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An Immediate Derivation of Maximum Principle in Banach spaces, Assuming Reflexive Input and State Spaces.

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ABSTRACT

We consider a standard setting for the norm optimal problem in Banach spaces and show that with a simple argument which invokes some appropriately selected powerful general Theorems for Banach spaces a straightforward derivation of the Maximum Principle is obtained.

RESUMEN

Consideramos una formulación estándar para el problema de norma optimal en espacios de Banach y mostramos que con un argumento simple que invoca algunos fuertes teoremas generales de la teoría de espacios de Banach elegidos apropiadamente se deriva directamente el Principio del Máximo.

Keywords and Phrases: Linear Control Systems in Banach Spaces, Norm Optimal Control, Support of Closed Convex Sets

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1 Introduction

We consider an optimal control setting proposed by Fattorini [1], where the control functions are in the space $L_\infty([0,\Gamma],E_u)$, and E_u is a real Banach space. To simplify the matter we make the assumption that E_u and state space E (another real Banach space) be a reflexive Banach spaces (but, anyway, $L_\infty([0,\Gamma],E_u)$ is not reflexive). The spaces E_u and E are not assumed to be separable. As explained later on, the hypothesis on E_u has a justification, in that it is a possible way to make the setting to work, which requires $L_\infty([0,\Gamma],E_u)$ to be the adjoint of another space. The hypothesis on E make it easier on the semigroup front. Essentially this is one out of the many settings covered in [1], but handled with a different technique.

Targets states for which the Maximum Principle may hold must be support points of the set of states reachable under bounded norm. Applying the results in [8] we show that a weath of support points exists. Then, suitably blending certain ingredients and, more precisely: the properties of the setting, the celebrated Bishop Phelps Support Theorem, the determination and characterization of the radial kernel of the bounded norm reachable set, and, finally, an important technical Lemma by Fattorini, we obtain a *very simple* derivation of the Maximum Principle for a dense set of targets and functionals in E^*. In addition, we geometrically characterize the set of targets for which the Principle holds, along with the conditions, which decide whether the principle is only necessary or necessary and sufficient.

Naturally, all this simplicity is made possible by the profound and powerful results we invoke along the way. But, on the other hand, this is what powerful theorems are mainly useful for: make life easier.

There is clearly the need to investigate the connections of our analysis with recent research work on Maximum principle, mainly by Fattorini. We comment briefly on this in the conclusions, touching upon the open problems it poses.

2 Unconstrained and Constrained Reachable Sets

Consider the variation of constants formula:

$$x(t) = T(t)\overline{x} + \int_{[0,t]} T(t-\sigma)Bu(\sigma)ds$$

where $t \in [0,\Gamma]$, $\{T(t)\}$, is a C_0 semigroup on a real Banach space E, $B\colon E_u \to E$ is an operator on the real Banach space E_u, $u \in L_\infty([0,\Gamma],E_u)$, and the integral is a Bochner integral. The case $E_u = E$ and $B = I$ is referred to as full control case, but here we do not make the full control assumption. We assume both E and E_u reflexive. These hypotheses will be in force throughout the paper.

For simplicity, dealing with norm optimal control problems, we assume $\overline{x} = 0$. The general case where the system starts from a non-zero state $\overline{x}$ is dealt with considering the target $\zeta - T(\Gamma)\overline{x}$ in place of ζ (see [1]).

It is assumed that we can reach a certain target vector $\zeta \in E$ at fixed time $\Gamma > 0$ or:

$$\zeta = \int_{[0,\Gamma]} T(\Gamma - \sigma)Bu(\sigma)ds = \mathcal{L}_\Gamma u$$

and we look for the minimum norm input that does the job of reaching ζ. The relevant linear transformation $\mathcal{L}_\Gamma$: $L_\infty([0,\Gamma], E_u) \to E$ is well known to be continuous.

On the other hand, in our setting (see [1]):

$$L_\infty([0,\Gamma], E_u) = (L_1([0,\Gamma], E_u^*))^*$$

Remark 1. As stated by [1], if we had put $E_u = F$, and had taken X such that $F = X^*$ then for $L_\infty([0,\Gamma], F) = (L_1([0,\Gamma], X))^*$ to hold, barring separability assumptions, it remains to assume reflexivity of X. For simplicity, we have taken directly E_u reflexive which implies E_u^* reflexive. Thus our hypothesis on value space E_u is justified, as long as we are interested in a setting where the above duality relation on time function spaces holds. Reflexivity of E provides instead a simplification in terms of semigroup theory: in fact, in this case, the adjoint semigroup is C_0, and we are dispensed to invoke the Phillips dual, as in the general case (again [1]).

This setting has the advantage that we find ourselves on the dual side, where the situation is much more favorable. While in general, in view of the celebrated James' Theorem, the unit ball of a Banach space *is not* $weakly$ compact, in the dual, thanks to the Banach-Alaoglu Theorem ([6]), the unit ball *is* $weak^*$ compact.

This circumstance is obviously useful if we can prove that the operator $\mathcal{L}_\Gamma$ is $weak^*$ to $weak$ continuous.

We begin recalling that the assumed strong measurability of $u(.)$ implies weak measurability ([7]). Henceforth we use the symbol $< .,. >$ to denote the canonical pairing functionals. If different pair of spaces are involved we either use a suffixes, or leave distinctions to the context when we feel it is safe to do so.

Proposition 2. The operator $\mathcal{L}_\Gamma$ is $weak^*$ to $weak$ continuous.

Proof. Consider any $y \in E^*$. Recall that under our hypothesis that E be reflexive $\{T^*(.)\}$ is also a C_0 semigroup. Thus

$$g(.) = B^*T^*(\Gamma - .)y \in C([0,\Gamma], E_u^*) \subset L_1([0,\Gamma].E_u^*)$$

In particular it is weakly measurable. Next recall that continuous linear functionals and transformation can go in and out the Bochner integral ([7]), and that, since E is reflexive $\{T^*(.)\}$ is a C_0. semigroup. So, consider a $weak^*$ convergent net $\{u_\alpha\} \to u$ in $L_\infty([0,\Gamma], E_u)$ and write:

$$< y, \int_{[0,\Gamma]} T(\Gamma - \sigma)Bu_\alpha(\sigma)ds >=$$

$$\int_{[0,\Gamma]} < B^*T^*(\Gamma-\sigma)y, u_\alpha(\sigma) > ds =$$

$$< g(.), u_\alpha(.) >_{L_1 L_\infty}$$

Thus the net $\{< g(.), u_\alpha(.) >_{L_1 L_\infty}\}$ converges and the proof is finished. ∎

This result implies that $R_\rho = \mathcal{L}_\Gamma(B_\rho)$, the image under $\mathcal{L}_\Gamma$ of the unit ball B_ρ of $L_\infty([0,\Gamma], E_u)$, is weakly compact and hence weakly closed. Moreover, since R_ρ is convex, it is also strongly closed.

3 Properties of Bounded Norm Reachable Set.

Let $R_\Gamma = \mathcal{R}(\mathcal{L}_\Gamma)$, which is the reachable set in the interval $[0,\Gamma]$. We may assume, without restriction of generality that $R_\Gamma^- = E$. If this where not the case it suffices to consider R_Γ^- in lieu of E. Generality is not restricted because a closed subspace of a reflexive Banach space is reflexive.

For convenience we summarize the relevant properties of the constrained reachable set $\mathcal{L}_\Gamma(B_\rho) = R_\rho$.

- Convex and circled (and hence also symmetric). In particular $0 \in R_\rho$.
- $\mathcal{L}(R_\rho) = R_\Gamma$
- Weakly compact and both weakly and strongly closed.
- In general it has no interior.

In special cases R_ρ might well have interior, but in what follows we assume $R_\rho^i = \phi$.

We add to this list a further important property.

First we state the following:

Definition 3. We define:

$$R_\rho^\vee = \{z : z \in R_\rho, \inf\{\|u\|, \mathcal{L}_\Gamma(u) = z\} < \rho\}$$

and put:

$$R_\rho^\wedge = R_\rho \backslash R_\rho^\vee$$

Proposition 4. $R_\rho^\vee$ is the radial kernel of R_ρ in R_Γ. If $\zeta \in R_\rho^\wedge$ then $\exists \widetilde{u}$ such that $\mathcal{L}_\Gamma(\widetilde{u}) = \zeta$, $\|\widetilde{u}\| = \rho = \min\{\|u\| : \mathcal{L}_\Gamma(u) = \zeta\}$. A necessary condition for a state ζ to be a support point of R_ρ is that $\zeta \in R_\rho^\wedge$.

Proof. Clearly $0 \in R_\rho^\vee$. If $w \neq 0$ and $w \in R_\Gamma$, $\exists u \neq 0$ s.t. $\mathcal{L}_\Gamma(u) = w$. If $0 \leq \varepsilon < \rho$, then the non-zero state:

$$\mathcal{L}_\Gamma(\varepsilon \frac{u}{\|u\|}) \in R_\rho$$

thus R_ρ is radial at zero in R_ρ. Next, if $\xi \neq 0$, $\xi \in R_\rho^\vee$, then $\exists u$ s.t.$\|u\| = \rho'$, $0 < \rho' < \rho$ and $\mathcal{L}_\Gamma(u) = \xi$. Then for an arbitrary $z \neq \xi$, $0 \neq z \in R_\Gamma$, let $\mathcal{L}_\Gamma(u_z) = z - \xi$ and $\|u_z\| = \gamma \neq 0$. The state $\xi + \alpha(z - \xi)$, with $\alpha \leq \frac{\rho - \rho'}{\gamma}$, is reachable by the control $u + \alpha u_z$, whose norm is less or equal to ρ, by the triangle inequality. Thus $\xi + \beta(z - \xi) \in R_\rho$ for $0 \leq \beta \leq \alpha$ and R_ρ is radial at ξ. Next $\zeta \in R_\rho^\wedge$ implies

$$\inf\{\|u\|, \mathcal{L}_\Gamma(u) = \zeta\} \geq \rho$$

But, also, $\zeta \in R_\rho$ implies $\exists u$ s.t.$\|u\| \leq \rho$ and so the second statement is proved. The just proved radiality property prevents any state in $R_\rho^\vee$ to be a support point of R_ρ. In fact suppose that for some $\zeta \in R_\rho^\vee$ there exists a continuous linear functional f, such that $< f, R_\rho > \leq < f, \zeta >$. If it were $< f, \zeta >= 0$ then $< f, R_\rho >= \{0\} =< f, \mathcal{L}(R_\rho) >=< f, R_\Gamma >$. But the fact that R_Γ is contained in a closed hyperplane contradicts the fact that R_Γ is dense. If $< f, \zeta >= \alpha > 0$, then because by radiality, for some $\varepsilon > 0$, $\xi = (1 + \varepsilon)\zeta \in R_\rho$, we can write $< f, \xi >>< f, \zeta >$, contradicting separation. This concludes the proof. ∎

Remark 5. If R_ρ had interior then $R_\rho^\vee = R_\rho^i$ and $\mathcal{B}(R_\rho) = R_\rho^\wedge$, and all points of $R_\rho^\wedge$ would be support points. If not then, since R_ρ is closed, $\mathcal{B}(R_\rho) = R_\rho$, but this proposition tell us that we can find support points only in the proper subset $R_\rho^\wedge$. In this case we may view the above partition as a quasi-topological decomposition in which $R_\rho^\vee$ plays the role of quasi-interior and $R_\rho^\wedge$ as a quasi-boundary.

The results on the support problem given in [8] hold good here because they are surely true for Hausdorff complete locally convex spaces. We summarize the argument here. First notice that the tangent cone to a convex set at an extreme point is always pointed. Theorem 5 in [8] states that the closure of a pointed cone in a linear topological space is a proper cone. In the same paper Lemma 2 states that a closed proper cone is contained in a closed semispace; the statement is made for Hilbert spaces but it is obvious from it simple proof that it is indeed valid for Hausdorff complete locally convex spaces. It follows that all extreme points of a convex set are support points. Next, by the Krein-Milman Theorem, the set $ex(R_\rho)$ of extreme points of R_ρ is non-void (and generates R_ρ by closed convex extension). Clearly $ex(R_\rho) \subset R_\rho^\wedge$, because there cannot be radiality in an extreme point. As recalled, all points of $ex(R_\rho)$ are support points. Let us call the set of all support points S_ρ. We collect this remarks in the following:

Proposition 6. $ex(R_\rho) \subset S_\rho \subset R_\rho^\wedge$. In particular $S_\rho \neq \phi$.

4 Support for Void Interior Convex Sets

In a nutshell the maximum principle is about showing that a target is a support point for R_ρ and then translating the supporting condition in a pointwise in time condition for the optimal control.

The support condition is a special separation condition (separation between a singleton in a set and the set itself). One classical way to provide support points for a set is to invoke a topological separation theorem, which states that, in a linear topological space, given two convex sets A and B and assuming that A has interior, there is a continuous linear functional separating A and B iff $B \cap A^i = \phi$. One obviously uses this Theorem taking $B = \{\zeta\}$ with $\zeta \in \mathcal{B}(A)$ thereby obtaining a support point for A^-. The derivation of this Separation Theorem in [6] is essentially pre-topological and based on the theory of cones.

Despite the "iff" we are in presence of a masked sufficient condition, because of the presiding hypothesis that $A^i \neq \phi$. Thus we cannot exclude the possibility of finding support points for void interior sets.

If the dimension is finite, then every convex set has (relative) interior, so that the application of this separation theorem to find support targets is direct and general.

In infinite dimension, as already mentioned, R_ρ has no interior in general.

We see various possible techniques to overcome this hurdle.

The first consists in re-topologizing R_ρ in such a way that it has interior in the new topology. This is the technique introduced by Fattorini (see [1]), who has developed a very complete and advanced theory for both norm and time optimality. Because the new topology is stronger, in the larger dual space, singular functionals appear (non-zero functionals that are zero on the domain of the infinitesimal generator).

The second possibility (developed in [8]) consisted in introducing and applying a Support Theorem for extreme points of convex sets (like all other separation/support theorems, this too is based on theory of cones). This technique works well in a Hilbert space setting. In the present setting it allows us to exhibit an already large set of support points via the Krein-Milman Theorem. But it would be nice to tell more about the structure of the set of support points of R_ρ.

To this effect we apply a further tool: the celebrated Bishop-Phelps Theorem for closed sets in Banach spaces (see [9]).

We recall the relevant part of the Bishop-Phelps theorem:

Theorem 7. A closed convex set of a real Banach space has a non-void set of support points which is dense in its boundary.

Remark 8. Some authors have shown that the corresponding statement for complex Banach spaces does not hold (see e.g. [10]). This is completely irrelevant here.

Basically the idea of the proof of the Bishop Phelps Theorem is to observe that the cone generated by a translated ball not containing the origin is a pointed cone with interior. If we can place, by translation, the apex of this cone on a point of the convex set in such a way that the apex is the only point in their intersection, then the cone is separated from the convex set. Note the ingenuous swap of roles: here it is the cone in charge of insuring that at least one of the convex sets has non-void interior, so that the convex set, for which support is sought, is allowed to have void interior. But such separation and the fact that the second set is a cone, imply that the point in question is a support point for the convex set.

5 Norm Optimality

Putting together the Bishop-Phelps Theorem and Proposition 4, we obtain the following result:

Theorem 9. The set S_ρ of all support points for R_ρ, which we proved to be non-void and contained in $R_\rho^\wedge$, is dense in $R_\rho^\wedge$. For all points $\zeta \in S_\rho$, the minimum norm of controls that steer the system from 0 (at $t = 0$) to ζ (at $t = \Gamma$) exists and is ρ.

The proof of this Theorem is contained in the previous analysis and can be omitted. Notice that if the interior of R_ρ is void (as we are assuming) then S_ρ is dense in the whole R_ρ and hence also in the set $R_\rho^\wedge$ into which is contained. If the interior is non-void, then S_ρ is dense in $\mathfrak{B}(R_\rho)$, which, in such case, coincides with $R_\rho^\wedge$.

Let's write down the support condition for $\zeta \in S_\rho$. There exists a continuous linear functional $f \in E^*$ such that:

$$< f, \zeta > \geq < f, z > , \forall z \in R_\rho$$

The same condition holds, however, for all $\xi \in R_\rho$ such that:

$$< f, \xi > = < f, \zeta >$$

Define

$$\Omega = \{\xi : \xi \in R_\rho, \ < f, \xi > = < f, \zeta >\}$$

Note that Ω is a closed convex set, as a matter of facts it is a closed exposed face of R_ρ and obviously:

$$\Omega \subset S_\rho$$

It is well possible that $\Omega = \{\zeta\}$, in which case ζ is an exposed extreme point. But it may happen, as well, that Ω is a proper superset of $\{\zeta\}$.

Next, to move back to the control space, we begin recalling the following well known proposition ([6]):

Proposition 10. Let T be a linear transformation $E \to G$ and C be a convex subset of E. If A is a face of $T(C)$ then $T^{-1}(A) \cap C$ is a face of C.

It follows that $\mathcal{L}_\Gamma^{-1}(\Omega) \cap B_\rho$ is a closed face of B_ρ. It is also an exposed face because we can retrieve it as a face of B_ρ, generated by the support functional

$$g = \mathcal{L}_\Gamma^* f = B^* T^* (\Gamma - .) f \in C([0, \Gamma], E_u^*) \subset L_1([0, \Gamma], E_u^*)$$

that "pushes back" the support functional $f \in E^*$.

In fact for each $\xi \in \Omega$, for all the corresponding $u_\xi \in \mathcal{L}_\Gamma^{-1}(\Omega) \cap B_\rho$ it must be (in the next formula there are two different pairing functionals, but we leave unchanged the symbol):

$$< f, \xi > = < f, \mathcal{L}_\Gamma u_\xi > = < \mathcal{L}_\Gamma^* f, u_\xi > \geq < f, \mathcal{L}_\Gamma u > = < \mathcal{L}_\Gamma^* f, u > , \forall u \in B_\rho$$

If instead $u \in B_\rho \backslash \mathcal{L}_\Gamma^{-1}(\Omega) \cap B_\rho$:

$$< g, u_\xi > > < g, u >$$

Thus we have proved that the functional g is a support functional for B_ρ at all points of $\mathcal{L}_\Gamma^{-1}(\Omega) \cap B_\rho$ and this set is a closed and exposed face of B_ρ.

Remark 11. A consequence of James' Theorem insures that, for a non-reflexive Banach spaces there exist some continuous linear functionals, that do not attain their supremum on the unit ball (but, of course, by the Separation Theorem, there is also a profusion of continuous linear functionals that do attain their supremum on the unit ball). Clearly g is is not one of those pathological continuous linear functionals, because we have proved that it attains its supremum on B_ρ.

If we use the above condition to characterize u_ξ, we have a necessary condition for the optimum controls corresponding to the target ζ. The condition is not sifficient if $\Omega \backslash \{\zeta\} \neq \phi$. If, by the contrary, no other target is involved, or $\Omega = \{\zeta\}$, the condition becomes sufficient (independently of the fact that the optimum control be unique). Reall that $\Omega = \{\zeta\}$means that ζ is an exposed extreme point. The proof is implicit in the above discussion illustrating the correspondence between the two exposed faces of R_ρ and B_ρ. We register this fact in the following:

Theorem 12. The condition on $\overline{u}$:

$$< \mathcal{L}_\Gamma^* f, \overline{u} > = \max\{< \mathcal{L}_\Gamma^* f, u > : u \in B_\rho\}$$

is necessary for a control $\overline{u}$ to be the norm optimal for the target ζ The condition becomes also sufficient if $\Omega = \{\zeta\}$, or, equivalently, if ζ is an exposed extreme point.

6 Maximum Principle

To state the Maximum Principle, we need to recast the support condition in an equivalent condition on the optimal input, which characterizes the input pointwise in time.

Given the reflexivity assumptions in force, the function

$$\mathcal{L}_\Gamma^* f(.) = B^* T^* (\Gamma - .) f$$

is continuous thus Borel and weakly measurable. Also continuity of the norm implies that $\|\mathcal{L}_\Gamma^* f(.)\|_{E^*}$ is a continuous and thus measurable function. The support condition for the optimum control $\overline{u}$ yields:

$$< \mathcal{L}_\Gamma^* f, \overline{u} >_{L_1 L_\infty} = \int_{[0,\Gamma]} < \mathcal{L}_\Gamma^* f, \overline{u} >_{E_u^* E_u} (\sigma) d\sigma =$$

$$= \max\left\{ \int\limits_{[0,\Gamma]} < \mathcal{L}_\Gamma^* f, u > (\sigma) d\sigma \colon\ u \in B_\rho \right\} \leq \rho \int\limits_{[0,\Gamma]} \|\mathcal{L}_\Gamma^* f\|_{E_u^*}(\sigma) d\sigma =$$

$$= \int\limits_{[0,\Gamma]} \max\{< \mathcal{L}_\Gamma^* f, \nu >\colon \|\nu\|_{E_u} \leq \rho\}(\sigma) d\sigma < \infty$$

On the other hand, by Lemma 2.2.10 in [1], the inequality can be substituted by equality (indeed Fattorini proved this for arbitrary Banach spaces) and so the support condition implies (and clearly is implied by):

$$< B^* T^*(\Gamma - \sigma) f, \overline{u}(\sigma) >= \max\{< B^* T^*(\Gamma - \sigma) f, \nu >\colon \|\nu\|_{E_u} \leq \rho\}$$

a.e. for $\sigma \in [0, \Gamma]$. This characterization of the optimal control is the Maximum Principle. It yields a set of optimal controls. In view of the equivalence with the support condition, our consideration on whether necessity or necessity and sufficiency prevail hold good as well for the Maximum Principle. Thus if ζ is a an exposed extreme points all controls defined by the principle are optimal (or the condition is necessary and sufficient). Otherwise we can only say that the optimal controls are among those functions satysfying this condition (the condition is necessary).

7 Conclusions

The following considerations are inspired by the cited work by Fattorini, including a private communication.

One open problem is the relationship between the dense subset of targets for which the Maximum Principle holds for functionals in E^*, that we have shown to exists, and the domain of A.

In [1] the Maximum Principle is proved for targets in $\mathcal{D}(A)$ using functionals in a linear space Z larger than E^*.

On the other hand [3] shows that, in general, if the target is in $\mathcal{D}(A)$, then the functional in E^*does not always exist.

He also conjectures that this implication may fail even under the assumption that the semigroup is selfadjoint and the state space is a Hilbert space, albeit the implication has been found to hold for the left translation semigroup and $E = L_2([0, \infty))$ ([4]).

Thus he puts the question of finally determining a condition, stronger than target $\zeta \in \mathcal{D}(A)$, ensuring that the functionals appearing in the Maximum Principle exists in E^*.

This question is of course crucial but it is open at the moment.

In some cases this whole issue is known to be connected the other important aspect of regularity of optimal control (e.g. [2]). A further motivation for orientating research toward these open problems.

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References

[1] H.O. Fattorini "*Infinite Dimensional Linear Control Systems*", Elsevier, Amsterdam 2005.

[2] H.O. Fattorini, "*Strong Regularity of Time and Norm Optimal Controls*", Submitted

[3] H.O. Fattorini, "*Linear Control Systems In Sequence Spaces*" Functional Analysis and evolution equations, The Gunter Lumer Volume 2007, pp 273-290.

[4] H.O. Fattorini, "*Regular and Strongly Regular Time and Norm Optimal Controls*" Submitted.

[5] H.O. Fattorini, Private Communication.

[6] J.L. Kelley and I. Namioka, *Linear Topological Spaces*, Springer, New York, 1963

[7] K. Yosida, "Functional Analysis", Springer-Verlag, New York, 1974.

[8] P. d'Alessandro, "Closure of Pointed Cones and Maximum Principle in Hilbert Spaces", CUBO a Mathematical Journal, Vol.13, No.2 (73-84), June 2011.

[9] R.R. Phelps, "Support Cones in Banach Spaces and Their Application", Adv. in Math. 13 (1974), 1-19.

[10] W.B. Johnson, J. Lindenstrauss Eds, "Handbook of The Geometry of Banach Spaces", Vol. 1, Elsevier, Amsterdam, 2001.