3.2}$$

Since we assume that Ran Ω_- $\subset$ Ran F_+, we have $F_+\Omega_- = \Omega_-$, so from (3.2), $S^*S = F_-$, an orthogonal projection. If $H_0 = -\Delta$, this operator F_- is the identity operator on $\mathcal{H}$. □

Since Ran $S \subset$ Ran $\Omega_+^* \subset E_{\mathrm{cont}}(H_0)\mathcal{H}$, we have that $S : E_{\mathrm{cont}}(H_0)\mathcal{H} \to E_{\mathrm{cont}}(H_0)\mathcal{H}$. An essential property of the S-operator is that it commutes with the free time evolution, as stated in the following proposition.

Proposition 7. The S-operator commutes with the free time evolution: $[S, U_0(t)] \equiv SU_0(t) - U_0(t)S = 0$. Consequently, the S-operator satisfies $E_0(I)S = SE_0(I)$, where $E_0(I)$ is the spectral projector for H_0 and any Lebesgue measurable $I \subset \mathbb{R}$.

Proof. This follows from the definition $S = \Omega_+^*\Omega_-$ and the intertwining properties (2.14) of the wave operators. We compute:

$$SU_0(t) = \Omega_+^* U_V(t)\Omega_- = (U_V(-t)\Omega_+)^*\Omega_- = (\Omega_+ U_0(-t))^*\Omega_- = U_0(t)S. \tag{3.3}$$

□

It follows from Proposition 7 that for a wide class of reasonable functions ϕ, we have the general result $S\phi(H_0) = \phi(H_0)S$.

The key property of the equality of the ranges of the wave operators (part of asymptotic completeness) has important consequences for the S-operator.

Theorem 8. Suppose that for a pair of self-adjoint operators (H_0, H_V), we have Ran $\Omega_-(H_V, H_0) =$ Ran $\Omega_+(H_V, H_0)$. Then, the S-operator is a unitary operator on $L^2(\mathbb{R}^d)$.

To prove the unitarity of the S-operator, we recall from (3.2) that, in general, $S^*S = \Omega_-^* F_+ \Omega_-$. If Ran Ω_- = Ran Ω_+, we have $F_+\Omega_- = \Omega_-$. Furthermore, under our hypotheses, we have $\Omega_-^*\Omega_- = 1_{L^2(\mathbb{R}^d)}$, so that $S^*S = 1$. As for SS^*, a similar calculation gives $SS^* = \Omega_+^* F_- \Omega_+$. It could happen that Ran Ω_+ is strictly larger that Ran Ω_-. In this case, the kernel of SS^* is nontrivial and consists of any element of Ran Ω_+ orthogonal to Ran Ω_-. In this case, SS^* is not invertible. Our condition that Ran Ω_- = Ran Ω_+ eliminates this possibility and we find $SS^* = \Omega_+^* F_- \Omega_+ = \Omega_+^*\Omega_+ = 1$. Hence, the S-operator S is invertible and $S^{-1} = S^*$.

3.2 The S-matrix

Because the S-operator commutes with spectral family for H_0, both operators admit a simultaneous spectral decomposition. This is achieved with the Fourier transform. We define the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ by

$$(Ff)(k) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ik\cdot x} f(x)\, d^d x. \tag{3.4}$$

The inverse Fourier transform is defined, for any $g \in \mathcal{S}(\mathbb{R}^d)$, by

$$(F^{-1}g)(x) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ik\cdot x} g(k)\ d^d k. \tag{3.5}$$

The Fourier transform extends to a unitary map on $L^2(\mathbb{R}^d)$. Note that for $H_0 = -\Delta$, and $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$(F(H_0 f))(k) = |k|^2 (Ff)(k). \tag{3.6}$$

It is convenient to write $k = \lambda\omega \in \mathbb{R}^d$, where $\lambda \in [0,\infty)$ and $\omega \in \mathbb{S}^{d-1}$. With this decomposition a function $f(k)$ may be viewed as a function on $\mathbb{S}^{d-1}$ parameterized by $\lambda \in [0,\infty)$.

We need a family of maps from $L^2(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$ parameterized by the energy λ. These maps $E_\pm(\lambda)$ can be defined via the Fourier transform (3.4). For $\lambda \in \mathbb{R}$, and any $f \in \mathcal{S}(\mathbb{R}^d)$, we define

$$(E_\pm(\lambda)f)(\omega) \equiv (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{\pm i\lambda x\cdot\omega} f(x)\ d^d x, \quad \omega \in \mathbb{S}^{d-1}. \tag{3.7}$$

The transpose of these maps, ${}^tE_\pm(\lambda) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{R}^d)$.

The formula for the S-matrix involves the resolvent $R_V(\lambda) \equiv (H_V - \lambda^2)$ of H_V. We will study the resolvent in detail in section 4. Provided $\Im\lambda^2 \neq 0$ and $-\lambda^2$ is not an eigenvalue of H_V, the resolvent $R_V(\lambda)$ is a bounded operator. We need to understand the behavior of $VR_V(\lambda + i\epsilon)V$, for $\lambda \in \mathbb{R}$, in the limit as $\epsilon \to 0$. That this limit exists as a compact operator is part of the *limiting absorption principle* that is discussed in section 4.1. We will write $VR_V(\lambda + i0)V$ for this limit. Recall from section 3.2 that the singular values of a compact operator A are given by $\mu_j(A) = \sqrt{\lambda_j(A^*A)}$, where $\{\lambda_j(B)\}$ are the eigenvalues of B, and that K is in the *trace class* if $\sum_j \mu_j(K) < \infty$.

Theorem 9. Assume that the pair (H_0, H_V) is asymptotically complete with $H_0 = -\Delta$. Then the S-matrix is the unitary family of operators $S(\lambda)$, for $\lambda \in \mathbb{R}$, on $L^2(\mathbb{S}^{d-1})$ given by

$$S(\lambda) = 1_{L^2(\mathbb{S}^{d-1})} - \pi i\lambda^{d-2} E_-(\lambda)(V - VR_V(\lambda + i0)V)\,{}^tE_+(\lambda) = 1_{L^2(\mathbb{S}^{d-1})} - A(\lambda). \tag{3.8}$$

The operator $A(\lambda)$ is the scattering amplitude. It is given by

$$A(\lambda) \equiv -\pi i\lambda^{d-2} E_-(\lambda)(V - VR_V(\lambda + i0)V)\,{}^tE_+(\lambda), \tag{3.9}$$

and is in the trace class.

We can also express the S-matrix in terms of localization operators in the case the support of V is compact. We assume that $\operatorname{supp} V \subset B(0, R_1)$. We choose two other length scales so that $0 < R_1 < R_2 < R_3 < \infty$. Let $0 \leq \chi_j \in C_0^2(\mathbb{R}^d)$ have the property that $\chi_j V = V$ and $\operatorname{supp}\chi_2 \subset B(0, R_2)$ and $\operatorname{supp}\chi_3 \subset B(0, R_3)$. Finally, let $W(\phi)$ denote the commutator $W(\phi) \equiv [-\Delta, \phi]$, for any $\phi \in C^2(\mathbb{R}^d)$. The following representation is due to Petkov and Zworski [28].

Theorem 10. Let $V \in C_0^2(\mathbb{R}^d)$ and consider the S-matrix $S(\lambda)$, $\lambda \in \mathbb{R}$, as a unitary operator on $L^2(\mathbb{S}^{d-1})$. Then, the S-matrix has the form

$$S(\lambda) = 1_{L^2(\mathbb{S}^{d-1})} + A(\lambda), \quad \lambda \in \mathbb{R}, \tag{3.10}$$

where $A(\lambda)$ is in the trace class. Explicitly, the scattering amplitude $A(\lambda)$ has the form

$$A(\lambda) = c_d \lambda^{d-2} E_-(\lambda) W(\chi_2) R_V(\lambda) W(\chi_1)^t E_+(\lambda), \tag{3.11}$$

where the constant $c_d = -i(2\pi)^{-d} 2^{(1-d)/2}$.

4 The resolvent and resonances

We now switch our perspective and return to the study of the resolvent of the Schrödinger operator H_V. We will connect these results with the S-matrix in section 4.5. We recall from section 2 that the *spectrum* of a self-adjoint operator A, denoted by $\sigma(A)$, is a closed subset of the real line. The *discrete spectrum* of A, denoted $\sigma_{\text{disc}}(A)$, is the subset of the spectrum consisting of all isolated eigenvalues with finite multiplicity. The complement of the spectrum is called the *resolvent set* of A, denoted by $\rho(A) \equiv \mathbb{C}\backslash\sigma(A)$. The resolvent of a self-adjoint operator A is defined, for any $z \in \rho(A)$, as the bounded operator $R_A(z) = (A - z)^{-1}$. It is a bounded operator-valued analytic function on $\rho(A)$. This means that about any point $z_0 \in \rho(A)$, the resolvent $R_A(z)$ has a norm convergent power series of the form

$$R_A(z) = \sum_{j=0}^{\infty} A_j (z - z_0)^j, \tag{4.1}$$

for bounded operators A_j depending on z_0. We note that for a self-adjoint operator A, the set $\mathbb{C}\backslash\mathbb{R}$ is always in the resolvent set.

For a Schrödinger operator $H_V = -\Delta + V$, we reparameterized the spectrum by setting $z = \lambda^2$ and write $R_{H_V}(z) = R_V(\lambda)$. Under this change of energy parameter, the spectrum in the complex λ-plane is the union of the line $\Im\lambda = 0$ and at most finitely-many points of the form $i\lambda_j$ on the positive imaginary axis $\lambda_j > 0$. These points correspond to the negative eigenvalues of H_V so that $z = -\lambda_j^2 \in \sigma_{\text{disc}}(A)$.

Let $\chi_V \in C_0^\infty(\mathbb{R})$ be a compactly-supported function so that $\chi_V V = V$. We are most concerned with the properties of the localized resolvent $\mathcal{R}_V(\lambda) \equiv \chi_V R_V(\lambda) \chi_V$. The operator-valued function $\mathcal{R}_V(\lambda)$ is defined for $\Im\lambda > 0$ and $\lambda \neq i\lambda_j$, with $\lambda_j > 0$ and $-\lambda_j^2$ an eigenvalue of H_V. We would like to find the largest region in the complex λ-plane on which $\mathcal{R}_V(\lambda)$ can be defined.

4.1 Limiting absorption principle

One might first ask if the bounded operator $\mathcal{R}_V(\lambda)$ has a limit as $\Im\lambda \to 0$, from $\Im\lambda > 0$. That is, does the boundary-value of this operator-valued meromorphic function exist as a bounded operator for $\lambda \in \mathbb{R}$? Because of the weight functions χ_V the answer to this question is yes. In more general settings, this result is part of what is referred to as the *limiting absorption principle (LAP)*. The LAP plays an important role in scattering theory.

Theorem 11. The meromorphic bounded operator-valued function $\mathcal{R}_V(\lambda)$ on the open set $\rho_+(H_V) \equiv \{\lambda \in \mathbb{C} \mid \Im\lambda > 0, -\lambda^2 \notin \sigma_{\rm disc}(H_V)\}$ admits continuous boundary values $\mathcal{R}_V(\lambda)$ for $\lambda \in \mathbb{R}$, except possibly at $\lambda = 0$. That is, $\lim_{\epsilon\to 0^+} \mathcal{R}_V(\lambda + i\epsilon)$ exists for all $\lambda \in \mathbb{R}\backslash\{0\}$, and is a bounded, continuous operator-valued function on that set.

The proof of this is given for more general potentials and N-body Schrödinger operators in, for example, [9, chapter 4]. The key ingredient is a *local commutator estimate* called the Mourre estimate, due to E. Mourre [22]. Let $A = (1/2)(x\cdot\nabla + \nabla\cdot x)$ be the generator of the unitary group implementing the dilations $x \to e^{\theta}x$, for $\theta \in \mathbb{R}$, on $L^2(\mathbb{R}^d)$. One formally computes the following commutator, assuming ∇V exists:

$$[H_V, A] = 2H_0 - x\cdot\nabla V = 2H_V - (2V + x\cdot\nabla V). \tag{4.2}$$

Let $I \subset \mathbb{R}$ be a closed interval. Let $E_V(I)$ be the projector for H_V and the interval I. We conjugate the commutator in (4.2) by this spectral projector:

$$E_V(I)[H_V, A]E_V(I) = 2E_V(I)H_VE_V(I) - K(V, I), \tag{4.3}$$

where $K(V, I) \equiv E_V(I)(2V + x\cdot\nabla V)E_V(I)$ is a compact, self-adjoint operator due to the properties of V.

We now assume that there are no eigenvalues of H_V in the interval I. For $I \subset \mathbb{R}^+$, this means that there are no positive eigenvalues of H_V. In our situation, this is true (see [9, chapter 4]). Then, the spectral theorem implies that $s - \lim_{|I|\to 0} E_V(I) = 0$. Since $K(V, I)$ is a compact operator and $K(V, I) = K(V, I)E_V(I)$, it follows that $\lim_{|I|\to 0} \|K(V, I)\| = 0$. Furthermore, if $I = [E_1, E_2]$, then $2E(I)H_VE(I) \geq 2E_1$. Given any $\epsilon > 0$, we choose I so that $|I|$ is so small that $\|K(V, I)\| \leq \epsilon$. Consequently, the commutator on the left in (2.9) is strictly nonnegative and bounded below:

$$E_V(I)[H_V, A]E_V(I) \geq (2E_1 - \epsilon)E_V(I) \geq 0, \quad |I| = E_2 - E_1 \ \text{ sufficiently small.} \tag{4.4}$$

One of the main results of Mourre theory is that for any interval I for which a positive commutator estimate of the form (4.4) holds, the boundary value of the weighted resolvent exists. More precisely, for any $\alpha > 1$, one has

$$\lim_{\epsilon\to 0^+} \left\{ \sup_{E\in I} \|(A^2 + 1)^{-\alpha/2}(H_V - E - i\epsilon)^{-1}(A^2 + 1)^{-\alpha/2}\| \right\} < \infty. \tag{4.5}$$

This technical estimate is the heart of the LAP. Estimate (4.5) is proved using a differential inequality-type argument. In our case, the function χ_V serves as the weight for the resolvent. One also proves that the limit in (4.5) is continuous in $E \in I$. If there are no embedded eigenvalues, as in our case, this holds for all $E > 0$.

Let us summarize what we have proved so far. The cut-off resolvent $\mathcal{R}_V(\lambda)$ is meromorphic on $\mathbb{C}^+$ with poles having finite-rank residues at at most finitely-many values $i\lambda_j$, with $\lambda_j > 0$ such that $-\lambda_j^2$ is an eigenvalue of H_V. Using the LAP, we can extend the cut-off resolvent $\mathcal{R}_V(\lambda)$ onto the real axis as a bounded operator $\mathcal{R}_V(\lambda)$, for $\lambda \in \mathbb{R}\backslash\{0\}$. This extension is continuous in λ. Hence, the cut-off resolvent is meromorphic on $\mathbb{C}^+$ and continuous on $\overline{\mathbb{C}^+}\backslash\{0\}$.

4.2 Analytic continuation of the cut-off resolvent of H_0

Our cut-off resolvent $\mathcal{R}_V(\lambda)$ is meromorphic on $\mathbb{C}^+$ and continuous on the real axis, except possibly at zero. It is now natural to ask if the operator has a meromorphic extension to the entire complex λ-plane as a bounded operator. We first consider the simpler case when $V = 0$. In this case, let $\chi \in C_0^\infty(\mathbb{R}^d)$ be any compactly-supported cut-off function and consider the compact operator $\mathcal{R}_0(\lambda) \equiv \chi R_0(\lambda)\chi$. We mention that the kernel of this operator is known explicitly:

$$\mathcal{R}_0(\lambda)(x,y) = \frac{i}{4}\chi(x)\left(\frac{\lambda}{2\pi|x-y|}\right)^{(d-2)/2} H^{(1)}_{(d-2)/2}(\lambda|x-y|)\chi(y), \tag{4.6}$$

where $H_j^{(1)}(s)$ is the Hankel function of the first kind with index j. We remark that the LAP is not necessary in order to construct an analytic continuation of the free cut-off resolvent $\mathcal{R}_0(\lambda)$. An alternate and very nice method, based on the explicit formula (4.6), is presented in Vodev's review article [44].

We are tempted to define the continuation $\tilde{\mathcal{R}}_0(\lambda)$ of $\mathcal{R}_0(\lambda)$ for $\Im\lambda < 0$ as the operator $\chi R_0(-\lambda)\chi$ since, if $\lambda \in \mathbb{C}^-$, then $-\lambda \in \mathbb{C}^+$ and $\chi R_0(-\lambda)\chi$ is well defined away from $\sigma_{\text{disc}}(H_V)$. Clearly, $\tilde{\mathcal{R}}_0(\lambda) \equiv \chi R_0(-\lambda)\chi$ for $\Im\lambda < 0$ is a meromorphic function in $\mathbb{C}^-$. The problem with this extension is that the two functions $\mathcal{R}_0(\lambda)$ and $\tilde{\mathcal{R}}_0(\lambda)$ do not match up on the real axis.

In order to understand this, recall that in the z-plane, the resolvent $(H_0 - z)^{-1}$ is analytic on $\mathbb{C}\backslash[0,\infty)$. For $\lambda_0 > 0$ and $\epsilon > 0$, we are interested in the discontinuity of the resolvent across the positive z-axis at the point $\lambda_0^2 > 0$. We can measure this by computing the following limit of the difference of the resolvents from above and below the point $\lambda_0^2 > 0$:

$$(H_0 - (\lambda_0^2 + i\epsilon))^{-1} - (H_0 - (\lambda_0^2 - i\epsilon))^{-1}, \tag{4.7}$$

as $\epsilon \to 0$. The point $z_+ = \lambda_0^2 + i\epsilon$ has two square roots in the λ-plane. Let $\tilde{\lambda}_0 \equiv \sqrt{\lambda_0^4 + \epsilon^2}$. For z_+, let θ be the angle in the first quadrant so that $0 \le \theta < \pi/2$. Then, the the two square roots are $\pm\tilde{\lambda}_0[\cos(\theta/2) + i\sin(\theta/2)]$. The positive square root lies in $\mathbb{C}^+$ in the λ-plane so we work with this root $\lambda_+(\epsilon) \equiv \tilde{\lambda}_0[\cos(\theta/2) + i\sin(\theta/2)]$. Similarly, the point $z_- = \lambda_0^2 - i\epsilon$ has two square roots $\pm\tilde{\lambda}_0[\cos(\theta/2) - i\sin(\theta/2)]$. Note that because z_- lies in the fourth quadrant, the imaginary part is negative. We choose the negative square root of z_- because it lies in $\mathbb{C}^+$ and call it $\lambda_-(\epsilon)$. Finally, for ϵ small, we may write $\lambda_\pm(\epsilon) = \pm\lambda_0 + i\epsilon \in \mathbb{C}^+$. Consequently, the jump discontinuity in (4.7) across $\mathbb{R}^+$ in the z-plane corresponds to studying

$$\lim_{\epsilon\to 0}[R_0(\lambda_0 + i\epsilon) - R_0(-\lambda_0 + i\epsilon)], \tag{4.8}$$

in the λ-plane. Both terms in (4.8) are well-defined since the points $\pm\lambda + i\epsilon$ have positive imaginary parts $\epsilon > 0$.

We will compute the limit as $\epsilon \to 0$ in (4.8) and show that it is nonzero. Furthermore, we will see that the limit extends to an analytic function on $\mathbb{C}$. This is the term that must be added to

$\mathcal{R}_0(-\lambda)$, for $\Im\lambda < 0$, in order to obtain a function that is continuous (and actually analytic) across $\Im\lambda = 0$. We follow a calculation in [19, sections 1.5-1.6]. For $f \in C_0^\infty(\mathbb{R}^d)$ and $\Im\lambda > 0$, we have

$$(R_0(\lambda)f)(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\xi\cdot x} \frac{(Ff)(\xi)}{(\xi^2 - \lambda^2)}\, d^d\xi. \tag{4.9}$$

The Fourier transform Ff is a Schwartz function so it decays rapidly in $|\xi|$ (see, for example, [30, section IX.1, Theorem IX.1]). Since $\Im\lambda > 0$, this guarantees that the integral in (4.9) is absolutely convergent. Switching to polar coordinates $\xi = \rho\omega$, with $\rho \geq 0$ and $\omega \in \mathbb{S}^{d-1}$, we obtain for the integral

$$\int_{\mathbb{R}^d} e^{i\xi\cdot x} \frac{(Ff)(\xi)}{(\xi^2 - \lambda^2)}\, d^d\xi = \int_{\mathbb{S}^{d-1}} d\omega \int_0^\infty d\rho\; e^{i\rho\omega\cdot x} \frac{\rho^{n-1}(Ff)(\rho\omega)}{(\rho^2 - \lambda^2)}. \tag{4.10}$$

In order to compute $R_0(\lambda_0 + i\epsilon)$, we deform the ρ-contour into the lower-half complex ρ-plane in a small, counter-clockwise oriented semicircle centered at λ_0. The Fourier transform Ff extends to an analytic function (see, for example [30, section IX.3]) so there is no difficulty with this. Similarly, in order to compute $R_0(-\lambda_0 + i\epsilon)$, we note that this is the same as computing the integral in (4.10) with $\lambda = \lambda_0 - i\epsilon$. This allows us to deform the ρ-integral into the upper-half complex ρ-plane and integrate around a small, clockwise semicircle centered at λ_0. Subtracting the two terms as in (4.8), we obtain

$$R_0(\lambda_0 + i\epsilon) - R_0(-\lambda_0 + i\epsilon) = \int_{\mathbb{S}^{d-1}} d\omega \int_{\Gamma(\lambda_0)} d\rho\; e^{i\rho\omega\cdot x} \frac{\rho^{d-1}(Ff)(\rho\omega)}{(\rho^2 - (\lambda_0 + i\epsilon)^2)} \tag{4.11}$$

where $\Gamma(\lambda_0)$ is a counter-clockwise oriented circle about $\lambda_0 > 0$. Evaluating the integral by the residue theorem, we obtain for $\lambda_0 > 0$,

$$\lim_{\epsilon\to 0}[(R_0(\lambda_0 + i\epsilon) - R_0(-\lambda_0 + i\epsilon))f)(x)] = \frac{i}{2}\frac{\lambda_0^{d-2}}{(2\pi)^{(d-1)/2}} \int_{\mathbb{S}^{d-1}} d\omega\; (Ff)(\lambda_0\omega)\; e^{i\lambda_0\,\omega\cdot x}. \tag{4.12}$$

We define the kernel $M(\lambda; x, y)$ by

$$M(\lambda; x, y) \equiv \frac{i}{2}\frac{1}{(2\pi)^{d-1/2}} \int_{\mathbb{S}^{d-1}} d\omega\; e^{i\lambda\omega\cdot(x-y)}. \tag{4.13}$$

Undoing the Fourier transform in (4.12), we can write the limit in (4.12) as

$$\lim_{\epsilon\to 0}[(R_0(\lambda_0 + i\epsilon) - R_0(-\lambda_0 + i\epsilon))f)(x)] = \lambda_0^{d-2} \int_{\mathbb{R}^d} M(\lambda_0; x, y) f(y)\; d^d y. \tag{4.14}$$

Because the integration is over a compact set, the sphere, the kernel $M(\lambda; x, y)$ extends to an analytic function on $\mathbb{C}$. Furthermore, recalling that we have compactly supported cut-off functions, the localized kernel

$$\mathcal{M}(\lambda; x, y) \equiv \chi(x) M(\lambda; x, y)\chi(y), \tag{4.15}$$

is square integrable for any $\lambda \in \mathbb{C}$. Hence, the operator $\mathcal{M}(\lambda)$ is an analytic, operator-valued function on $\mathbb{C}$ with values in the Hilbert-Schmidt class of operators (see [29, section VI.6]).

We can now define an extension $\tilde{\mathcal{R}}_0(\lambda)$ of the cut-off resolvent $\mathcal{R}_0(\lambda)$ from $\Im\lambda > 0$ to $\mathbb{C}^-\backslash(-\infty, 0]$ by

$$\tilde{\mathcal{R}}_0(\lambda) \equiv \chi\tilde{R}_0(\lambda)\chi = \chi R_0(-\lambda)\chi + \lambda^{d-2}\chi M(\lambda)\chi, \quad \Im\lambda < 0. \tag{4.16}$$

We then have for $\lambda > 0$,

$$\begin{aligned} \lim_{\epsilon\to 0} \chi\tilde{R}_0(\lambda - i\epsilon)\chi &= \lim_{\epsilon\to 0}[\chi R_0(-\lambda + i\epsilon)\chi + \lambda^{d-2}\chi M(\lambda - i\epsilon)\chi] \\ &= \chi R_0(\lambda)\chi, \end{aligned} \tag{4.17}$$

and thus we have continuity across the positive λ half-axis. It can be checked that this actually gives analyticity in a neighborhood of $\mathbb{R}\backslash(-\infty, 0]$. As for the open negative real axis $(-\infty, 0)$, we note that $M(-\lambda) = M(\lambda)$ since the sphere is invariant under the antipodal map $\omega \to -\omega$. A similar analysis can be performed for $d \geq 2$ even. We summarize the main results on the analytic continuation for the free cut-off resolvent.

Proposition 12. Suppose that the dimension $d \geq 3$ is odd. The cut-off resolvent $\chi R_0(\lambda)\chi$ of the Lapalcian admits an analytic continuation as a compact operator-valued function to the entire complex plane. In the case $d = 1$, there is an isolated pole of order one at $\lambda = 0$. When the dimension $d \geq 4$ is even, the cut-off resolvent admits an analytic continuation as a compact operator-valued function to the infinite-sheeted Riemann surface of the logarithm Λ. In the case $d = 2$, there is a logarithmic singularity at $\lambda = 0$.

4.3 Meromorphic continuation of the cut-off resolvent of H_V

We can use Proposition 12 and the second resolvent formula to obtain a meromorphic continuation of the resolvent $\mathcal{R}_V(\lambda)$. First, we write the second resolvent equation for $\lambda \in \mathbb{C}^+$,

$$R_V(\lambda) = R_0(\lambda) - R_V(\lambda)VR_0(\lambda). \tag{4.18}$$

Conjugating this equation by the cut-off function χ_V and using the fact that $\chi_V V = V$, we obtain

$$\mathcal{R}_V(\lambda) = \chi R_0(\lambda)\chi - \mathcal{R}_V(\lambda)V\chi R_0(\lambda)\chi. \tag{4.19}$$

Solving this for $\mathcal{R}_V(\lambda)$, we obtain

$$\mathcal{R}_V(\lambda)(1 + V\chi_V R_0(\lambda)\chi_V) = \chi_V R_0(\lambda)\chi_V. \tag{4.20}$$

We use this equality in order to construct the meromorphic continuation of $\mathcal{R}_V(\lambda)$.

The right side of (4.20) has an analytic continuation as does the second factor on the left. We need to prove that this factor $(1 + V\chi_V R_0(\lambda)\chi_V)$ has a continuation that is boundedly invertible, at least away from a discrete set of λ.

Recall that an operator of the form $1 + K$, for a bounded operator K, is boundedly invertible if, for example, $\|K\| < 1$. The inverse is constructed as a norm convergent geometric series.

There is another sufficient condition for invertibility. If the operator K is compact, then the Fredholm Alternative Theorem [14, Theorem 9.12] states that either K has an eigenvalue -1, and consequently, the operator $1 + K$ is not injective, or $1 + K$ is boundedly invertible. It follows from section 4.2 that the operator $K(\lambda) = V\chi_V R_0(\lambda)\chi_V$ in our expression (4.20) extends to a compact operator-valued analytic function. In this setting, the Analytic Fredholm Theorem [29, Theorem VI.14] is most useful.

Theorem 13. Suppose that $K(\lambda)$ is a compact operator-valued analytic function on a open connected set $\Omega \subset \mathbb{C}$. Then, either the operator $1+K(\lambda)$ is not invertible for any $\lambda \in \Omega$, or else it is boundedly invertible on Ω except possibly on a discrete set D of points having no accumulation point in Ω. The operator is meromorphic on $\Omega\backslash D$ At those points, the inverse has a residue that is a finite-rank operator.

This theorem tells us that $1+K(\lambda)$, the first factor on the right of (4.20), is boundedly invertible for $\lambda \in \mathbb{C}$ except at a discrete set of points. Since we know that $R_\chi(\lambda)$ is invertible for $\Im\lambda > 0$, except for a finite number of points on the positive imaginary axis corresponding to eigenvalues, it also follows from (4.20) that the discrete set of points at which $1 + K(\lambda)$ fails to be invertible lies in $\mathbb{C}^-$ if d is odd, or on $\Lambda\backslash\mathbb{C}^+$ if d is even. Consequently, the Analytic Fredholm Theorem allows us to establish the existence of a meromorphic extension of $\mathcal{R}_V(\lambda)$.

Proposition 14. Let $V \in C_0^2(\mathbb{R}^d)$ be a real-valued potential and let $\chi_V \in C_0^\infty(\mathbb{R}^d)$ be any function such that $\chi_V V = V$. Then the cut-off resolvent $\mathcal{R}_V(\lambda)$ admits a meromorphic extension to $\mathbb{C}$ if d is odd and to Λ if d is even. The poles have finite-rank residues.

4.4 Resonances of H_V

Having constructed the meromorphic continuation of the cut-off resolvent $\mathcal{R}_V(\lambda)$, we can now define the resonances of H_V.

Definition 15. Let $V \in C_0^2(\mathbb{R}^d)$ be a real-valued potential. The resonances of H_V are the poles of the meromorphic continuation of the compact operator $\mathcal{R}_V(\lambda)$ occurring in $\mathbb{C}^-$ for d odd, or on $\Lambda\backslash\mathbb{C}^+$ for d even.

This definition can also be extended to complex-valued potentials. The residues of the extension of $\mathcal{R}_V(\lambda)$ at the poles are finite rank operators. If $\lambda_0 \in \mathbb{C}^-$ is a resonance, then a resonance state $\psi_{\lambda_0} \in \mathcal{H}$ corresponding to λ_0 is a solution to

$$(1 + V\chi_V R_0(\lambda_0)\chi_V)\psi_{\lambda_0} = 0. \tag{4.21}$$

The poles are independent of the cut-off function used provided it has compact support and satisfies $\chi V = V$.

4.5 Meromorphic continuation of the S-matrix

The meromorphic continuation of the cut-off resolvent $\mathcal{R}_V(\lambda)$ permits us to mermorphically continue the S-matrix $S(\lambda)$ as a bounded operator on $L^2(\mathbb{S}^d)$ from $\lambda \in \mathbb{C}^+$ to all of $\mathbb{C}$ or Λ depending on the parity of d. This follows from formula (3.8) of Theorem 9. Because of the compactness of the support of V, the operators $E_\pm(\lambda)$, and their transposes, admit analytic continuations. This property, together with the continuation properties of $\mathcal{R}_V(\lambda)$ and formula (3.8), establish the meromorphic continuation of $S(\lambda)$. For complex λ, the S-matrix is no longer unitary. The relation $S(\lambda)S(\lambda)^* = 1$, however, does continue to hold for $\lambda \in \mathbb{C}$ (or Λ).

Theorem 16. The S-matrix $S(\lambda)$ admits a mermorphic continuation to $\mathbb{C}$ if d is odd, or to the Riemann surface Λ, if d is even, with poles precisely at the resonances of H_V. The order of the poles are the same as the order of the poles for H_V and the residues at these poles have the same finite rank.

For the Schrödinger operator H_V, the resonances may be defined as the poles of the meromorphic continuation of the cut-off resolvent $\mathcal{R}_V(\lambda)$, or in terms of the meromorphic continuation of the S-matrix $S(\lambda)$. From formula (3.8), it follows that the poles of the meromorphic continuation of the S-matrix are included in the poles of the continuation of the resolvent. It is not always true that the scattering poles, defined via the S-matrix, are the same as the resolvent poles. A striking example where the scattering poles differ from the resolvent poles occurs for hyperbolic spaces. However, in the Schrödinger operator case considered here, these are the same. A proof is given by Shenk and Thoe [37].

5 Resonances: Existence and the counting function

The resonance set $\mathcal{R}_V$ for a Schrödinger operator was defined in Definition 15 as the poles of the meromorphic continuation of the cut-off resolvent $\mathcal{R}_V(\lambda)$ to $\mathbb{C}$ for $d \geq 3$ odd or to the Riemann surface Λ for $d \geq 4$ even, together with their multiplicities. There are two basic questions that arise:

(1) Existence: Do resonances exist for Schrödinger operators H_V with our class of potentials?

(2) Counting: How many resonances exist?

5.1 Existence of resonances

There are many different proofs of the existence of resonances for various quantum mechanical systems. Resonances are considered as *almost bound states* or *long-lived states* that eventually decay to spatial infinity. To understand this physical description, let us consider the time evolution of a resonance state ψ_0 corresponding to a resonance energy $z_0 = E_0 - i\Gamma$ (in the z-parametrization), with $\Gamma > 0$. A *resonance state* ψ_0 solves the partial differential equation $H_V\psi_0 = z_0\psi_0$ and is

purely outgoing. Since V has compact support, the function ψ_0 satisfies $-\Delta\psi_0 = z_0\psi_0$ for $|x|$ large enough. The outgoing condition means that the radial behavior of a component of ψ_0 with angular momentum $\ell \geq 0$ is a Hankel function of the first kind $H^{(1)}_{(d-2)/2+\ell}(\sqrt{z_0}|x|)$. Such a function ψ_0 grows exponentially as $|x| \to \infty$ and is not in $\mathcal{H}$.

We can formally compute $U_V(t)\psi_0$ by expressing the time evolution group as an integral of the resolvent over the energy

$$U_V(t)\psi_0 = \frac{-1}{2\pi i} \int_{\mathbb{R}} e^{-itE} R_V(E)\, dE. \tag{5.1}$$

Performing a deformation of the contour to capture the pole of the resolvent at z_0 and applying the residue theorem, one finds that the time evolution behaves like $e^{-iz_0 t}\psi_0 = e^{-\Gamma t}e^{-iE_0 t}\psi_0$. The factor $e^{-itE_0}\psi_0$ has an oscillatory time evolution similar to that of a bound state with energy E_0, whereas the factor $e^{-t\Gamma}$ is an exponentially decaying amplitude. The *lifetime* of the state is $\tau = \Gamma^{-1}$. This is, roughly, the time it takes the amplitude to decay to e^{-1} times its original size.

As noted above, there is no such state $\psi_0 \in \mathcal{H}$ corresponding to a resonance z_0 in the sense that $H_V\psi_0 = z_0\psi_0$. Since H_V is self-adjoint and z_0 has a nonzero imaginary part, the solutions of this eigenvalue equation are not in $\mathcal{H}$. There are, however, approximate resonance states obtained by truncating such ψ_0 to bounded regions, say $K \subset \mathbb{R}^d$. The truncated state $\chi_K\psi_0 \in \mathcal{H}$ has an approximate time evolution like $e^{-\Gamma t}e^{-iE_0 t}\chi_K\psi_0$ showing that the amplitude of the resonance state in the bounded region K decays exponentially to zero.

A typical situation for which resonances are expected to exist is the hydrogen atom Hamiltonian $H_V = -\Delta - |x|^{-1}$ acting on $L^2(\mathbb{R}^3)$, perturbed by an external constant electric field $V_{pert}(x) = \mathcal{E}x_1$ in the x_1-direction. The total Stark hydrogen Schrödinger operator is $H_V(\mathcal{E}) = -\Delta - |x|^{-1} + \mathcal{E}x_1$. When $\mathcal{E} = 0$, the spectrum of H_V consists of an infinite sequence of eigenvalues $E_n = -1/4n^2$ plus the half line $[0, \infty)$. When $\mathcal{E}$ is turned on, the spectrum of $H_V(\mathcal{E})$ is purely absolutely continuous and equal to exactly the entire real line. There are no eigenvalues!

It is expected that the bound states E_n of the hydrogen atom have become resonances for $\mathcal{E} \neq 0$. These finite-lifetime states are observed in the laboratory. These resonances, in the z-parametrization, have their real parts close to the eigenvalues E_n. Their imaginary parts are exponentially small behaving like $e^{-\alpha/\mathcal{E}}$. This means their lifetime is very long.

The proof of the existence of these resonances for the the Stark hydrogen Hamiltonian was given by Herbst [13] in 1979. The method of proof is perturbative in that the electric field strength is assumed to be very small.

More generally, there are various models for which one can prove the existence of resonances using the smallness of some parameter. The semiclassical approximation is the most common regime. The quantum Hamiltonian is written as $H_V(h) = -h^2\Delta + V_0 + V_1$ and h is considered as a small parameter. For a discussion of resonances in the semiclassical regime, see, for example, [14, Chapters 20–23]. For more information on the semiclassical approximation for eigenvalues, eigenfunctions and resonances, see, for example, the monographs [10, 18, 33].

If we inquire about the existence of resonances for the models studied here, $H_V = -\Delta + V$, with $V \in C_0^\infty(\mathbb{R}^d)$, with no parameters, the proof is much harder and requires different techniques.

Melrose [19] gave perhaps the first proof of the existence of infinity many resonances for such H_V. The proof holds for smooth, real-valued, compactly-supported potentials $V \in C_0^\infty(\mathbb{R}^d)$, for $d \geq 3$ odd. The proof requires two ingredients that will be presented here without proof.

5.1.1 Small time expansion of the wave trace

The wave group $W_V(t)$ associated with the Schrödinger operator H_V is defined as follows. Let ∂_t denote the partial derivative $\partial/\partial t$. Consider the wave equation associated with H_V:

$$(\partial_t^2 - H_V)u = 0,\ u(t=0) = u_0,\ \ \partial_t u(t=0) = u_1. \tag{5.2}$$

The solution can be expressed in terms of the initial conditions (u_0, u_1). The time evolution occurs on a direct sum of two Hilbert spaces $\mathcal{H}_{FE} = \{(u_0, u_1) \mid \int[|\nabla u_0|^2 + |u_1|^2] < \infty\}$. This is the space of finite energy solutions. In two-by-two matrix notation, the time evolution acts as

$$W_V(t)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} u \\ \partial_t u \end{pmatrix} \tag{5.3}$$

The infinitesimal generator of the wave group $W_V(t)$ is the two-by-two matrix-valued operator

$$A_V \equiv \begin{pmatrix} 0 & 1 \\ H_V & 0 \end{pmatrix}. \tag{5.4}$$

The evolution group $W_V(t)$ is unitary on $\mathcal{H}_{FE}$. Similarly, the free wave group $W_0(t)$ is generated by A_0 that is expressed as in (5.4) with $V = 0$. If $H_V \equiv -\Delta + V \geq 0$, then this operator can be diagonalized. The diagonal form is

$$\begin{pmatrix} \sqrt{H_V} & 0 \\ 0 & -\sqrt{H_V} \end{pmatrix}. \tag{5.5}$$

In this case, the wave group $W_V(t)$ can be considered as two separate unitary groups $e^{\pm i\sqrt{H_V}t}$ each acting on a single component Hilbert space.

The basic fact that we need is that the map $t \in \mathbb{R} \to \mathrm{Tr}[W_V(t) - W_0(t)]$ is a distribution. This means that for any $\rho(t)$, a smooth, compactly-supported function, the integral

$$\int_{\mathbb{R}} dt\ \rho(t)\mathrm{Tr}[W_V(t) - W_0(t)] \tag{5.6}$$

is finite and bounded above by an appropriate sum of semi-norms of ρ. The distribution has a singularity at $t = 0$ and the behavior of the distribution at $t = 0$ has been well-studied. For $d \geq 3$

odd, the wave trace has the following expansion as $t \to 0$:

$$\begin{aligned} \mathrm{Tr}[W_V(t) - W_0(t)] &= \sum_{j=1}^{(d-1)/2} w_j(V)(-i)^{d-1-2j}\delta^{(d-1-2j)}(t) \\ &\quad + \sum_{j=(d+1)/2}^{N} w_j(V)|t|^{2j-d} + r_N(t), \end{aligned} \tag{5.7}$$

where the remainder $r_N(t) \in C^{2N-d}(\mathbb{R})$. The first sum consists of derivatives of the delta function $\delta(t)$ at zero. We recall that for any smooth function f, these distributions are defined as $\langle \delta^j, f\rangle = (-1)^j f^{(j)}(0)$. The second part of the sum consists of distributions that are polynomial in t. The coefficients $w_j(V)$ are integrals of the potential V and its derivatives. These are often called the 'heat invariants'. The first three are:

$$\begin{aligned} w_1(V) &= c_{1,d}\int_{\mathbb{R}^d} V(x)\ d^dx \\ w_2(V) &= c_{2,d}\int_{\mathbb{R}^d} V^2(x)\ d^dx \\ w_3(V) &= c_{3,d}\int_{\mathbb{R}^d} (V^3(x) + |\nabla V(x)|^2)\ d^dx, \end{aligned} \tag{5.8}$$

where the constants $c_{j,d}$ are nonzero and depend only on the dimension d.

For some insight as to why the trace in (5.7) exists, note that for $\rho \in C_0^\infty(\mathbb{R})$, a formal calculation gives

$$\int_{\mathbb{R}} \rho(t)\mathrm{Tr}[W_V(t) - W_0(t)]\ dt = \mathrm{Tr}(\ (F\rho)(A_V) - (F\rho)(A_0)). \tag{5.9}$$

The Fourier transform $F\rho$ is a smooth, rapidly decreasing function. Because V has compact support, the difference $(F\rho)(A_V) - (F\rho)(A_0)$ is in the trace class. This follows from the fact that the difference of the resolvents $R_V(z)^k - R_0(z)^k$ is in the trace class for $\Im z \neq 0$ and $k > d/2$.

5.1.2 Poisson formula

The key formula that links the resonances with the trace of the difference of the wave groups is the Poisson formula. In our context it was proved by Melrose [20]. It is named this because of the analogy with the classical Poisson summation formula. Let $f \in C^\infty(\mathbb{R}^d)$ be a Schwarz function meaning that the function and all of its derivatives decay faster that $\langle \|x\| \rangle^{-N}$, for any $N \in \mathbb{N}$. The classical Poisson summation formula states that

$$\sum_{k\in\mathbb{Z}^d} f(x+k) = \sum_{k\in\mathbb{Z}^d} (Ff)(k)e^{2\pi i x\cdot k}. \tag{5.10}$$

The Poisson formula for the wave group states that

$$\mathrm{Tr}[W_V(t) - W_0(t)] = \sum_{\xi\in\mathcal{R}_V} m(\xi)e^{i|t|\xi}, \quad t \neq 0, \tag{5.11}$$

where $m(\xi)$ is the algebraic multiplicity of the resonance ξ. This multiplicity is defined as the rank of the residue of the resolvent at the pole ξ or, equivalently, by the rank of the contour integral:

$$m(\xi) = \text{ Rank}\left(\int_{\gamma_\xi} R(s)\ ds\right), \tag{5.12}$$

where γ_ξ is a small contour enclosing only the pole ξ of the resolvent. It is important to note that the Poisson formula (5.11) is not valid at $t = 0$.

5.1.3 Melrose's proof of the existence of resonances

Melrose [19, section 4.3] observed that the Poisson formula (5.11) and the trace formula (5.7) can be used together to prove the existence of infinitely many resonances for Schrödinger operators.

Theorem 17. Let us suppose that $d \geq 3$ is odd and that $V \in C_0^\infty(\mathbb{R}^d;\mathbb{R})$. Suppose also that $w_j(V) \neq 0$ for some $j \geq (d+1)/2$. Then H_V has infinitely many resonances. In particular, for $d = 3$, since $w_2(V) = c_2 \int V^2(x)\ dx$, for a positive constant $c_2 > 0$, if $V \in C_0^\infty(\mathbb{R}^3;\mathbb{R})$ is nonzero, then H_V has an infinite number of resonances.

Proof. 1. Suppose that H_V has no resonances. Then the right side of the Poisson formula (5.11) is zero. On the other hand, it follows from the small time expansion (5.7) and the assumption that $w_j(V) \neq 0$ for some $j \geq (d+1)/2$ that for $t > 0$ the right side of the expansion (5.7) is nonzero. Note that for $t > 0$ all the contributions from the delta functions vanish. Hence we obtain a contradiction. Consequently, there must be at least one resonance.

2. If there are only finitely-many resonances, then the sum on the right in (5.11) is finite and the formula can be extended to $t = 0$. In particular, at $t = 0$, it is a finite positive number greater than or equal to the number of resonances. On the other hand, looking at the trace formula (5.7), if only one or more of the coefficients $w_j(V) \neq 0$ for $j > (d+1)/2$, then the trace is zero at $t = 0$ (the coefficients of the derivatives of the delta functions being zero), so we get a contradiction. Hence, at least one of the coefficients of a delta function term is nonzero. Then the trace formula indicates that the distribution $\text{Tr}[W_V(t) - W_0(t)]$ is not continuous at $t = 0$ whereas the Poisson formula indicates that it is continuous through $t = 0$, and we again obtain a contradiction. Consequently, there must be an infinite number of resonances. □

We remark that in the even dimensional case for $d \geq 4$, Sá Barreto and Tang [36] proved the existence of at least one resonance for a real-valued, compactly-supported, smooth nontrivial potential. Having settled the question of existence, we now turn to counting the number of resonances.

5.2 The one-dimensional case: Zworski's asymptotics

As with many problems, the one-dimensional case is special since many techniques of ordinary differential equations can be used. The most complete result on resonances for $H_V = -d^2/dx^2 + V$ on $L^2(\mathbb{R})$ with a compactly-supported potential was proven by Zworski [46].

Theorem 18. Let $V \in L_0^\infty(\mathbb{R})$. Then the number of resonances $N_V(r)$ with modulus less that $r > 0$ satisfies:

$$N_V(r) = \frac{2}{\pi}\left(\sup_{x,y\in \mathrm{supp}\, V} |x-y|\right) r + o(r). \tag{5.13}$$

There are extensions of this result to a class of super-exponentially decaying potentials due to R. Froese [11]. We will not comment further on the one-dimensional case.

5.3 Estimates on the number of resonances: Upper bounds

The *resonance counting function* counts the number of poles, including multiplicities, of the meromorphic continuation of the cut-off resolvent in $\mathbb{C}^-$ for d odd, and on Λ for d even. We will concentrate on the odd d-dimensional case, although we will give comments on the even dimensional case too. For any $r > 0$, we define $N_V(r)$ as

$$N_V(r) = \#\{j \mid \lambda_j(V) \text{ satisfies } |\lambda_j(V)| \leq r \text{ and } \Im\lambda_j(V) < 0\}. \tag{5.14}$$

This function is monotone increasing in r. It is the analogue of the eigenvalue counting function $N_{\mathcal{M}}(r)$ studied by Weyl to count the number of eigenvalues of the Laplace-Beltrami operator on a compact Riemannian manifold $\mathcal{M}$ with size less than $r > 0$. The Weyl upper bound on the eigenvalue counting function is

$$N_{\mathcal{M}}(r) \leq c_d \mathrm{Vol}(\mathcal{M})\langle r\rangle^d, \tag{5.15}$$

where $\langle r\rangle = \sqrt{1+r^2}$.

It is natural to ask if the resonance counting function $N_V(r)$ satisfies a similar upper bound. Since Melrose's early work [21], many people have established upper bounds on $N_V(r)$ with increasing optimality. Zworski [49] presents a good survey of the state-of-the-art up to 1994. The optimal upper bound, having the same polynomial behavior as Weyl's eigenfunction counting function (5.15), was achieved by Zworski [47]. A significant simplification of the proof was given by Vodev [41].

Theorem 19. For $d \geq 3$ odd, the resonance counting function $N_V(r)$ satisfies

$$N_V(r) \leq C(d, V)\langle r\rangle^d, \tag{5.16}$$

for a constant $0 < C(d, V) < \infty$ depending on d and V.

A sketch of the proof of this theorem will be given following the beautiful exposition of Zworski [49, section 5], using Vodev's simplification [41]. One basic idea of the proof is to find a suitable

analytic or meromorphic function that has zeros exactly at the resonances. Suppose $h(\lambda)$ is one such function analytic on $\mathbb{C}$. Then one can count the number of zeros using Jensen's formula. This formula relates the number of zeros of h to growth properties of h. If a circle of radius $r > 0$ crosses no zero of h, if $h(0) \neq 0$, and if a_k are the zeros of h inside the circle, then Jensen's formula states that

$$\sum_{k=1}^{N_h(r)} \log\left(\frac{r}{|a_k|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log|h(re^{i\theta})|\, d\theta - \log|h(0)|. \tag{5.17}$$

If we only sum over those zeros inside the circle of radius $r/2$, we have that $\log(r/|a_k|) \geq \log 2$, so that

$$N_h(r/2)[\log 2] \leq \frac{1}{2\pi}\int_0^{2\pi} |\log|h(re^{i\theta})||\, d\theta + |\log|h(0)||. \tag{5.18}$$

This inequality shows that it suffices to bound h on circles of radius $2r$ in order to count the number of zeros inside the circle of radius $r > 0$. We will use some inequalities for singular values, the proofs of which can be found in [39].

Proof. 1. The first observation is that the operator $(VR_0(\lambda)\chi_V)^{d+1}$ is in the trace class for $\Im\lambda \geq 0$. Consequently, the following determinant is well-defined:

$$h(\lambda) \equiv \det(1 - (VR_0(\lambda)\chi_V)^{d+1}). \tag{5.19}$$

This function is analytic on $\mathbb{C}^+$ with at most a finite number of zeros corresponding to the eigenvalues of H_V. It follows from section 4.3 that this function has an analytic continuation to all of $\mathbb{C}$. Furthermore, the zeros of this function for $\Im\lambda < 0$ include with the resonances of H_V that are given as the zeros of the analytic continuation of $1 + VR_0(\lambda)\chi_V$ according to (4.20). The problem, then, is to count the number of zeros of the analytic function $h(\lambda)$ inside a ball of radius $r > 0$ in $\mathbb{C}$. By Jensen's inequality (5.18), it suffices to obtain a growth estimate on h of the form

$$|h(\lambda)| \leq C_1 e^{c_2|\lambda|^d}. \tag{5.20}$$

2. We first estimate h in the half space $\Im\lambda \geq 0$ using the fact that V has compact support contained inside of a bounded region Ω. Let $-\Delta_\Omega \geq 0$ denote the Dirichlet Laplacian on Ω. By Weyl's bound (5.15), the j^{th} eigenvalue $\lambda_j(\Omega)$ of $-\Delta_\Omega$ grows like $\lambda_j(\Omega) \sim j^{2/d}$. Furthermore, we have $\Delta_\Omega V = \Delta V$. Using these ideas and the simple inequality for the singular values $\mu_j(AB) \leq \|A\|\mu_j(B)$, we have

$$\begin{aligned}
\mu_j(\chi_V R_0(\lambda)\chi_V) &= \mu_j((-\Delta_\Omega + 1)^{-1/2}(-\Delta_\Omega + 1)^{1/2}\chi_V R_0(\lambda)\chi_V) \\
&\leq \|(-\Delta_\Omega + 1)^{1/2}\chi_V R_0(\lambda)\chi_V\|\ \mu_j((-\Delta_\Omega + 1)^{-1/2}) \\
&\leq Cj^{-1/d}.
\end{aligned} \tag{5.21}$$

It is important to note that the operator $\chi_V R_0(\lambda)\chi_V : L^2(\mathbb{R}^d) \to H^1(\mathbb{R}^d)$ is bounded uniformly in λ, for $\Im\lambda \geq 0$. Consequently, the norm $\|(-\Delta_\Omega + 1)^{1/2}\chi_V R_0(\lambda)\chi_V\|$ is bounded uniformly in λ in the upper half space. Upon squaring this norm, the operator $-\Delta_\Omega$ can be replaced by $-\Delta$

because of the support of χ_V. Since $\mu_{m+k-1}(AB) \leq \mu_k(A)\mu_m(B)$, we have $\mu_{2j-1}(A^2) \leq \mu_j(A)^2$, and consequently, for all large j

$$\mu_j((\chi_V \mathbb{R}_0(\lambda)\chi_V)^{d+1}) \leq C j^{-(d+1)/d}. \tag{5.22}$$

It follows that $|h(\lambda)| \leq C$ for $\Im\lambda \geq 0$.

3. For $\Im\lambda < 0$, we make use of the following formula from scattering theory used already in section 4.2. For $\lambda \in \mathbb{R}$, we have

$$\chi_V(R_0(\lambda) - R_0(-\lambda))\chi_V = c_d(\lambda^{d-2})\ {}^tE_\chi(\lambda)E_\chi(\lambda), \tag{5.23}$$

where $E_\chi(\lambda) : L^2(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$ is given by

$$(E_\chi(\lambda)g)(\omega) \equiv \int_{\mathbb{R}^d} e^{i\lambda\omega\cdot x}\chi_V(x)g(x)\ d^dx, \tag{5.24}$$

and ${}^tE_\chi(\lambda)$ denotes the transpose of this operator. This formula can be extended to all of $\mathbb{C}$. We compute the singular values of the operator on the left in (5.23):

$$\mu_j(\chi_V(R_0(\lambda) - R_0(-\lambda))\chi_V) \leq C|\lambda|^{d-2}e^{c_2|\lambda|}\mu_j(E_\chi(\lambda)). \tag{5.25}$$

Since $E_\chi(\lambda)^* : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{R}^d)$, the operator $E_\chi(\lambda)^*E_\chi(\lambda) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$. This is a crucial observation since the operator acts on a $d-1$ dimensional space. Without this reduction, one obtains an upper bound but with exponent $d+1$ rather than the optimal exponent d. In a manner similar to (5.21), we compute for any $m > 0$,

$$\begin{aligned}
\mu_j(E_\chi(\lambda)) &\leq \mu_j((-\Delta_{\mathbb{S}^{d-1}} + 1)^{-m}(-\Delta_{\mathbb{S}^{d-1}} + 1)^m E_\chi(\lambda)) \\
&\leq \|(-\Delta_{\mathbb{S}^{d-1}} + 1)^m E_\chi(\lambda))\|_{L^2(\mathbb{S}^{d-1})}\ \mu_j((-\Delta_{\mathbb{S}^{d-1}} + 1)^{-m}) \\
&\leq C^m(2m)!j^{-2m/(d-1)}e^{c|\lambda|}.
\end{aligned} \tag{5.26}$$

This follows from the explicit formula for the kernel of $E_\chi(\lambda)$),

$$E_\chi(\lambda)(\omega, x) = e^{-i\lambda\omega\cdot x}\chi_V(x). \tag{5.27}$$

In particular, the factor $(2m)!$ comes from differentiating the exponential factor. Using Stirling's formula for the factorial, we obtain from (5.25)–(5.26)

$$\mu_j(\chi_V(R_0(\lambda) - R_0(-\lambda))\chi_V) \leq |\lambda|^{d-2}e^{c_2|\lambda|}C^m(2m+1)^{2m+(1/2)}(j^{-1/(d-1)})^{2m}. \tag{5.28}$$

We now optimize over the free parameter m by choosing $m \sim j^{-1/(d-1)}$. As a result, we obtain

$$\mu_j(\chi_V(R_0(\lambda) - R_0(-\lambda))\chi_V) \leq e^{c|\lambda|}e^{-cj^{1/(d-1)}}. \tag{5.29}$$

4. We now combine (5.21) with (5.29). For this, we need Fan's inequality for singular values [39, Theorem 1.7] that states that

$$\mu_{n+m+1}(A + B) \leq \mu_{m+1}(A) + \mu_{n+1}(B). \tag{5.30}$$

For $\Im\lambda < 0$, we write

$$\mu_j(\chi_V R_0(\lambda)\chi_V) = \mu_j([\chi_V(R_0(\lambda) - R_0(-\lambda))\chi_V] + \chi_V R_0(-\lambda)\chi_V). \tag{5.31}$$

Applying Fan's inequality (5.30) to the right side of (5.31), we find that for $\Im\lambda \leq 0$, the singular values satisfy

$$\mu_j(\chi_V R_0(\lambda)\chi_V) \leq e^{c|\lambda|}e^{-cj^{1/(d-1)}} + cj^{-1/d}. \tag{5.32}$$

Taking the $(d+1)^{st}$ power of the operators, as in (5.22), we find

$$\mu_j((\chi_V R_0(\lambda)\chi_V)^{d+1}) \leq e^{c|\lambda|}e^{-cj^{1/(d-1)}} + cj^{-(d+1)/d}, \tag{5.33}$$

for a constant $c > 0$. As $j \to \infty$, the first term dominates until $j \sim [|\lambda|^{d-1}]$, where $[\cdot]$ denotes the integer part. We then use the Weyl estimate for the determinant (see [39]), factorize the product using the first estimate in (5.33) for $j \leq [d|\lambda|^{d-1}]$, to obtain

$$\begin{aligned} |h(\lambda)| &\leq |\det(1 + (VR_0(\lambda)\chi_V)^{d+1})| \\ &\leq \Pi_{j=1}^{\infty}(1 + \mu_j((VR_0(\lambda)\chi_V)^{d+1})) \\ &\leq \left(\Pi_{j=1}^{[d|\lambda|^{d-1}]}e^{C|\lambda|}\right)\ \left(\Pi_{j\geq[d|\lambda|^{d-1}]}(1 + c_2 j^{-(d+1)/d})\right) \\ &\leq ce^{c|\lambda|^d}. \end{aligned} \tag{5.34}$$

This establishes (5.20) so by Jensen's inequality (5.18) we obtain the optimal upper bound on the resonance counting function. □

Upper bounds for super-exponentially decaying potentials in $d \geq 3$ odd dimensions were proved by R. Froese [12]. There are fewer results in even dimensions. We refer to [7] for a discussion and the papers [15, 42, 43].

5.4 Estimates on the number of resonances: Lower bounds

One might hope to have a lower bound on the number of resonances of the form

$$N_V(r) \geq C_d r^d. \tag{5.35}$$

This is known to hold in two cases. The first case is Zworski's result for $d = 1$. The second is for a class of spherically symmetric potentials in dimension $d \geq 3$ odd. Zworski proved that if $V(r)$ has the property that $V'(a) \neq 0$, where $a > 0$ is the radius of the support of V, then an asymptotic expansion holds for the number of resonances:

$$N_V(r) = c_d a^d r^d + o(r^d), \quad d \geq 3 \text{ and odd}. \tag{5.36}$$

In general, for $V \in L_0^\infty(\mathbb{R}^d)$ (or, even $V \in C_0^\infty(\mathbb{R}^d)$), there is presently no known proof of the optimal lower bound (5.35). There are some partial results for $d \geq 3$ odd. These include nonoptimal lower bounds, estimates on the number of purely imaginary poles for potentials with fixed sign, and counterexamples made from certain *complex potentials.*

5.4.1 Nonoptimal lower bounds

For the case of $d \geq 3$ odd, the first quantitative lower bounds for the resonance counting function for nontrivial, smooth, real-valued $V \in C_0^\infty(\mathbb{R}^d)$, not of fixed sign, were proved in [2]. In particular, it was proved there that

$$\limsup_{r\to\infty} \frac{n_V(r)}{r(\log r)^{-p}} = \infty, \tag{5.37}$$

for all $p > 1$. For the same family of potentials, Sá Barreto [34] improved this to

$$\limsup_{r\to\infty} \frac{n_V(r)}{r} > 0. \tag{5.38}$$

We mention that, in particular, all these lower bounds require the potential to be smooth.

Concerning lower bounds in the even dimensional case for $d \geq 4$, Sá Barreto [35] studied the resonance counting function $N_{SaB}(r)$ defined to be the number of resonances λ_j with $1/r < |\lambda_j| < r$ and $|\arg \lambda_j| < \log r$. As $r \to \infty$, this region in the Riemann surface Λ opens like $\log r$. Sá Barreto proved that for even $d \geq 4$,

$$\limsup_{r\to\infty} \frac{N_{SaB}(r)}{(\log r)(\log\log r)^{-p}} = \infty, \tag{5.39}$$

for all $p > 1$.

5.4.2 Purely imaginary poles

Lax and Phillips [17] noticed that for odd dimensions $d \geq 3$, the wave operator associated with exterior obstacle scattering has an infinite number of *purely imaginary resonances*. They remarked that their proof held for Schrödinger operators with nonnegative, compactly-supported, nontrivial potentials. Vasy [40] used their method to prove that a Schrödinger operator H_V with a compactly-supported, bounded, real-valued potential with fixed sign (either positive or negative) has an infinite number of purely imaginary resonances. These resonances are poles of the meromorphic continuation of the resolvent of the form $-i\mu_j(V)$, with $\mu_j(V) > 0$. In the $z = \lambda^2$ plane, these are located on the second Riemann sheet of the square root function. Furthermore, Vasy is able to count these poles and prove the following lower bound

$$N_V(r) \geq C_d r^{d-1}. \tag{5.40}$$

This is not an optimal lower bound on the total number of resonances.

Recently, the author and T. J. Christiansen [7] proved that in *even dimension* there are no purely imaginary resonances on any sheet for H_V with bounded, positive, real-valued potentials with compact support.

5.4.3 Complex potentials

Most surprisingly, Christiansen [3] gave examples of compactly supported, bounded *complex-valued* potentials *having no resonances* in any dimension $d \geq 2$! This result, while interesting in its own

right, means that any technique that provides a result of the type (5.35) must be sensitive to whether the potential is real- or complex-valued.

6 Maximal order of growth is generic for the resonance counting function

There is one general result that is a weak form of (5.35) due to the author and T. J. Christiansen [5]. This result states that for almost all potentials $V \in L_0^\infty(K)$, for a compact subset $K \subset \mathbb{R}^d$, real- or complex-valued, the lower bound holds in the following sense as determined by the *order of growth* of the resonance counting function $N_V(r)$.

Definition 20. The order of growth of the monotone increasing function $N_V(r)$ is defined by

$$\rho_V \equiv \lim_{r\to\infty} \frac{\log N_V(r)}{\log r}, \tag{6.1}$$

if the limit exists and is finite.

Because of the upper bound (5.16), the order of growth of the resonance counting function is bounded from above as $\rho_V \leq d$. We say that the order of growth is *maximal* for a potential V if $\rho_V = d$. By "almost all potentials" referred to above, we mean that the set of potentials in $L_0^\infty(K)$, for a fixed compact subset $K \subset \mathbb{R}^d$ with nonempty interior, for which the resonance counting function has maximal order of growth, is a dense G_δ-set. Recall that a G_δ-set is a countable intersection of open sets. One sometimes says that a property that holds for all elements in a dense G_δ-set is *generic*. (Added in proof: For some recent developments, see Dinh and Vu arXiv:1207.4273v1.)

6.1 Generic behavior: odd dimensions

The basic theorem on generic behavior is the following.

Theorem 21. [5] Let $d \geq 3$ be odd and $K \subset \mathbb{R}^d$ be a fixed, compact set with nonempty interior. Let $\mathcal{M}_F(K) \subset L_0^\infty(K)$, for $F = \mathbb{R}$ or $F = \mathbb{C}$, be the set of all real-valued, respectively, complex-valued potentials in $L_0^\infty(K)$ such that the resonance counting function $N_V(r)$ has maximal order of growth. Then, the set $\mathcal{M}_F(K)$ is a dense G_δ set for $F = \mathbb{R}$ or $F = \mathbb{C}$.

This holds for both real-valued and complex-valued potentials. By [3], we know there are complex potentials with zero order of growth. An interesting open question is whether there exist real-valued potentials in $L_0^\infty(\mathbb{R}^d)$ for which the resonance counting function has less than maximal order of growth.

The proof of this theorem uses the S-matrix and its continuation to the entire complex plane. In section 3, we defined the scattering matrix for the pair $H_0 = -\Delta$ and $H_V = H_0 + V$. The S-matrix $S(\lambda)$, acting on $L^2(\mathbb{S}^{d-1})$, is the bounded operator defined in (3.8). In the case that V is

real-valued, this is a unitary operator for $\lambda \in \mathbb{R}$. Under the assumption that supp V is compact, the scattering amplitude $A(\lambda) : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$, defined in (3.9), is a trace class operator. Hence, the function

$$f_V(\lambda) \equiv \det S(\lambda), \tag{6.2}$$

is well-defined, at least for $\Im\lambda > 0$ sufficiently large.

What are the meromorphic properties of $f_V(\lambda)$? As proved in Theorem 16, the S-matrix has a meromorphic continuation to the entire complex plane with finitely many poles for $\Im\lambda > 0$ corresponding to eigenvalues of H_V, and poles in $\Im\lambda < 0$ corresponding to resonances. We recall that if $\Im\lambda_0 \geq c_0\langle\|V\|_{L^\infty}\rangle$, the multiplicity of λ_0, as a zero of $\det S_V(\lambda)$, and of $-\lambda_0$, as a pole of the cut-off resolvent $\mathcal{R}_V(\lambda)$, coincide. Consequently, the function $f_V(\lambda)$ is holomorphic for $\Im\lambda > c_0\langle\|V\|_{L^\infty}\rangle$, and well-defined for $\Im\lambda \geq 0$ with finitely many poles corresponding to the eigenvalues of H_V. Hence, the problem of estimating the number of zeros of $f_V(\lambda)$ in the upper half plane is the same as estimating the number of resonances in the lower half plane.

The estimates on $f_V(\lambda)$ are facilitated in the odd dimensional case by the well-known representation of $f_V(\lambda)$ in terms of canonical products. Let $G(\lambda; p)$ be defined for integer $p \geq 1$, by

$$G(\lambda; p) = (1-\lambda)e^{\lambda+\lambda^2/2+\cdots+\lambda^p/p}, \tag{6.3}$$

and define

$$P(\lambda) = \Pi_{\lambda_j \in \mathcal{R}_V, \lambda_j \neq 0}\ G(\lambda/\lambda_j; d-1). \tag{6.4}$$

Then the function $f_V(\lambda)$ may be written as

$$f_V(\lambda) = \alpha e^{ig(\lambda)}\frac{P(-\lambda)}{P(\lambda)}, \tag{6.5}$$

for some constant $\alpha > 0$ and where $g(\lambda)$ is a polynomial of order at most d. Careful study of the scattering matrix and the upper bound of Theorem 19 may be used to show that $f_V(\lambda)$ is of order at most d in the half-plane $\Im\lambda > c_0\langle\|V\|_\infty\rangle$, see [48].

We consider a fixed compact set $K \subset \mathbb{R}^d$ with nonempty interior. Let $\mathcal{M}(K)$ be the subset of potentials in $L^\infty_0(K)$ having a resonance counting function with maximal order of growth. We can separately consider real- or complex-valued potentials. The proof of Theorem 21 requires that we prove 1) that $\mathcal{M}(K)$ is a G_δ-set, and 2) that $\mathcal{M}(K)$ is dense in $L^\infty_0(K)$. The proof that $\mathcal{M}(K)$ is a G_δ-set uses standard estimates on the S-matrix as in [5]. For $N, M, j \in \mathbb{N}$ with $j > 2N+1$, and for $q > 0$, we define sets of potentials $A(N, M, q, j) \subset L^\infty_0(K)$ by

$$\begin{aligned} A(N, M, q, j) \quad \equiv \quad & \{V \in L^\infty_0(K) \mid \|V\|_{L^\infty} \leq N,\ \ \log|\det(S_V(\lambda))| \leq M|\lambda|^q, \\ & \text{for } \Im\lambda \geq 2N+1 \text{ and } |\lambda| \leq j\} \end{aligned} \tag{6.6}$$

One proves that these sets are closed. More importantly, we can use these sets to characterize the set of potentials having a resonance counting function with an order of growth *strictly less that* d.

For this, we define sets $B(N, M, q)$ by

$$B(N, M, q) \equiv \bigcap_{j \geq 2N+1} A(N, M, q, j). \tag{6.7}$$

One proves that if $N_V(r)$ has order of growth strictly less than d, then one can find $(N, M, \ell) \in \mathbb{N}^3$ so that $V \in B(N, M, d - 1/\ell)$. Since the sets $A(N, M, j, q)$ are closed, so are the sets $B(N, M, j)$. One notes that $\cup_{(N,M,j)\in\mathbb{N}^3} B(N, M, j)$ is an F_σ set. The final step of the proof is to show that $\mathcal{M}(K)$ is the complement of this set. It follows that $\mathcal{M}(K)$ is a G_δ-set.

The proof of the density of $\mathcal{M}(K)$ is more involved and relies on machinery from complex analysis as developed in [4]. The basic idea is to consider a wider family of potentials $V(x; z)$ holomorphic in the complex variable $z \in \Omega \subset \mathbb{C}$, for some open subset Ω. The construction of the S-matrix goes through for these complex-valued potentials. The key result is that if for some $z_0 \in \Omega$ the order of growth $\rho_{V(z_0)}$ for $N_{V(z_0)}$ is equal to d, then there is a pluripolar subset $E \subset \Omega$ so that the order of growth for all potentials $V(z)$, with $z \in \Omega\backslash E$, is equal to d. A pluripolar set is very small, in particular, the Lebesgue measure of $E \cap \mathbb{R}$ is zero.

How do we know there is a potential for which $N_V(r)$ has maximal order of growth? For $d \geq 3$ odd, we can use the result of Zworski [47]. As mentioned in section 5.4, Zworski proved the an asymptotic expansion for $N_V(r)$ for a class of radially symmetric potentials with compact support. Let V_0 be one of these potentials so that $V_0 \in \mathcal{M}(K)$. To prove the density of $\mathcal{M}(K)$ in $L^\infty_0(\mathbb{R}^d)$, we take any $V_1 \in L^\infty_0(K)$ and form $V(z) = zV_0 + (1 - z)V_1$. This is a holomorphic function of z for $z \in \Omega = \mathbb{C}$. We apply the result described above to this family of holomorphic potentials. In particular, for $z_0 = 1$, we have $V(z_0) = V_0$ and $\rho_{V(z_0)} = d$ by Zworski's result. Let $E \subset \mathbb{C}$ be the pluripolar set so that for $z \in \mathbb{C}\backslash E$, the order of growth $\rho_{V(z)} = d$. Since the Lebesgue measure of $E \cap \mathbb{R}$ is zero, we can find $z \in \mathbb{R}\backslash(E \cap \mathbb{R})$, with $|z|$ as small as desired, for which $\rho_{V(z)} = d$. So, given $\epsilon > 0$, we take $\tilde{z} \in \mathbb{R}\backslash(E \cap \mathbb{R})$ so that $|\tilde{z}| \leq \epsilon(1 + \|V_1\|_{L^\infty} + \|V_0\|_{L^\infty})^{-1}$. With this choice, we have

$$\|V_1 - V(\tilde{z})\|_{L\infty} \leq |\tilde{z}| \, (\|V_1\|_{L^\infty}\|V_0\|_{L^\infty}) \leq \epsilon. \tag{6.8}$$

This proves the density of $\mathcal{M}(K)$ in $L^\infty_0(\mathbb{R}^d)$. Note that we can take V_0 real-valued and so $V(\tilde{z})$ is real-valued.

We remark that the representation (6.5) is not available in the even dimensional case.

6.2 Generic behavior: even dimensions

We now summarize the corresponding results in the even dimensional case. Let $\chi_V \in C^\infty_0(\mathbb{R}^d)$ be a smooth, compactly-supported function satisfying $\chi_V V = V$, and denote the resolvent of H_V by $R_V(\lambda) = (H_V - \lambda^2)^{-1}$. In the even-dimensional case, the operator-valued function $\chi_V R_V(\lambda)\chi_V$ has a meromorphic continuation to the infinitely-sheeted Riemann surface of the logarithm Λ. We denote by Λ_m the m^{th} open sheet consisting of $z \in \Lambda$ with $m\pi < \arg z < (m + 1)\pi$. The physical sheet corresponds to Λ_0 and it is identified with the upper half complex plane. We denote the number

of the poles $N_{V,m}(r)$ of the meromorphic continuation of the truncated resolvent $\chi_V R_V(\lambda)\chi_V$ on each sheet Λ_m, counted with multiplicity, and with modulus at most $r > 0$.

The *order of growth* of the resonance counting function $N_{V,m}(r)$ for H_V on the m^{th}-sheet is defined by

$$\rho_{V,m} \equiv \limsup_{r\to\infty} \frac{\log N_{V,m}(r)}{\log r}. \tag{6.9}$$

It is known that $\rho_{V,m} \leq d$ for $d \geq 2$ even [41, 42]. As in the odd dimensional case, it is proved that generically (in the sense of Baire typical) the resonance counting function has the maximal order of growth d on each non-physical sheet.

Theorem 22. Let $d \geq 2$ be even, and let $K \subset \mathbb{R}^d$ be a fixed, compact set with nonempty interior. Let $\mathcal{M}_F(K) \subset L^\infty_0(K)$, for $F = \mathbb{R}$ or $F = \mathbb{C}$, be the set of all real-valued, respectively, complex-valued potentials in $L^\infty_0(K)$ such that the resonance counting functions $N_{V,m}(r)$, for $m \in \mathbb{Z}\backslash\{0\}$, have maximal order of growth. Then, the set $\mathcal{M}_F(K)$ is a dense G_δ set for $F = \mathbb{R}$ or $F = \mathbb{C}$.

This theorem states that for a generic family of real or complex-valued potentials in $L^\infty_0(K)$, the order of growth of the resonance counting function is maximal on each sheet, $\rho_{V,m} = d$, for $m \in \mathbb{Z}\backslash\{0\}$. This implies that there are generically infinitely many resonances on each nonphysical sheet.

There are two challenges in proving Theorem 22. The first is to construct a function whose analytic extension to the m^{th}-sheet Λ_m has zeros at the resonances of H_V. This function will substitute for (6.2). The second problem is prove a lower bound (5.35) for some potential in $L^\infty_0(K)$ in even dimensions.

To resolve the first problem, we use the following key identity, that follows from (4.16) and the formulas for the meromorphic continuation of Hankel functions (see [6, section 6] or [23, chapter 7]), relating the free resolvent on Λ_m to that on Λ_0, for any $m \in \mathbb{Z}$,

$$R_0(e^{im\pi}\lambda) = R_0(\lambda) - m(d)T(\lambda), \quad \text{where } m(d) = \begin{cases} m \mod 2 & d \text{ odd} \\ m & d \text{ even.} \end{cases} \tag{6.10}$$

The operator $T(\lambda)$ on $L^2(\mathbb{R}^d)$ has a Schwartz kernel

$$T(\lambda; x, y) = i\pi(2\pi)^{-d}\lambda^{d-2}\int_{\mathbb{S}^{d-1}} e^{i\lambda(x-y)\cdot\omega}\, d\omega, \tag{6.11}$$

see [19, Section 1.6]. This operator is related to $M(\lambda)$ introduced in section 4.2 in (4.13) (see also (5.24)). We note that for any $\chi \in C^\infty_0(\mathbb{R}^d)$, $\chi T(\lambda)\chi$ is a holomorphic trace-class operator for $\lambda \in \mathbb{C}$. The operator $T(\lambda)$ has a kernel proportional to $|x-y|^{(-d+2)/2}J_{(d-2)/2}(\lambda|x-y|)$ when d is odd, and to $|x-y|^{(-d+2)/2}N_{(d-2)/2}(\lambda|x-y|)$ when d is even. The different behavior of the free resolvent for d odd or even is encoded in (6.10).

By the second resolvent formula (4.20), the poles of $\mathcal{R}_V(\lambda)$ with multiplicity, correspond to the zeros of $I + VR_0(\lambda)\chi_V$. We can reduce the analysis of the zeros of the continuation of $I + VR_0(\lambda)\chi_V$

to Λ_m to the analysis of zeros of a related operator on Λ_0 using (6.10). If $0 < \arg\lambda < \pi$ and $m \in \mathbb{Z}$, then $e^{im\pi}\lambda \in \Lambda_m$, and

$$\begin{aligned} I + VR_0(e^{im\pi}\lambda)\chi &= I + V(R_0(\lambda) - mT(\lambda))\chi_V \\ &= (I + VR_0(\lambda)\chi_V)(I - m(I + VR_0(\lambda)\chi_V)^{-1}VT(\lambda)\chi_V). \end{aligned}$$

For any fixed $V \in L^\infty_0(\mathbb{R}^d)$, there are only finitely many poles of $(I+VR_0(\lambda)\chi_V)^{-1}$ with $0 < \arg\lambda < \pi$. Thus

$$f_{V,m}(\lambda) = \det(I - m(I + VR_0(\lambda)\chi_V)^{-1}VT(\lambda)\chi_V) \tag{6.12}$$

is a holomorphic function of λ when $0 < \arg\lambda < \pi$ and $|\lambda| > c_0\langle\|V\|_{L^\infty}\rangle$. Moreover, with at most a finite number of exceptions, the zeros of $f_{V,m}(\lambda)$, with $0 < \arg\lambda < \pi$ correspond, with multiplicity, to the poles of $\mathcal{R}_V(\lambda)$ with $m\pi < \arg\lambda < (m+1)\pi$. Henceforth, we will consider the function $f_{V,m}(\lambda)$, for $m \in \mathbb{Z}^* \equiv \mathbb{Z}\backslash\{0\}$, on Λ_0. For d odd, we are only interested in $m = -1$. In this case, the zeros of $f_{V,-1}(\lambda)$, for $\lambda \in \Lambda_0$, correspond to the resonances. This provides an alternative to the S-matrix formalism, as presented in section 6.1, for estimating the resonance counting function in the odd dimensional case.

The second problem in even dimensions is to prove that there are some potentials in $L^\infty_0(K)$ for which the resonance counting function has the correct lower bound on each sheet. This is done by an explicit calculation. We prove (5.35) in even dimensions $d \geq 2$ for Schrödinger operators H_V with radial potentials $V(x) = V_0\chi_{B_R(0)}(x)$, with $V_0 > 0$, using separation of variables and uniform asymptotics of Bessel and Hankel functions due to Olver [23, 24, 25]. This method can also be used in odd dimensions as an alternative to [47] thus providing examples as required in section 6.1.

7 Topics not covered and some literature

This notes focussed on perturbations of the Laplacian on $\mathbb{R}^d$ by real-valued, smooth, compactly supported potentials. This is just one family of examples where resonances arise. Other topics concerning resonances include:

(1) Complex-spectral deformation method and resonances

(2) Obstacle scattering

(3) Resonance free regions

(4) Resonances for the wave equation

(5) Resonances for elastic media

(6) Resonances for manifolds hyperbolic at infinity

(7) Semiclassical theory of resonances

(8) Description of resonance wave functions

(9) Approximate exponential decay of resonance states

(10) Local energy decay estimates

There are some reviews on resonances that cover many aspects of the theory in this list. These reviews include:

(1) The long discussion by M. Zworski [49] that covers the complex scaling method developed by Sjöstrand and Zworski (inspired by the Baslev-Combes method) and its applications.

(2) G. Vodev has written an expository article in *Cubo* [44]. Many aspects of resonances for elastic bodies and obstacle scattering are described there.

(3) The proof of the generic properties of the resonance counting function for even and odd dimensions is described in Christiansen and Hislop [8], an expository summary written for *les Journées EDP 2008 Evian* available on the arXiv and from Cedram.

(4) Text book versions of spectral deformation and quantum resonances, with an emphasis on the semiclassical regime, can be found in [9] and [14].

Finally, for a lighter and intuitive discussion of resonances, the reader is referred to Zworski's article *Resonances in physics and geometry* that appeared in the *Notices of the American Mathematical Society* [50].

7.1 Acknowledgements

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8 Appendix: Assorted results

Two groups of results that are related to material in the text are summarized here. The first is a synopsis of the spectral theory of linear self-adjoint operators. The second is an analysis of the time evolution of states lying in various spectral subspaces of a self-adjoint operator. Detailed discussions of these topics may be found in the Reed-Simon series [29]-[32], for example, and many other texts.

8.1 Spectral theory

Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Then, there is a direct sum decomposition $\mathcal{H} = \mathcal{H}_{ac}(A) \oplus \mathcal{H}_{sc}(A) \oplus \mathcal{H}_{pp}(A)$ into three orthogonal subspaces that are A-invariant in that $A : D(A) \cap \mathcal{H}_X(A) \to \mathcal{H}_X(A)$ for $X = ac, sc, pp$. The pure point subspace $\mathcal{H}_{pp}(A)$ is the closure of the span of all the eigenfunctions of A. The continuous subspace $\mathcal{H}_{con}(A) \equiv \mathcal{H}_{ac}(A) \oplus \mathcal{H}_{sc}(A)$ is the orthogonal complement of $\mathcal{H}_{pp}(A)$. For most Schrödinger operators $H_V = -\Delta + V$, one has $\mathcal{H}_{sc}(H_V) = \emptyset$. The proof of the absence of singular continuous spectrum is one of the main applications of the Mourre estimate, see the discussion in section 4.1, [9, chapter 4], and the original paper [22]. As the names suggest, there is a measure associated with a self-adjoint operator and this measure has a Lebesgue decomposition into pure point and continuous measures. The continuous measure admits a decomposition relative to Lebesgue measure into a singular continuous and absolutely continuous parts.

8.2 The RAGE Theorem

The RAGE Theorem (Ruelle, Amrein, Georgescu, Enss) (see, for example, [9, section 5.4]) is a general result about the averaged time evolution of states in the continuous subspace $\mathcal{H}_{cont}(A)$ of a self-adjoint operator A.

Theorem 23. Let A be a self-adjoint operator and $\phi \in \mathcal{H}_{cont}(A)$, where $\mathcal{H}_{cont}(A)$ is the continuous spectral subspace of A. Suppose that C is a bounded operator and that $C(A + i)^{-1}$ is compact. Then, we have

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|Ce^{-itA}\phi\| \, dt = 0. \tag{8.1}$$

Furthermore, if $\phi \in \mathcal{H}$ satisfies (8.1), then $\phi \in \mathcal{H}_{cont}(A)$.

Let $A = H_V$ be a Schrödinger operator of the type considered here, and $C = \chi_{B_R(0)}$, the characteristic function on a ball of radius $R > 0$ centered at the origin. The RAGE Theorem (8.1) states that a state, initially localized near the origin, and in the continuous spectral subspace of H_V, will eventually leave this neighborhood of the origin in this time-averaged sense. The continuous spectral subspace $\mathcal{H}_{cont}(H_V)$ has a further decomposition into the singular and absolutely continuous subspaces. It is the possible recurrent behavior of states in the singular continuous subspace that requires the time averaging in (8.1).

Corollary 24. Let H_V be a self-adjoint operator on $L^2(\mathbb{R}^d)$. Let $\phi \in \mathcal{H}_{ac}(H_V)$, where $\mathcal{H}_{ac}(H_V)$ is the absolutely continuous spectral subspace of H_V. Let χ_K be the characteristic function for a compact subset $K \subset \mathbb{R}^d$. Then, we have

$$\lim_{t\to\infty} \|\chi_K U_V(t)\phi\| = 0. \tag{8.2}$$

As one might expect, if ϕ is a finite linear combination of eigenfunctions, then the state $\chi_K U_V(t)\phi$ remains localized for all time. Indeed, for any $\epsilon > 0$, there is a compact subset $K_\epsilon \subset \mathbb{R}^d$ so that $\|\chi_{\mathbb{R}^d \setminus K_\epsilon} U_V(t)\phi\| < \epsilon$, for all t.

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