Some generalized difference double sequence spaces defined by a sequence of Orlicz-functions

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ABSTRACT

In the present paper we introduce some generalized difference double sequence spaces defined by a sequence of Orlicz-functions. We study some topological properties and some inclusion relations between these spaces. We also make an effort to study these properties over n-normed spaces.

RESUMEN

En este artículo introducimos algunos espacios de sucesiones doble-diferencia generalizadas definidas por una sucesión de funciones de Orlicz. Estudiamos algunas propiedades topológicas y algunas relaciones de inclusión entre estos espacios. Además, hacemos un esfuerzo para estudiar estas propiedades en espacios n-normados.

1 Introduction and Preliminaries

The initial works on double sequences is found in Bromwich [4]. Later on, it was studied by Hardy [6], Moricz [17], Moricz and Rhoades [18], Tripathy ([33], [34]), Basarir and Sonalcan [2] and many others. Hardy[6] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [36] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [21] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesaro summable double sequences. Nextly, Mursaleen [19] and Mursaleen and Edely [22] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{mn})$ into one whose core is a subset of the M-core of x. More recently, Altay and Basar [1] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(\nu)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Now, recently Basar and Sever [3] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Let w^2 denote the set of all double sequences of complex numbers. By the convergence of a double sequence we mean the convergence of the Pringsheim sense i.e. a double sequence $x = (x_{kl})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\epsilon > 0$ there exists $n \in N$ such that $|x_{kl} - L| < \epsilon$ whenever $k, l > n$ see [26]. We shall write more briefly as P-convergent. We shall denote the space of all P-convergent sequences by c^2. The double sequence $x = (x_{kl})$ is bounded if there exists a positive number M such that $|x_{kl}| < M$ for all k and l. Let l_∞^2 the space of all bounded double sequence such that $\|x_{kl}\|_{\infty,2} = \sup_{kl} |x_{kl}| < \infty$. For more details about double sequence spaces see ([30], [31],[32]) and references therein.

The notion of difference sequence spaces was introduced by Kızmaz [13], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et. and Çolak [5] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let m, s be non-negative integers, then for $Z = l_\infty$, c, c_0 we have sequence spaces

$$Z(\Delta_s^m) = \{x = (x_k) \in w : (\Delta_s^m x_k) \in Z\},$$

where $\Delta_s^m x = (\Delta_s^m x_k) = (\Delta_s^{m-1} x_k - \Delta_s^{m-1} x_{k+1})$ and $\Delta_s^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_s^m x_k = \sum_{v=0}^m (-1)^v \binom{m}{v} x_{k+sv}.$$

Taking $s = 1$, we get the spaces which were studied by Et and Çolak [5]. Taking $m = s = 1$, we get the spaces which were introduced and studied by Kızmaz [13].

An Orlicz function $M : [0, \infty) \to [0, \infty)$ is a continuous, non-decreasing and convex function such that $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$. Lindenstrauss and Tzafriri [14] used the idea of Orlicz function to define the following sequence space:

$$l_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \right\}$$

which is called as an Orlicz sequence space. Also l_M is a Banach space with the norm

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1 \right\}.$$

Also, it was shown in [14] that every Orlicz sequence space l_M contains a subspace isomorphic to $l_p (p \geq 1)$. The Δ_2- condition is equivalent to $M(Lx) \leq LM(x)$, for all L with $0 < L < 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t) dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0, \eta$ is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

Let X be a linear metric space. A function $p : X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$,

(2) $p(-x) = p(x)$, for all $x \in X$,

(3) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$,

(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [35], Theorem 10.4.2, P-183). For more details about sequence spaces see ([12], [15], [20], [23], [24], [25], [27]) and references therein.

Let $\mathcal{M} = (M_{k,l})$ be a sequence of orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers. Let X be a semi-normed space over the complex field $\mathbb{C}$ with the seminorm q. Now we define the following classes of sequences in the present paper:

$$c^2(\Delta_n^m, \mathcal{M}, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l}(kl)^{-s} u_{k,l}\left[M_{k,l}\left(q\left(\frac{\Delta_n^m x_{k,l} - L}{\rho}\right)\right)\right]^{p_{k,l}} = 0, \right.$$

$$\left. \text{for some } \rho > 0, L \text{ and } s \geq 0 \right\},$$

$$c_0^2(\Delta_n^m, \mathcal{M}, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[M_{k,l} \left(q\left(\frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l}} = 0, \right.$$

$$\left. \text{for some } \rho > 0 \text{ and } s \geq 0 \right\}$$

and

$$l_\infty^2(\Delta_n^m, \mathcal{M}, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : \sup_{k,l}(kl)^{-s} u_{k,l} \left[M_{k,l} \left(q\left(\frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l}} < \infty, \right.$$

$$\left. \text{for some } \rho > 0 \text{ and } s \geq 0 \right\}.$$

If we take $\mathcal{M}(x) = x$, we get

$$c^2(\Delta_n^m, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[q\left(\frac{\Delta_n^m x_{k,l} - L}{\rho} \right) \right]^{p_{k,l}} = 0, \right.$$

$$\left. \text{for some } \rho > 0, L \text{ and } s \geq 0 \right\},$$

$$c_0^2(\Delta_n^m, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[q\left(\frac{\Delta_n^m x_{k,l}}{\rho} \right) \right]^{p_{k,l}} = 0, \right.$$

$$\left. \text{for some } \rho > 0 \text{ and } s \geq 0 \right\}$$

and

$$l_\infty^2(\Delta_n^m, u, p, q, s) = \left\{ x = (x_{k,l}) \in w^2 : \sup_{k,l}(kl)^{-s} u_{k,l} \left[q\left(\frac{\Delta_n^m x_{k,l}}{\rho} \right) \right]^{p_{k,l}} < \infty, \right.$$

$$\left. \text{for some } \rho > 0 \text{ and } s \geq 0 \right\}.$$

If we take $p = (p_{k,l}) = 1$, we get

$$c^2(\Delta_n^m, \mathcal{M}, u, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[M_{k,l} \left(q\left(\frac{\Delta_n^m x_{k,l} - L}{\rho} \right) \right) \right] = 0, \right.$$

$$\left. \text{for some } \rho > 0, L \text{ and } s \geq 0 \right\},$$

$$c_0^2(\Delta_n^m, \mathcal{M}, u, q, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[M_{k,l} \left(q\left(\frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right] = 0, \right.$$

$$\left. \text{for some } \rho > 0 \text{ and } s \geq 0 \right\}$$

and

$$l_\infty^2(\Delta_n^m, \mathcal{M}, u, q, s) = \left\{ x = (x_{k,l}) \in w^2 : \sup_{k,l}(kl)^{-s} u_{k,l} \left[M_{k,l} \left(q\left(\frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right] < \infty, \right.$$

$$\left. \text{for some } \rho > 0 \text{ and } s \geq 0 \right\}.$$

If we take $m = n = 0$ and $q(x) = |x|$, then we get new double sequence spaces as follows :

$$c^2(\mathcal{M}, u, p, s) = \left\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[M_{k,l} \left(\frac{|x_{k,l} - L|}{\rho} \right) \right]^{p_{k,l}} = 0, \right.$$

$$\text{for some } \rho > 0, L \text{ and } s \geq 0 \Big\},$$

$$c_0^2(\mathcal{M}, u, p, s) = \Big\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l}(kl)^{-s} u_{k,l} \Big[M_{k,l}\Big(\frac{|x_{k,l}|}{\rho}\Big)\Big]^{p_{k,l}} = 0,$$

$$\text{for some } \rho > 0 \text{ and } s \geq 0 \Big\}$$

and

$$l_\infty^2(\mathcal{M}, u, p, s) = \Big\{ x = (x_{k,l}) \in w^2 : \sup_{k,l}(kl)^{-s} u_{k,l} \Big[M_{k,l}\Big(\frac{|x_{k,l}|}{\rho}\Big)\Big]^{p_{k,l}} < \infty,$$

$$\text{for some } \rho > 0 \text{ and } s \geq 0 \Big\}.$$

If we take $m = n = 1$ and $q(x) = |x|$, then we get new double sequence spaces as follows :

$$c^2(\Delta_n^m, \mathcal{M}, u, p, s) = \Big\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l}(kl)^{-s} u_{k,l} \Big[M_{k,l}\Big(\frac{|\Delta x_{k,l} - L|}{\rho}\Big)\Big]^{p_{k,l}} = 0,$$

$$\text{for some } \rho > 0, L \text{ and } s \geq 0 \Big\},$$

$$c_0^2(\Delta_n^m, \mathcal{M}, u, p, s) = \Big\{ x = (x_{k,l}) \in w^2 : P - \lim_{k,l}(kl)^{-s} u_{k,l} \Big[M_{k,l}\Big(\frac{|\Delta x_{k,l}|}{\rho}\Big)\Big]^{p_{k,l}} = 0,$$

$$\text{for some } \rho > 0 \text{ and } s \geq 0 \Big\}$$

and

$$l_\infty^2(\Delta_n^m, \mathcal{M}, u, p, s) = \Big\{ x = (x_{k,l}) \in w^2 : \sup_{k,l}(kl)^{-s} u_{k,l} \Big[M_{k,l}\Big(\frac{|\Delta_n^m x_{k,l}|}{\rho}\Big)\Big]^{p_{k,l}} < \infty,$$

$$\text{for some } \rho > 0 \text{ and } s \geq 0 \Big\}.$$

The following inequality will be used throughout the paper. Let $p = (p_{k,l})$ be a double sequence of positive real numbers with $0 < p_{k,l} \leq \sup_{k,l} = H$ and let $K = \max\{1, 2^{H-1}\}$. Then for the factorable sequences $\{a_{k,l}\}$ and $\{b_{k,l}\}$ in the complex plane, we have

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq K(|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}). \tag{1.1}$$

The main goal of this paper is to extend a few known results in the literature from single difference sequence spaces to double difference sequence spaces. We also make an effort to study some topological properties and inclusion relations between above defined sequence spaces.

2   Main Results

Theorem 2.1 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers, then the classes of sequences $c_0^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$, $c^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$ and $l_\infty(\Delta_n^m, \mathcal{M}, u, p, q, s)$ are linear spaces over the field of complex numbers $\mathbb{C}$.*

Proof. Let $x = (x_{k,l})$, $y = (y_{k,l}) \in c_0^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\lim_{k,l}(kl)^{-s} u_{k,l}\Big[M_{k,l}\Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho_1}\Big)\Big)\Big]^{p_{k,l}} = 0, \ \text{ for some } \ \rho_1 > 0$$

and

$$\lim_{k,l}(kl)^{-s} u_{k,l}\Big[M_{k,l}\Big(q\Big(\frac{\Delta_m^n y_{k,l}}{\rho_2}\Big)\Big)\Big]^{p_{k,l}} = 0, \ \text{ for some } \ \rho_2 > 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_{k,l})$ is non-decreasing convex function and so by using inequality (1.1), we have

$$\lim_{k,l}(kl)^{-s} u_{k,l}\Big[M_{k,l}\Big(q\Big(\frac{\Delta_n^m(\alpha x_{k,l} + \beta y_{k,l})}{\rho_3}\Big)\Big)\Big]^{p_{k,l}}$$

$$\begin{aligned}
&= \lim_{k,l}(kl)^{-s} u_{k,l}\Big[M_{k,l}\Big(q\Big(\frac{\alpha\Delta_n^m x_{k,l}}{\rho_3}\Big) + q\Big(\frac{\beta\Delta_n^m y_{k,l}}{\rho_3}\Big)\Big)\Big]^{p_{k,l}}\\
&\leq \ K\lim_{k,l}\frac{1}{2^{p_{k,l}}}(kl)^{-s} u_{k,l}\Big[M_{k,l}\Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho_1}\Big)\Big)\Big]^{p_{k,l}}\\
&+ \ K\lim_{k,l}\frac{1}{2^{p_{k,l}}}(kl)^{-s} u_{k,l}\Big[M_{k,l}\Big(q\Big(\frac{\Delta_n^m y_{k,l}}{\rho_2}\Big)\Big)\Big]^{p_{k,l}}\\
&\leq \ K\lim_{k,l}(kl)^{-s} u_{k,l}\Big[M_{k,l}\Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho_1}\Big)\Big)\Big]^{p_{k,l}}\\
&+ \ K\lim_{k,l}(kl)^{-s} u_{k,l}\Big[M_{k,l}\Big(q\Big(\frac{\Delta_n^m y_{k,l}}{\rho_2}\Big)\Big)\Big]^{p_{k,l}}\\
&= \ 0.
\end{aligned}$$

So, $\alpha x + \beta y \in c_0^2(\Delta_m^n, \mathcal{M}, u, p, q, s)$. Hence $c_0^2(\Delta_m^n, \mathcal{M}, u, p, q, s)$ is a linear space. Similarly, we can prove that $c^2(\Delta_m^n, \mathcal{M}, u, p, q, s)$ and $l_\infty^2(\Delta_m^n, \mathcal{M}, u, p, q, s)$ are linear spaces.

Theorem 2.2 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers. For $Z^2 = l_\infty^2, c^2$ and c_0^2, the spaces $Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$ are paranormed spaces, paranormed by*

$$g(x) = \sum_{k,l=1}^{nm} q(x_{k,l}) + \inf\Big\{\rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho}\Big)\Big) \leq 1\Big\}$$

where $H = \max(1, \sup_{k,l} p_{k,l})$.

Proof.   Clearly $g(-x) = g(x)$, $g(0) = 0$. Let $(x_{k,l})$ and $(y_{k,l})$ be any two sequences belong to any one of the spaces $Z^2(\Delta_m^n, \mathcal{M}, u, p, q, s)$, for $Z^2 = c_0^2$, c^2 and l_∞^2. Then, we get ρ_1, $\rho_2 > 0$ such that

$$\sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho_1}\Big)\Big) \leq 1$$

and

$$\sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\Big(q\Big(\frac{\Delta_n^m y_{k,l}}{\rho_2}\Big)\Big) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then by convexity of $\mathcal{M} = (M_{k,l})$, we have

$$
\begin{aligned}
\sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\Big(q\Big(\frac{\Delta_n^m(x_{k,l} + y_{k,l})}{\rho}\Big)\Big) \quad &\leq \quad \Big(\frac{\rho_1}{\rho_1 + \rho_2}\Big) \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho_1}\Big)\Big) \\
&+ \quad \Big(\frac{\rho_2}{\rho_1 + \rho_2}\Big) \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\Big(q\Big(\frac{\Delta_n^m y_{k,l}}{\rho_2}\Big)\Big) \\
&\leq \quad 1.
\end{aligned}
$$

Hence we have,

$$
\begin{aligned}
g(x+y) \quad = \quad & \sum_{k,l=1}^{mn} q(x_{k,l} + y_{k,l}) \\
+ \quad & \inf\Big\{\rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\Big(q\Big(\frac{\Delta_n^m(x_{k,l} + y_{k,l})}{\rho}\Big)\Big) \leq 1\Big\} \\
\leq \quad & \sum_{k,l=1}^{mn} q(x_{k,l}) + \inf\Big\{\rho_1^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho_1}\Big)\Big) \leq 1\Big\} \\
+ \quad & \sum_{k,l=1}^{mn} q(y_{k,l}) + \inf\Big\{\rho_2^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\Big(q\Big(\frac{\Delta_n^m y_{k,l}}{\rho_2}\Big)\Big) \leq 1\Big\}.
\end{aligned}
$$

This implies that

$$g(x+y) \leq g(x) + g(y).$$

The continuity of the scalar multiplication follows from the following inequality

$$
\begin{aligned}
g(\mu x) \quad = \quad & \sum_{k,l=1}^{mn} q(\mu x_{k,l}) + \inf\Big\{\rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\Big(q\Big(\frac{\Delta_n^m \mu x_{k,l}}{\rho}\Big)\Big) \leq 1\Big\} \\
= \quad & |\mu| \sum_{k,l=1}^{mn} q(x_{k,l}) + \inf\Big\{(t|\mu|)^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\Big(q\Big(\frac{\Delta_n^m x_{k,l}}{t}\Big)\Big) \leq 1\Big\},
\end{aligned}
$$

where $t = \frac{\rho}{|\mu|}$. Hence the space $Z^2(\Delta_m^n, \mathcal{M}, u, p, q, s)$, for $Z^2 = c_0^2, c^2$ and l_∞^2 is a paranormed space, paranormed by g.

Theorem 2.3   *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers.*

For $Z^2 = l_\infty^2, c^2$ *and* c_0^2, *the spaces* $Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$ *are complete paranormed spaces, para-normed by*

$$g(x) = \sum_{k,l=1}^{nm} q(x_{k,l}) + \inf\left\{\rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s}u_{k,l}M_{k,l}\left(q\left(\frac{\Delta_n^m x_{k,l}}{\rho}\right)\right) \leq 1\right\},$$

where $H = \max(1, \sup_{k,l} p_{k,l})$.

Proof. We prove the result for the space $l_\infty^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$. Let $(x_{k,l}^i)$ be any Cauchy sequence in $l_\infty^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$. Let $\epsilon > 0$ be given and for $t > 0$, choose x_0 be fixed such that $u_{k,l}M_{k,l}\left(\frac{tx_0}{2}\right) \geq 1$, then there exists a positive integer $n_0 \in \mathbb{N}$ such that $g(x_{k,l}^i - x_{k,l}^j) < \frac{\epsilon}{x_0 t}$, for all $i, j \geq n_0$. Using the definition of paranorm, we get

$$\sum_{k,l=1}^{mn} q(x_{k,l}^i - x_{k,l}^j) + \inf\left\{\rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s}u_{k,l}M_{k,l}\left(q\left(\frac{\Delta_n^m(x_{k,l}^i - x_{k,l}^j)}{\rho}\right)\right)\right\} < \frac{\epsilon}{x_0 t},$$

$$\text{for all } i, j \geq n_0 \qquad (2.1).$$

Hence we have,

$$\sum_{k,l=1}^{mn} q(x_{k,l}^i - x_{k,l}^j) < \epsilon, \text{ for all } i, j \geq n_0.$$

This implies that

$$q(x_{k,l}^i - x_{k,l}^j) < \epsilon, \text{ for all } i, j \geq n_0 \text{ and } 1 \leq k \leq nm.$$

Thus $(x_{k,l}^i)$ is a Cauchy sequence in $\mathbb{C}$ for $k, l = 1, 2, ..., nm$. Hence $(x_{k,l}^i)$ is convergent in $\mathbb{C}$ for $k, l = 1, 2, ..., nm$. Let $\lim_{i\to\infty} x_{k,l}^i = x_{k,l}$, say for $k, l = 1, 2, ..., nm$. $\qquad (2.2)$
Again from equation (2.1) we have,

$$\inf\left\{\rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s}u_{k,l}M_{k,l}\left(q\left(\frac{\Delta_n^m(x_{k,l}^i - x_{k,l}^j)}{\rho}\right)\right) \leq 1\right\} < \epsilon, \text{ for all } i, j \geq n_0.$$

Hence we get

$$\sup_{k,l}(kl)^{-s}u_{k,l}M_{k,l}\left(q\left(\frac{\Delta_n^m(x_{k,l}^i - x_{k,l}^j)}{g(x^i - x^j)}\right)\right) \leq 1, \text{ for all } i, j \geq n_0.$$

It follows that $(kl)^{-s}u_{k,l}M_{k,l}\left(q\left(\frac{\Delta_n^m(x_{k,l}^i - x_{k,l}^j)}{g(x^i - x^j)}\right)\right) \leq 1$, for each $k, l \geq 1$ and for all $i, j \geq n_0$.
For $t > 0$ with $(kl)^{-s}u_{k,l}M_{k,l}(\frac{tx_0}{2}) \geq 1$, we have

$$(kl)^{-s}u_{k,l}M_{k,l}\left(q\left(\frac{\Delta_n^m(x_{k,l}^i - x_{k,l}^j)}{g(x^i - x^j)}\right)\right) \leq (kl)^{-s}u_{k,l}M_{k,l}\left(\frac{tx_0}{2}\right).$$

This implies that

$$q(\Delta_n^m x_{k,l}^i - \Delta_n^m x_{k,l}^j) < \frac{tx_0}{2}\frac{\epsilon}{tx_0} = \frac{\epsilon}{2}.$$

Hence $q(\Delta_n^m x_{k,l}^i)$ is a Cauchy sequence in $\mathbb{C}$ for all $k, l \in \mathbb{N}$. This implies that $q(\Delta_n^m x_{k,l}^i)$ is convergent in $\mathbb{C}$ for all $k, l \in \mathbb{N}$. Let $\lim\limits_{i \to \infty} q(\Delta_n^m x_{k,l}^i) = y_{k,l}$ for each $k, l \in \mathbb{N}$. Let $k, l = 1$, then we have

$$\lim_{i \to \infty} q(\Delta_n^m x_{1,1}^i) = \lim_{i \to \infty} \sum_{\nu=0}^{m} (-1)^\nu \left(\begin{array}{c} m \\ \nu \end{array} \right) x_{1+n\nu, 1+m\nu}^i = y_{1,1}. \tag{2.3}$$

We have by equation (2.2) and equation (2.3) $\lim\limits_{i \to \infty} x_{mn+1}^i = x_{mn+1}$, exists. Proceeding in this way inductively, we have $\lim\limits_{i \to \infty} x_{k,l}^i = x_{k,l}$ exists for each $k, l \in \mathbb{N}$. Now we have for all $i, j \geq n_0$,

$$\sum_{k,l=1}^{mn} q(x_{k,l}^i - x_{k,l}^j) + \inf \left\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l} \left(q\left(\frac{\Delta_n^m (x_{k,l}^i - x_{k,l}^j)}{\rho} \right) \right) \leq 1 \right\} < \epsilon.$$

This implies that

$$\lim_{j \to \infty} \left\{ \sum_{k,l=1}^{mn} q(x_{k,l}^i - x_{k,l}^j) + \inf \left\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l} \left(q\left(\frac{\Delta_n^m (x_{k,l}^i - x_{k,l}^j)}{\rho} \right) \right) \leq 1 \right\} \right\} < \epsilon,$$

for all $i \geq n_0$. Using the continuity of $M_{k,l}$, we have

$$\sum_{k,l=1}^{mn} q(x_{k,l}^i - x_{k,l}) + \inf \left\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l} \left(q\left(\frac{\Delta_n^m x_{k,l}^i - \Delta_n^m x_{k,l}}{\rho} \right) \right) \leq 1 \right\} < \epsilon,$$

for all $i \geq n_0$. It follows that $(x^i - x) \in l_\infty^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$. Since $x^i \in l_\infty^2(\Delta_n^n, \mathcal{M}, u, p, q, s)$ and $l_\infty^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$ is a linear space, so we have $x = x^i - (x^i - x) \in l_\infty^2(\Delta_m^n, \mathcal{M}, u, p, q, s)$. This completes the proof. Similarly, we can prove that $c^2(\Delta_m^n, \mathcal{M}, u, p, q, s)$ and $c_0^2(\Delta_m^n, \mathcal{M}, u, p, q, s)$ are complete paranormed spaces in view of the above proof.

Theorem 2.4 *Let* $m \geq 1$, *then for all* $0 < i \leq m$, $Z^2(\Delta_n^i, \mathcal{M}, u, p, q, s) \subset Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$, *where* $Z^2 = c^2, c_0^2$ *and* l_∞^2.

Proof. We will prove it for only $c_0^2(\Delta_n^{m-1}, \mathcal{M}, u, p, q, s)$. Let $x = (x_{k,l}) \in c_0^2(\Delta_n^{m-1}, \mathcal{M}, u, p, q, s)$. Then

$$P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[M_{k,l} \left(\frac{q(\Delta_n^{m-1} x_{k,l})}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \text{ and } s \geq 0 \tag{2.4}$$

Then from (2.4) we have

$$P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[M_{k,l} \left(q\left(\frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right]^{p_{k,l+1}} = 0,$$

$$P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[M_{k,l} \left(q\left(\frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right]^{p_{k+1,l}} = 0$$

and

$$P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[M_{k,l} \left(q\left(\frac{\Delta_n^m x_{k,l}}{\rho} \right) \right) \right]^{p_{k+1,l+1}} = 0.$$

Now for

$$\Delta_n^m x = (\Delta_n^m x_{k,l}) = (\Delta_n^{m-1} x_{k,l} - \Delta_n^{m-1} x_{k,l+1} - \Delta_n^{m-1} x_{k+1,l} + \Delta_n^{m-1} x_{k+1,l+1}),$$

we have

$$(kl)^{-s} u_{k,l} \Big[M_{k,l} \Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho}\Big) \Big) \Big]^{p_{k,l}}$$

$$\leq (kl)^{-s} u_{k,l} \Big[M_{k,l} \Big(q\Big(\frac{\Delta_n^{m-1} x_{k,l}}{\rho}\Big) + q\Big(\frac{\Delta_n^{m-1} x_{k,l+1}}{\rho}\Big)$$

$$+ q\Big(\frac{\Delta_n^{m-1} x_{k+1,l}}{\rho}\Big) + q\Big(\frac{\Delta_n^{m-1} x_{k+1,l+1}}{\rho}\Big) \Big) \Big]^{p_{k,l}}$$

$$\leq K^2 (kl)^{-s} u_{k,l} \Big\{ \Big[M_{k,l} \Big(q\Big(\frac{\Delta_n^{m-1} x_{k,l}}{\rho}\Big) \Big) \Big]^{p_{k,l}} + u_{k,l} \Big[M_{k,l} \Big(q\Big(\frac{\Delta_n^{m-1} x_{k+1,l}}{\rho}\Big) \Big) \Big]^{p_{k,l}}$$

$$+ u_{k,l} \Big[M \Big(q\Big(\frac{\Delta_n^{m-1} x_{k,l+1}}{\rho}\Big) \Big) \Big]^{p_{k,l}} + u_{k,l} \Big[M_{k,l} \Big(q\Big(\frac{\Delta_n^{m-1} x_{k+1,l+1}}{\rho}\Big) \Big) \Big]^{p_{k,l}} \Big\}$$

$$\leq K^2 \Big\{ \Big[(kl)^{-s} u_{k,l} M_{k,l} \Big(q\Big(\frac{\Delta_n^{m-1} x_{k,l}}{\rho}\Big) \Big) \Big]^{p_{k,l}} + \Big[(kl)^{-s} u_{k,l} M_{k,l} \Big(q\Big(\frac{\Delta_n^{m-1} x_{k+1,l}}{\rho}\Big) \Big) \Big]^{p_{k+1,l}}$$

$$+ \Big[(kl)^{-s} u_{k,l} M_{k,l} \Big(q\Big(\frac{\Delta_n^{m-1} x_{k,l+1}}{\rho}\Big) \Big) \Big]^{p_{k,l+1}} + u_{k,l} \Big[(kl)^{-s} M_{k,l} \Big(q\Big(\frac{\Delta_n^{m-1} x_{k+1,l+1}}{\rho}\Big) \Big) \Big]^{p_{k+1,l+1}} \Big\}$$

from this it follows that $x = (x_{k,l}) \in c_0^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$ and hence $c_0^2(\Delta_n^{m-1}, \mathcal{M}, u, p, q, s) \subset c_0^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$. On applying the principle of induction, it follows that $c_0^2(\Delta_n^i, \mathcal{M}, u, p, q, s) \subset c_0^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$ for $i = 0, 1, 2, \cdots, m-1$. Similarly, we can prove the other cases.

Theorem 2.5 *(a) If $0 < \inf\limits_{k,l} p_{k,l} \leq p_{k,l} < 1$, then $Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s) \subset Z^2(\Delta_n^m, \mathcal{M}, u, q, s)$,*
(b) If $1 < p_{k,l} \leq \sup\limits_{k,l} p_{k,l} < \infty$, then $Z^2(\Delta_n^m, \mathcal{M}, u, q, s) \subset Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$,
where $Z^2 = c^2, c_0^2$ and l_∞^2.

Proof. (i) Let $x = (x_{k,l}) \in l_\infty^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$. Since $0 < \inf p_{k,l} \leq 1$, we have

$$\sup_{k,l} (kl)^{-s} u_{k,l} \Big[M_{k,l} \Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho}\Big) \Big) \Big] \leq \sup_{k,l} (kl)^{-s} u_{k,l} \Big[M_{k,l} \Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho}\Big) \Big) \Big]^{p_{k,l}},$$

and hence $x = (x_{k,l}) \in l_\infty^2(\Delta_n^m, \mathcal{M}, u, p, q, s)$.

(ii) Let $p_{k,l}$ for each (k, l) and $\sup\limits_{k,l} p_{k,l} < \infty$. Let $x = (x_{k,l}) \in l_\infty^2(\Delta_n^m, \mathcal{M}, u, q, s)$. Then, for each $0 < \epsilon < 1$, there exists a positive integer $\mathbb{N}$ such that

$$\sup_{k,l} (kl)^{-s} u_{k,l} \Big[M_{k,l} \Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho}\Big) \Big) \Big] \leq \epsilon < 1,$$

for all $m, n \in \mathbb{N}$. This implies that

$$\sup_{k,l} (kl)^{-s} u_{k,l} \Big[M_{k,l} \Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho}\Big) \Big) \Big]^{p_{k,l}} \leq \sup_{k,l} (kl)^{-s} u_{k,l} \Big[M_{k,l} \Big(q\Big(\frac{\Delta_n^m x_{k,l}}{\rho}\Big) \Big) \Big].$$

Thus $x = (x_{k,l}) \in l^2_\infty(\Delta^m_n, \mathcal{M}, u, p, q, s)$ and this completes the proof.

Theorem 2.6 *Let $\mathcal{M}' = (M'_{k,l})$ and $\mathcal{M}'' = (M''_{k,l})$ be two sequences of Orlicz functions satisfying Δ_2-condition. If $\beta = \lim\limits_{t\to\infty} \dfrac{M''_{k,l}(t)}{t} \geq 1$, then $Z^2(\Delta^m_n, \mathcal{M}', u, p, q, s) = Z^2(\Delta^m_n, \mathcal{M}'' \circ \mathcal{M}', u, p, q, s)$, where $Z^2 = c^2, c^2_0$ and l^2_∞.*

Proof. We prove it for $Z^2 = c^2$ and the other cases will follows on applying similar techniques. Let $x = (x_{k,l}) \in c^2(\Delta^m_n, \mathcal{M}', u, p, q, s)$, then

$$P - \lim_{k,l}(kl)^{-s}\left[M'_{k,l}\left(q\left(\frac{\Delta^m_n x_{k,l} - L}{\rho}\right)\right)\right]^{p_{k,l}} = 0.$$

Let $0 < \epsilon < 1$ and δ with $0 < \delta < 1$ such that $M''_{k,l}(t) < \epsilon$ for $0 \leq t < \delta$. Let

$$y_{k,l} = M'_{k,l}\left(q\left(\frac{\Delta^m_n x_{k,l} - L}{\rho}\right)\right)$$

and consider

$$[M''_{k,l}(y_{k,l})]^{p_{k,l}} = [M''_{k,l}(y_{k,l})]^{p_{k,l}} + [M''_{k,l}(y_{k,l})]^{p_{k,l}} \tag{2.5}$$

where the first term is over $y_{k,l} \leq \delta$ and the second is over $y_{k,l} > \delta$. From the first term in (2.5), we have

$$(kl)^{-s}[M''_{k,l}(y_{k,l})]^{p_{k,l}} < (kl)^{-s}[M''_{k,l}(2)]^H[(y_{k,l})]^{p_{k,l}} \tag{2.6}$$

On the other hand, we use the fact that

$$y_{k,l} < \frac{y_{k,l}}{\delta} < 1 + \frac{y_{k,l}}{\delta}.$$

Since $(M''_{k,l})$ for each k, l is non-decreasing and convex, it follows that

$$M''_{k,l}(y_{k,l}) < M''_{k,l}\left(1 + \frac{y_{k,l}}{\delta}\right) < \frac{1}{2}M''_{k,l}(2) + \frac{1}{2}M''_{k,l}\left(\frac{2y_{k,l}}{\delta}\right).$$

Since $(M''_{k,l})$ for each k, l satisfies Δ_2-condition, we have

$$M''_{k,l}(y_{k,l}) < \frac{1}{2}K\frac{y_{k,l}}{\delta}M''_{k,l}(2) + \frac{1}{2}K\frac{y_{k,l}}{\delta}M''_{k,l}(2) = K\frac{y_{k,l}}{\delta}M''_{k,l}(2).$$

Hence, from the second term in (2.5), it follows that

$$(kl)^{-s}[M''(y_{k,l})]^{p_{k,l}} \leq \max\left(1, (KM''(2)\delta^{-1})^H\right)(kl)^{-s}[(y_{k,l})]^{p_{k,l}} \tag{2.7}$$

By the inequalities (2.6) and (2.7), taking limit in the Pringsheim sense, we have $x = (x_{k,l}) \in c^2(\Delta^m_n, \mathcal{M}'' \circ \mathcal{M}', u, p, q, s)$. Observe that in this part of the proof we did not need $\beta \geq 1$. Now, let $\beta \geq 1$ and $x = (x_{k,l}) \in c^2(\mathcal{M}', \Delta^r_n, u, q, p)$. Then, we have $M''_{k,l}(t) \geq \beta(t)$ for all $t \geq 0$. It follows that $x = (x_{k,l}) \in c^2(\Delta^m_n, \mathcal{M}'' \circ \mathcal{M}', u, p, q, s)$ implies $x = (x_{k,l}) \in c^2(\Delta^m_n, \mathcal{M}', u, p, q, s)$.

This implies $c^2(\Delta_n^m, \mathcal{M}', u, p, q, s) = c^2(\Delta_n^m, \mathcal{M}'' \circ \mathcal{M}', u, p, q, s)$.

Theorem 2.7 *Let $\mathcal{M}' = (M'_{k,l})$ and $\mathcal{M}'' = (M''_{k,l})$ be two sequences of Orlicz functions, q, q_1 and q_2 be seminorms and s, s_1 and s_2 be positive real numbers. Then*

(1) $Z^2(\Delta_n^m, \mathcal{M}', u, p, q, s) \cap Z^2(\Delta_n^m, \mathcal{M}'', u, p, q, s) \subset Z^2(\Delta_n^m, \mathcal{M}' + \mathcal{M}'', u, p, q, s),$

(2) $Z^2(\Delta_n^m, \mathcal{M}, u, p, q_1, s) \cap Z^2(\Delta_n^m, \mathcal{M}, u, p, q_2, s) \subset Z^2(\Delta_n^m, \mathcal{M}, u, p, q_1 + q_2, s),$

(3) *If q_1 is stronger than q_2, then* $Z^2(\Delta_n^m, \mathcal{M}, u, p, q_1, s) \subset Z^2(\Delta_n^m, \mathcal{M}, u, p, q_2, s)$

(4) *If $s_1 \leq s_2$, then* $Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s_1) \subset Z^2(\Delta_n^m, \mathcal{M}, u, p, q, s_2),$

where $Z^2 = c^2, c_0^2$ and l_∞^2.

Proof. (1) Let $x = (x_{k,l}) \in c^2(\Delta_n^m, \mathcal{M}', u, p, q, s) \cap c^2(\Delta^m, \mathcal{M}'', u, p, q, s)$. Then

$$P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[M'_{k,l} \left(q \left(\frac{\Delta_n^m x_{k,l} - L}{\rho_1} \right) \right) \right]^{P_{k,l}} = 0, \text{ for some } \rho_1 > 0,$$

$$P - \lim_{k,l}(kl)^{-s} u_{k,l} \left[M''_{k,l} \left(q \left(\frac{\Delta_n^m x_{k,l} - L}{\rho_2} \right) \right) \right]^{P_{k,l}} = 0, \text{ for some } \rho_2 > 0.$$

Let $\rho = \max(\rho_1, \rho_2)$. The result follows from the following inequality

$(kl)^{-s} \left[(\mathcal{M}' + \mathcal{M}'') \left(q \left(\frac{\Delta_n^m x_{k,l} - L}{\rho} \right) \right) \right]^{P_{k,l}}$

$$\leq K \left\{ (kl)^{-s} u_{k,l} \left[\mathcal{M}' \left(q \left(\frac{\Delta_n^m x_{k,l} - L}{\rho_1} \right) \right) \right]^{P_{k,l}} + (kl)^{-s} u_{k,l} \left[\mathcal{M}'' \left(q \left(\frac{\Delta_n^m x_{k,l} - L}{\rho_2} \right) \right) \right]^{P_{k,l}} \right\}.$$

The proofs of (2), (3) and (4) follows by same pattern.

Theorem 2.8 *For any sequence of orlicz functions, if $q_1 \equiv$ (equivalent to) q_2, then $Z^2(\Delta_n^m, \mathcal{M}, u, p, q_1, s) = Z^2(\Delta_n^m, \mathcal{M}, u, p, q_2, s)$, where $Z^2 = c^2, c_0^2$ and l_∞^2.*

Proof. It is easy to prove so we omit the details.

3 Some generalized difference double sequence spaces over n-normed spaces

The concept of 2-normed spaces was initially developed by Gähler[8] in the mid of 1960's, while that of n-normed spaces one can see in Misiak[16]. Since then, many others have studied this concept and obtained various results, see Gunawan ([9],[10]), Gunawan and Mashadi [11] and many others.

Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $\|\cdot, \cdots, \cdot\|$ on X^n satisfying the following four conditions:

(1) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;

(2) $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutation;

(3) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \; \|x_1, x_2, \cdots, x_n\|$ for any $\alpha \in \mathbb{K}$, and

(4) $\|x + x', x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|x', x_2, \cdots, x_n\|$

is called an n-norm on X and the pair $(X, \|\cdot, \cdots, \cdot\|)$ is called a n-normed space over the field $\mathbb{K}$. For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $\|x_1, x_2, \cdots, x_n\|_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \cdots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$ and $\|.\|_E$ denotes the Euclidean norm. Let $(X, \|\cdot, \cdots, \cdot\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot, \cdots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \cdots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \cdots, x_{n-1}, a_i\| : i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.
A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} \|x_k - L, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k,p \to \infty} \|x_k - x_p, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space. For more details about sequence spaces see ([28], [29]) and references therein.
Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of positive reals such that $u_{k,l} \neq 0$ for all k, then we define the following sequences spaces in the present paper:

$c_0^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|) =$

$$\left\{ x = (x_{k,l}) \in w^2 : \lim_{k,l \to \infty} (kl)^{-s} u_{k,l} \left[M_{k,l} \left(\|\frac{\Delta_n^m x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\| \right) \right]^{p_{k,l}} = 0, \right.$$

$$\text{for some } \rho > 0 \text{ and } s \geq 0\Big\},$$

$$c^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|) =$$

$$\Big\{x = (x_{k,l}) \in w^2 : \lim_{k,l \to \infty} (kl)^{-s} u_{k,l} \Big[M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} = 0,$$

$$\text{for some } \rho > 0, \ L \text{ and } s \geq 0\Big\},$$

and
$$l_\infty^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|) =$$

$$\Big\{x = (x_{k,l}) \in w^2 : \sup_{k,l \geq 1} (kl)^{-s} u_{k,l} \Big[M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} < \infty,$$

$$\text{for some } \rho > 0 \text{ and } s \geq 0\Big\}.$$

In this section of the present paper we shall study the topological properties and some interesting inclusion relation between the spaces $c^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|)$, $c_0^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|)$ and $l_\infty^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|)$.

Theorem 3.1 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers, then the spaces $c_0^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|)$, $c^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|)$ and $l_\infty^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|)$ are linear spaces.*

Proof. Let $x = (x_{k,l})$, $y = (y_{k,l}) \in c_0^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive number ρ_1 and ρ_2 such that

$$\lim_{k,l \to \infty} (kl)^{-s} u_{k,l} \Big[M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l}}{\rho_1}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} = 0, \ \text{ for some } \rho_1 > 0$$

and

$$\lim_{k,l \to \infty} (kl)^{-s} u_{k,l} \Big[M_{k,l}\Big(\|\frac{\Delta_n^m y_{k,l}}{\rho_2}, z_1, \cdots, z_{n-1}\|\Big)\Big]^{p_{k,l}} = 0, \ \text{ for some } \rho_2 > 0.$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $\mathcal{M} = (M_{k,l})$ is non-decreasing convex function and so by using inequality (1.1), we have

$$\lim_{k,l\to\infty}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m(\alpha x_{k,l}+\beta y_{k,l})|}{\rho_3},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$= \lim_{k,l\to\infty}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\alpha\Delta_n^m x_{k,l}}{\rho_3},z_1,\cdots,z_{n-1}\|+\|\frac{\beta\Delta_n^m y_{k,l}}{\rho_3},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$\leq K\lim_{k,l\to\infty}\frac{1}{2^{p_{k,l}}}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l}}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$+ K\lim_{k,l\to\infty}\frac{1}{2^{p_{k,l}}}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m y_{k,l}}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$\leq K\lim_{k,l\to\infty}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l}}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$+ K\lim_{k,l\to\infty}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m y_{k,l}}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$= 0.$$

So, $\alpha x+\beta y\in c_0^2(\mathcal{M},\Delta_m^n,p,u,s,\|\cdot,\cdots,\cdot\|)$. Hence $c_0^2(\mathcal{M},\Delta_m^n,p,u,s,\|\cdot,\cdots,\cdot\|)$ is a linear space. Similarly, we can prove that $c^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$ and $l_\infty^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$ are linear spaces.

Theorem 3.2 *Let $\mathcal{M}=(M_{k,l})$ be a sequence of Orlicz functions, $p=(p_{k,l})$ be a bounded sequence of positive real numbers and $u=(u_{k,l})$ be a sequence of strictly positive real numbers. For $Z^2=l_\infty^2,c^2$ and c_0^2, the spaces $Z^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$ are paranormed spaces, paranormed by*

$$g(x)=\sum_{k,l=1}^{nm}\|x_{k,l},z_1,\cdots,z_{n-1}\|+\inf\Big\{\rho^{\frac{p_{k,l}}{H}}:\sup_{k,l}(kl)^{-s}u_{k,l}M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\leq 1\Big\}$$

where $H=\max(1,\sup_{k,l}p_{k,l})$.

Proof. Clearly $g(-x)=g(x)$, $g(0)=0$. Let (x_k) and (y_k) be any two sequences belong to any one of the spaces $Z^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$, for $Z^2=c_0^2$, c^2 and l_∞^2. Then, we get ρ_1, $\rho_2>0$ such that

$$\sup_{k,l}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l}}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]\leq 1$$

and

$$\sup_{k,l}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m y_{k,l}}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]\leq 1.$$

Let $\rho=\rho_1+\rho_2$. Then by convexity of $\mathcal{M}=(M_{k,l})$, we have

$$\sup_{k,l}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m(x_{k,l}+y_{k,l})}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]$$

$$\leq \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\sup_{k,l}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l}}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]$$

$$+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\sup_{k,l}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m y_{k,l}}{\rho_2},z_1,\cdots,z_{n-1}\|\Big)\Big]$$

$$\leq 1.$$

Hence we have,

$$
\begin{aligned}
g(x+y) &= \sum_{k,l=1}^{mn} \|(x_{k,l}+y_{k,l}),z_1,\cdots,z_{n-1}\| \\
&+ \inf\left\{\rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s}u_{k,l}\left[M_{k,l}\left(\|\frac{\Delta_n^m(x_{k,l}+y_{k,l})}{\rho},z_1,\cdots,z_{n-1}\|\right)\right] \le 1\right\} \\
&\le \sum_{k,l=1}^{mn} \|x_{k,l},z_1,\cdots,z_{n-1}\| \\
&+ \inf\left\{\rho_1^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s}u_{k,l}\left[M_{k,l}\left(\|\frac{\Delta_n^m x_{k,l}}{\rho_1},z_1,\cdots,z_{n-1}\|\right)\right] \le 1\right\} \\
&+ \sum_{k,l=1}^{mn} \|y_{k,l},z_1,\cdots,z_{n-1}\| \\
&+ \inf\left\{\rho_2^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s}u_{k,l}\left[M_{k,l}\left(\|\frac{\Delta_n^m y_{k,l}}{\rho_2},z_1,\cdots,z_{n-1}\|\right)\right] \le 1\right\}.
\end{aligned}
$$

This implies that

$$ g(x+y) \le g(x)+g(y). $$

The continuity of the scalar multiplication follows from the following inequality

$$
\begin{aligned}
g(\mu x) &= \sum_{k,l=1}^{mn} \|\mu x_{k,l},z_1,\cdots,z_{n-1}\| \\
&+ \inf\left\{\rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s}u_{k,l}\left[M_{k,l}\left(\|\frac{\Delta_n^m \mu x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\right)\right] \le 1\right\} \\
&= |\mu| \sum_{k,l=1}^{mn} \|x_{k,l},z_1,\cdots,z_{n-1}\| \\
&+ \inf\left\{(t|\mu|)^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s}u_{k,l}\left[M_{k,l}\left(\|\frac{\Delta_n^m x_{k,l}}{t},z_1,\cdots,z_{n-1}\|\right)\right] \le 1\right\},
\end{aligned}
$$

where $t = \frac{\rho}{|\mu|}$. Hence the space $Z^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$, for $Z^2 = c_0^2, c^2$ and l_∞^2 is a paranormed space, paranormed by g.

Theorem 3.3 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers. For $Z^2 = l_\infty^2, c^2$ and c_0^2, the spaces $Z^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$ are complete paranormed spaces, paranormed by*

$$ g(x) = \sum_{k,l=1}^{nm} \|x_{k,l},z_1,\cdots,z_{n-1}\| + \inf\left\{\rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s}u_{k,l}M_{k,l}\left(\|\frac{\Delta_n^m x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\right) \le 1\right\}, $$

where $H = \max(1, \sup_{k,l} p_{k,l})$.

Proof. We prove the result for the space $l_\infty^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$. Let (x^i) be any Cauchy

sequence in $l^2_\infty(\mathcal{M}, \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|)$. Let $\epsilon > 0$ be given and for $t > 0$, choose x_0 be fixed such that $u_{k,l} M_{k,l}\left(\frac{tx_0}{2}\right) \geq 1$, then there exists a positive integer $n_0 \in \mathbb{N}$ such that $g(x^i_{k,l} - x^j_{k,l}) < \frac{\epsilon}{x_0 t}$, for all $i, j \geq n_0$. Using the definition of paranorm, we get

$$\sum_{k,l=1}^{mn} \|(x^i_{k,l} - x^j_{k,l}), z_1, \cdots, z_{n-1}\| + \inf\left\{\rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\left(\|\frac{\Delta^m_n(x^i_{k,l} - x^j_{k,l})}{\rho}, z_1, \cdots, z_{n-1}\|\right)\right\}$$

$$< \frac{\epsilon}{x_0 t}, \quad \text{for all } i, j \geq n_0 \qquad (3.1).$$

Hence we have,

$$\sum_{k,l=1}^{mn} \|(x^i_{k,l}) - x^j_{k,l}), z_1, \cdots, z_{n-1}\| < \epsilon, \quad \text{for all } i, j \geq n_0.$$

This implies that

$$\|(x^i_{k,l} - x^j_{k,l}), z_1, \cdots, z_{n-1}\| < \epsilon, \quad \text{for all } i, j \geq n_0 \text{ and } 1 \leq k \leq nm.$$

Thus $(x^i_{k,l})$ is a Cauchy sequence in $\mathbb{C}$ for $k, l = 1, 2, ..., nm$. Hence $(x^i_{k,l})$ is convergent in $\mathbb{C}$ for $k, l = 1, 2, ..., nm$. Let $\lim_{i \to \infty} x^i_{k,l} = x_{k,l}$, say for $k, l = 1, 2, ..., nm.$ \qquad (3.2)
Again from equation (3.1) we have,

$$\inf\left\{\rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\left(\|\frac{\Delta^m_n(x^i_{k,l} - x^j_{k,l})}{\rho}, z_1, \cdots, z_{n-1}\|\right) \leq 1\right\} < \epsilon, \quad \text{for all } i, j \geq n_0.$$

Hence we get

$$\sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l}\left(\|\frac{\Delta^m_n(x^i_{k,l} - x^j_{k,l})}{g(x^i - x^j)}, z_1, \cdots, z_{n-1}\|\right) \leq 1, \quad \text{for all } i, j \geq n_0.$$

It follows that $(kl)^{-s} u_{k,l} M_{k,l}\left(\|\frac{\Delta^m_n(x^i_{k,l} - x^j_{k,l})}{g(x^i - x^j)}, z_1, \cdots, z_{n-1}\|\right) \leq 1$, for each $k, l \geq 1$ and for all $i, j \geq n_0$.
For $t > 0$ with $(kl)^{-s} u_{k,l} M_{k,l}(\frac{tx_0}{2}) \geq 1$, we have

$$(kl)^{-s} u_{k,l} M_{k,l}\left(\|\frac{\Delta^m_n(x^i_{k,l} - x^j_{k,l})}{g(x^i - x^j)}, z_1, \cdots, z_{n-1}\|\right) \leq (kl)^{-s} u_{k,l} M_{k,l}(\frac{tx_0}{2}).$$

This implies that

$$\|(\Delta^m_n x^i_{k,l} - \Delta^m_n x^j_{k,l}), z_1, \cdots, z_{n-1}\| < \frac{tx_0}{2} \frac{\epsilon}{tx_0} = \frac{\epsilon}{2}.$$

Hence $(\Delta^m_n x^i_{k,l})$ is a Cauchy sequence in $\mathbb{C}$ for all $k, l \in \mathbb{N}$. This implies that $(\Delta^m_n x^i_{k,l})$ is convergent in $\mathbb{C}$ for all $k, l \in \mathbb{N}$. Let $\lim_{i \to \infty} \Delta^m_n x^i_{k,l} = y_{k,l}$ for each $k, l \in \mathbb{N}$. Let $k, l = 1$, then we have

$$\lim_{i \to \infty} \Delta^m_n x^i_{1,1} = \lim_{i \to \infty} \sum_{v=0}^{n} (-1)^v \binom{m}{v} x^i_{1+nv, mv+1} = y_1. \qquad (3.3)$$

We have by equation (3.2) and equation (3.3) $\lim_{i \to \infty} x^i_{mn+1} = x_{mn+1}$, exists. Proceeding in this way inductively, we have $\lim_{i \to \infty} x^i_{k,l} = x_{k,l}$ exists for each $k, l \in \mathbb{N}$. Now we have for all $i, j \geq n_0$,

$$\sum_{k,l=1}^{mn} \|(x^i_{k,l} - x^j_{k,l}), z_1, \cdots, z_{n-1}\|$$

$$+ \inf \left\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l} \left(\| \frac{\Delta^m_n(x^i_{k,l} - x^j_{k,l})}{\rho}, z_1, \cdots, z_{n-1} \| \right) \leq 1 \right\} < \epsilon.$$

This implies that

$$\lim_{j \to \infty} \left\{ \sum_{k,l=1}^{mn} \|(x^i_{k,l} - x^j_{k,l}), z_1, \cdots, z_{n-1}\| \right.$$

$$\left. + \inf \left\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l} \left(\| \frac{\Delta^m_n(x^i_{k,l} - x^j_{k,l})}{\rho}, z_1, \cdots, z_{n-1} \| \right) \leq 1 \right\} \right\} < \epsilon,$$

for all $i \geq n_0$. Using the continuity of $(M_{k,l})$, we have

$$\sum_{k,l=1}^{mn} \|(x^i_{k,l} - x_{k,l}), z_1, \cdots, z_{n-1}\|$$

$$+ \inf \left\{ \rho^{\frac{p_{k,l}}{H}} : \sup_{k,l}(kl)^{-s} u_{k,l} M_{k,l} \left(\| \frac{(\Delta^m_n x^i_{k,l} - \Delta^m_n x_{k,l})}{\rho}, z_1, \cdots, z_{n-1} \| \right) \leq 1 \right\} < \epsilon,$$

for all $i \geq n_0$. It follows that $(x^i - x) \in l^2_\infty(\mathcal{M}, \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|)$. Since $x^i \in l^2_\infty(\mathcal{M}, \Delta^n_m, p, u, s, \|\cdot, \cdots, \cdot\|)$ and $l^2_\infty(\mathcal{M}, \Delta^n_m, p, u, s, \|\cdot, \cdots, \cdot\|)$ is a linear space, so we have $x = x^i - (x^i - x) \in l^2_\infty(\mathcal{M}, \Delta^n_m, p, u, s, \|\cdot, \cdots, \cdot\|)$. This completes the proof. Similarly, we can prove that $c^2(\mathcal{M}, \Delta^n_m, p, u, \|\cdot, \cdots, \cdot\|)$ and $c^2_0(\mathcal{M}, \Delta^n_m, p, u, \|\cdot, \cdots, \cdot\|)$ are complete paranormed spaces in view of the above proof.

Theorem 3.4 *If $0 < p_{k,l} \leq q_{k,l} < \infty$ for each k, l, then $Z^2(\mathcal{M}, \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|) \subseteq Z^2(\mathcal{M}, \Delta^m_n, q, u, s, \|\cdot, \cdots, \cdot\|)$, for $Z^2 = c^2_0$ and c^2.*

Proof. Let $x = (x_{k,l}) \in c^2(\mathcal{M}, \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|)$. Then there exists some $\rho > 0$ and $L \in X$ such that

$$\lim_{k,l \to \infty} (kl)^{-s} u_{k,l} \left(M_{k,l} \left(\| \frac{\Delta^m_n x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \| \right) \right)^{p_{k,l}} = 0.$$

This implies that $(kl)^{-s} u_{k,l} M_{k,l} \left(\| \frac{\Delta^m_n x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \| \right) < \epsilon$ $(0 < \epsilon < 1)$ for sufficiently large k, l. Hence we get

$$\lim_{k,l \to \infty} (kl)^{-s} u_{k,l} \left(M_{k,l} \left(\| \frac{\Delta^m_n x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \| \right) \right)^{q_{k,l}}$$

$$\leq \lim_{k,l \to \infty} (kl)^{-s} u_{k,l} \left(M_{k,l} \left(\| \frac{\Delta^m_n x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \| \right) \right)^{p_{k,l}} = 0.$$

This implies that $x = (x_{k,l}) \in c^2(\mathcal{M}, \Delta^m_n, q, u, s, \|\cdot, \cdots, \cdot\|)$. This completes the proof. Similarly, we can prove for the case $Z^2 = c^2_0$.

Theorem 3.5 *If $\mathcal{M}' = (M'_{k,l})$ and $\mathcal{M}'' = (M''_{k,l})$ be a sequence of Orlicz functions. Then*
(i) $Z^2(\mathcal{M}', \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|) \subseteq Z^2(\mathcal{M}'' \circ \mathcal{M}', \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|),$
(ii)

$$Z^2(\mathcal{M}', \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|) \cap Z^2(\mathcal{M}'', \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|)$$
$$\subseteq Z^2(\mathcal{M}' + \mathcal{M}'', \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|),$$

for $Z^2 = l^2_\infty, c^2$ and c^2_0.

Proof. (i) We prove this part for $Z^2 = l^2_\infty$ and the rest of the cases will follow similarly. Let $(x_{k,l}) \in l^2_\infty(\mathcal{M}', \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|)$, then there exists $0 < U < \infty$ such that

$$(kl)^{-s} u_{k,l} \left(M'_{k,l} \left(\|\frac{\Delta^m_n x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\| \right) \right)^{p_{k,l}} \leq U, \text{ for all } k, l \in \mathbb{N}.$$

Let $y_{k,l} = (kl)^{-s} u_{k,l} M'_{k,l} \left(\|\frac{\Delta^m_n x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\| \right)$. Then $y_{k,l} \leq U^{\frac{1}{p_{k,l}}} \leq V$, say for all $k, l \in \mathbb{N}$. Hence we have

$$\left((M''_{k,l} \circ M'_{k,l}) \left(\|\frac{\Delta^m_n x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\| \right) \right)^{p_{k,l}} = (M''_{k,l}(y_{k,l}))^{p_{k,l}} \leq (M''_{k,l}(V))^{p_{k,l}} < \infty,$$

for all $k, l \in \mathbb{N}$.

Hence $\sup_{k,l} u_{k,l} \left((M''_{k,l} \circ M'_{k,l}) \left(\|\frac{\Delta^m_n x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\| \right) \right)^{p_{k,l}} < \infty$. Thus $x = (x_{k,l}) \in l^2_\infty(\mathcal{M}'' \circ \mathcal{M}', \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|)$.

(ii) We prove the result for the case $Z^2 = c^2$ and the rest of the cases will follow similarly. Let $x = (x_{k,l}) \in c^2(\mathcal{M}', \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|) \cap c^2(\mathcal{M}'', \Delta^m_n, p, u, s, \|\cdot, \cdots, \cdot\|)$, then there exist some $\rho_1, \rho_2 > 0$ and $L \in X$ such that

$$\lim_{k \to \infty} (kl)^{-s} u_{k,l} \left(M'_{k,l} \left(\|\frac{\Delta^m_n x_{k,l} - L}{\rho_1}, z_1, \cdots, z_{n-1}\| \right) \right)^{p_{k,l}} = 0$$

and

$$\lim_{k \to \infty} (kl)^{-s} u_{k,l} \left(M''_{k,l} \left(\|\frac{\Delta^m_n x_{k,l} - L}{\rho_2}, z_1, \cdots, z_{n-1}\| \right) \right)^{p_{k,l}} = 0.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$(kl)^{-s} u_{k,l} \left((M'_{k,l} + M''_{k,l}) \left(\|\frac{\Delta^m_n x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}\| \right) \right)^{p_{k,l}}$

$$\leq K \left[\left(\frac{\rho_1}{\rho_1 + \rho_2} \right) (kl)^{-s} u_{k,l} M'_{k,l} \left(\|\frac{\Delta^m_n x_{k,l} - L}{\rho_1}, z_1, \cdots, z_{n-1}\| \right) \right]^{p_{k,l}}$$
$$+ K \left[\left(\frac{\rho_2}{\rho_1 + \rho_2} \right) (kl)^{-s} u_{k,l} M''_{k,l} \left(\|\frac{\Delta^m_n x_{k,l} - L}{\rho_2}, z_1, \cdots, z_{n-1}\| \right) \right]^{p_{k,l}}.$$

This implies that

$$\lim_{k \to \infty} (kl)^{-s} u_{k,l} \left((M'_{k,l} + M''_{k,l}) \left(\|\frac{\Delta^m_n x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}\| \right) \right)^{p_{k,l}} = 0.$$

Thus $x = (x_{k,l}) \in c^2(\mathcal{M}' + \mathcal{M}'', \Delta_m^m, p, u, s, \|\cdot, \cdots, \cdot\|)$. This completes the proof.

Theorem 3.6 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be a sequence of strictly positive real numbers, then $Z^2(\mathcal{M}, \Delta_n^{m-1}, p, u, s, \|\cdot, \cdots, \cdot\|) \subset Z^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|)$, for $Z^2 = l_\infty^2, c^2$ and c_o^2.*

Proof.   We prove the result for the case $Z^2 = l_\infty^2$ and the rest of the cases will follow similarly. Let $x = (x_{k,l}) \in l_\infty^2(\mathcal{M}, \Delta_n^{m-1}, p, u, s, \|\cdot, \cdots, \cdot\|)$. Then we can have $\rho > 0$ such that

$$(kl)^{-s} u_{k,l} \Big(M_{k,l} \Big(\|\frac{\Delta_n^{m-1} x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\| \Big) \Big)^{p_{k,l}} < \infty, \quad \text{for all } k \in \mathbb{N}. \tag{3.4}$$

On considering 2ρ and using the convexity of $(M_{k,l})$, we have

$$(kl)^{-s} u_{k,l} M_{k,l} \Big(\|\frac{\Delta_n^m x_{k,l}}{2\rho}, z_1, \cdots, z_{n-1}\| \Big) \leq \frac{1}{2}(kl)^{-s} u_{k,l} M_{k,l} \Big(\|\frac{\Delta_n^{m-1} x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\| \Big)$$
$$+ \frac{1}{2}(kl)^{-s} u_{k,l} M_{k,l} \Big(\|\frac{\Delta_n^{m-1} x_{k+n,l+m}}{\rho}, z_1, \cdots, z_{n-1}\| \Big).$$

Hence we have
$$(kl)^{-s} u_{k,l} \Big(M_{k,l} \Big(\|\frac{\Delta_n^m x_{k,l}}{2\rho}, z_1, \cdots, z_{n-1}\| \Big) \Big)^{p_{k,l}}$$

$$\leq K \Big\{ (kl)^{-s} u_{k,l} \Big(\frac{1}{2} M_{k,l} \Big(\|\frac{\Delta_n^{m-1} x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\| \Big) \Big)^{p_{k,l}}$$
$$+ (kl)^{-s} u_{k,l} \Big(\frac{1}{2} M_{k,l} \Big(\|\frac{\Delta_n^{m-1} x_{k+n,l+m}}{\rho}, z_1, \cdots, z_{n-1}\| \Big) \Big)^{p_{k,l}} \Big\}.$$

Then using equation (3.4), we have

$$(kl)^{-s} u_{k,l} \Big(M_{k,l} \Big(\|\frac{\Delta_n^m x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\| \Big) \Big)^{p_{k,l}} < \infty, \quad \text{for all } k, l \in \mathbb{N}.$$

Thus $l_\infty^2(\mathcal{M}, \Delta_n^{m-1}, p, u, s, \|\cdot, \cdots, \cdot\|) \subset l_\infty^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|)$.

Theorem 3.7 *Let $\mathcal{M} = (M_{k,l})$ be a sequence of Orlicz functions. Then*

$$c_0^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|) \subset c^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|) \subset l_\infty^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|).$$

Proof.   It is obvious that $c_0^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|) \subset c^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|)$. We shall prove that $c^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|) \subset l_\infty^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|)$. Let
$x = (x_{k,l}) \in c^2(\mathcal{M}, \Delta_n^m, p, u, s, \|\cdot, \cdots, \cdot\|)$. Then there exists some $\rho > 0$ and $L \in X$ such that

$$\lim_{k,l \to \infty} (kl)^{-s} u_{k,l} \Big(M_{k,l} \Big(\|\frac{\Delta_n^m x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}\| \Big) \Big)^{p_{k,l}} = 0.$$

On taking $\rho = 2\rho_1$, we have

$$\sup_{k,l}(kl)^{-s}u_{k,l}\Big(M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\Big)\Big)^{p_{k,l}}$$

$$\leq \sup_{k,l}K\Big[\frac{1}{2}(kl)^{-s}u_{k,l}\Big(M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l}-L}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big)\Big]^{p_{k,l}}$$

$$+ \sup_{k,l}K\Big[\frac{1}{2}(kl)^{-s}u_{k,l}M_{k,l}\Big(\|\frac{L}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$\leq \sup_{k,l}K(\frac{1}{2})^{p_{k,l}}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l}-L}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$+ \sup_{k,l}K(\frac{1}{2})^{p_{k,l}}(kl)^{-s}\max(1,u_{k,l}\Big(M_{k,l}\Big(\|\frac{L}{\rho_1},z_1,\cdots,z_{n-1}\|\Big)\Big)^{H}),$$

where $H = \max(1,\sup p_{k,l})$. Thus we get $x = (x_{k,l}) \in l_\infty^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$. Hence

$$c_0^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|) \subset c^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|) \subset l_\infty^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|).$$

Theorem 3.8 *The sequence space* $l_\infty^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$ *is solid.*

Proof. Let $x = (x_{k,l}) \in l_\infty^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$, that is

$$\sup_{k,l\to\infty}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}} < \infty.$$

Let $(\alpha_{k,l})$ be a sequence of scalars such that $|\alpha_{k,l}| \leq 1$ for all $k,l \in \mathbb{N}$. Thus we have

$$\sup_{k,l\to\infty}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\alpha_{k,l}\Delta_n^m x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}}$$

$$\leq \sup_{k,l\to\infty}(kl)^{-s}u_{k,l}\Big[M_{k,l}\Big(\|\frac{\Delta_n^m x_{k,l}}{\rho},z_1,\cdots,z_{n-1}\|\Big)\Big]^{p_{k,l}} < \infty.$$

This shows that $(\alpha_{k,l}x_{k,l}) \in l_\infty^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$ for all sequences of scalars $(\alpha_{k,l})$ with $|\alpha_{k,l}| \leq 1$ for all $k,l \in \mathbb{N}$, whenever $(x_{k,l}) \in l_\infty^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$. Hence the space $l_\infty^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$ is a solid sequence space.

Theorem 3.9 *The sequence space* $l_\infty^2(\mathcal{M},\Delta_n^m,p,u,s,\|\cdot,\cdots,\cdot\|)$ *is monotone.*

Proof. The proof of the Theorem is obvious and so we omit it.

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