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Remarks on cotype and absolutely summing multilinear operators

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ABSTRACT

In this short note we present some new results concerning cotype and absolutely summing multilinear operators, extending recent results from different authors.

RESUMEN

En esta nota presentamos nuevos resultados sobre cotipo y suma absoluta de operadores multilineales, extendiendo resultados recientes de diferentes autores.

Keywords and Phrases: Absolutely p-summing multilinear operators, cotype.

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1 Introduction

In this note the letters $X_1, ..., X_n, X, Y$ will denote Banach spaces over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

From now on the space of all continuous n-linear operators from $X_1 \times \cdots \times X_n$ to Y will be denoted by $\mathcal{L}(X_1, ..., X_n; Y)$. If $1 \leq s < \infty$, the symbol s^* represents the conjugate of s. It will be convenient to adopt that $s/\infty = 0$ for any $s > 0$. For $1 \leq q < \infty$, we denote by $\ell_q^w(X)$ the set $\{(x_j)_{j=1}^\infty \subset X : \sup_{\varphi \in B_{X^*}} \sum_j |\varphi(x_j)|^q < \infty\}$.

If $0 < p, q_1, ..., q_n < \infty$ and

$$\frac{1}{p} \leq \frac{1}{q_1} + \cdots + \frac{1}{q_n},$$

a multilinear operator $T \in \mathcal{L}(X_1, ..., X_n; Y)$ is absolutely $(p; q_1, ..., q_n)$-summing if

$$(T(x_j^{(1)}, ..., x_j^{(n)}))_{j=1}^\infty \in \ell_p(Y)$$

for every $(x_j^{(k)})_{j=1}^\infty \in \ell_{q_k}^w(X_k)$, $k = 1, ..., n$. In this case we write $T \in \Pi_{p,q_1,...,q_n}^n(X_1, ..., X_n; Y)$. If $q_1 = \cdots = q_n = q$, we write $\Pi_{p,q}^n(X_1, ..., X_n; Y)$ instead of $\Pi_{p,q,...,q}^n(X_1, ..., X_n; Y)$. For details on the linear theory we refer to the excellent monograph [9] and for the multilinear theory we refer to [1, 7, 14] and references therein.

This paper deals with the connection between cotype and absolutely summing multilinear operators; this line of investigation begins with [4] and was followed by several recent papers (we refer, for example, to [5, 6, 8, 11, 12, 13, 15, 16] and for a full panorama we mention [14]). The following result appears in [10, Theorem 3 and Remark 2] and [16, Corollary 4.6] (see also [5, Theorem 3.8 (ii)] for a particular case):

Theorem 1.1 (Inclusion Theorem). *Let $X_1, ..., X_n$ be Banach spaces with cotype s and $n \geq 2$ be a positive integer:*

(i) If $s = 2$, then

$$\Pi_{q;q}^n(X_1, ..., X_n; Y) \subseteq \Pi_{p;p}^n(X_1, ..., X_n; Y)$$

holds true for $1 \leq p \leq q \leq 2$.

(ii) If $s > 2$, then

$$\Pi_{q;q}^n(X_1, ..., X_n; Y) \subseteq \Pi_{p;p}^n(X_1, ..., X_n; Y)$$

holds true for $1 \leq p \leq q < s^* < 2$.

As a consequence of results from [3] one can easily prove the following generalization of this result (see [2] for details):

Theorem 1.2. *If X_1 has cotype 2 and $1 \leq p \leq s \leq 2$, then*

$$\Pi_{s;s,q,...,q}^n(X_1, ..., X_n; Y) \subseteq \Pi_{p;p,q,....,q}^n(X_1, ..., X_n; Y)$$

for all $X_2, ..., X_n, Y$ *and all* $q \geq 1$. *In particular*

$$\Pi^n_{s;s}(X_1, ..., X_n; Y) \subseteqq \Pi^n_{p;p,s,\ldots,s}(X_1, ..., X_n; Y) \subseteqq \Pi^n_{p;p}(X_1, ..., X_n; Y). \tag{1.1}$$

A similar result, mutatis mutandis, holds if X_j *(instead of* X_1*) has cotype* 2.

In this note we remark that analogous results hold for other situations in which the spaces involved may have different cotypes and no space may have necessarily cotype 2.

2 Results

The following proposition can be found in [5]:

Proposition 2.1. *Let* $1 \leq p_1, ..., p_n, p, q_1, ..., q_n, q \leq \infty$ *such that* $1/t \leq \sum_{j=1}^n 1/t_j$ *for* $t \in \{p, q\}$. *Let* $0 < \theta < 1$ *and define*

$$\frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q} \text{ and } \frac{1}{r_j} = \frac{1-\theta}{p_j} + \frac{\theta}{q_j} \text{ for all } j = 1, ..., n$$

and let $T \in \mathcal{L}(X_1, ..., X_n; Y)$. *Then*

$$T \in \Pi^n_{p;p_1,\ldots,p_n} \cap \Pi^n_{q;q_1,\ldots,q_n} \text{ implies } T \in \Pi^n_{r;r_1,\ldots,r_n},$$

provided that for each $j = 1, ..., n$, *one of the following conditions holds:*

(i) X_j *is an* $\mathcal{L}_\infty$*-space;*

(ii) X_j *is of cotype* 2 *and* $1 \leq p_j, q_j \leq 2$*;*

(iii) X_j *is of finite cotype* $s_j > 2$ *and* $1 \leq p_j, q_j < s_j^*$*;*

(iv) $p_j = q_j = r_j$.

Next lemma appears in [13, Theorem 3.1] without proof. We present a proof for the sake of completeness:

Lemma 2.2. *Let* $s > 0$. *Suppose that* X_j *has cotype* s_j *for all* $j = 1, ..., n$ *and at least one of the* s_j *is finite. If*

$$\frac{1}{s} \leq \frac{1}{s_1} + \ldots + \frac{1}{s_n},$$

then

$$\mathcal{L}(X_1, ..., X_n; Y) = \Pi^n_{s;b_1,\ldots,b_n}(X_1, ..., X_n; Y)$$

for

$$b_j = 1 \text{ if } s_j < \infty \text{ and } b_j = \infty \text{ if } s_j = \infty.$$

Proof. Let $j_1,...,j_k \in \{1,...,n\}, k \le n$ such that $s_{j_1},...,s_{j_k}$ are finite and $s_j = \infty$ if $j \ne j_1,...,j_k$.

If $\left(x_i^{(j_l)}\right)_{i=1}^{\infty} \in \ell_1^w(X_{j_l})$ and $(x_i^{(j)})_{i=1}^{\infty} \in \ell_\infty(X_j), j \ne j_l,\ l = 1,...,k$, using Generalized Hölder Inequality, we obtain

$$\left(\sum_{i=1}^{\infty} \left\|T(x_i^{(1)},...,x_i^{(n)})\right\|^s\right)^{\frac{1}{s}} \le \|T\| \left(\sum_{i=1}^{\infty} \left(\left\|x_i^{(1)}\right\| \cdots \left\|x_i^{(n)}\right\|\right)^s\right)^{\frac{1}{s}}$$
$$\le C\,\|T\| \left(\sum_{i=1}^{\infty} \left\|x_i^{(j_1)}\right\|^{s_{j_1}}\right)^{1/s_{j_1}} \cdots \left(\sum_{i=1}^{\infty} \left(\left\|x_i^{(j_k)}\right\|\right)^{s_{j_k}}\right)^{1/s_{j_k}}$$

where C is such that

$$\prod_{j=1, j\ne j_1,...,j_n}^{n} \left\|x_i^{(j)}\right\| \le C$$

for all i. Since X_j has cotype s_j, for $j_1,...,j_k$, we have

$$\left(\sum_{i=1}^{\infty} \left\|T(x_i^{(1)},...,x_i^{(n)})\right\|^s\right)^{\frac{1}{s}} \le C\,\|T\| \left(\sum_{i=1}^{\infty} \left\|x_i^{(j_1)}\right\|^{s_{j_1}}\right)^{1/s_{j_1}} \cdots \left(\sum_{i=1}^{\infty} \left(\left\|x_i^{(j_k)}\right\|\right)^{s_{j_k}}\right)^{1/s_{j_k}}$$
$$= C\,\|T\| \prod_{t=1}^{k} \left(\sum_{i=1}^{\infty} \left\|id_{X_{j_t}}\left(x_i^{(j_t)}\right)\right\|^{s_{j_t}}\right)^{1/s_{j_t}} < \infty$$

and the result follows. □

The main result of this note is the following Theorem. At first glance it seems to have too restrictive assumptions, but Corollary 2.4 and Example 2.5 will illustrate its usefulness:

Theorem 2.3. *Let* k, n *be natural numbers,* $n \ge k \ge 2$ *and* $X_{k+1},...,X_n,Y$ *be arbitrary Banach spaces. If* X_j *has finite cotype* $s_j \ge 2$ *for* $j = 1,...,k$*, then*

$$\Pi^n_{p;p_1,...,p_k,q,...,q}(X_1,...,X_n;Y) \subseteqq \Pi^n_{r;r_1,...,r_k,q,...,q}(X_1,...,X_n;Y)$$

for any $(q,\theta) \in [1,\infty] \times (0,1)$,

$$1 \le p_j \le 2 \text{ (when } s_j = 2\text{)},$$
$$1 \le p_j < s_j^* \text{ (when } s_j > 2\text{)}$$

and $s \in [1,\infty)$ *so that*

$$\frac{1}{s} \le \frac{1}{s_1} + \cdots + \frac{1}{s_k},$$
$$\frac{1}{r} = \frac{1-\theta}{s} + \frac{\theta}{p},$$
$$\frac{1}{r_j} = \frac{1-\theta}{1} + \frac{\theta}{p_j},$$

for all $j = 1,...,k$.

Proof. Let $T \in \Pi^n_{p;p_1,\ldots,p_k,q,\ldots,q}(X_1,...,X_n;Y)$. By the previous lemma,

$$\mathcal{L}(X_1,...,X_n;Y) = \Pi^n_{s;1,\ldots,1,\infty,\ldots,\infty}(X_1,...,X_n;Y),$$

where 1 is repeated k times. A fortiori, we have

$$\mathcal{L}(X_1,...,X_n;Y) = \Pi^n_{s;1,\ldots,1,q,\ldots,q}(X_1,...,X_n;Y).$$

So,

$$T \in \Pi^n_{s;1,\ldots,1,q,\ldots,q}(X_1,...,X_n;Y) \cap \Pi^n_{p;p_1,\ldots,p_k,q,\ldots,q}(X_1,...,X_n;Y).$$

From Proposition 2.1 we get

$$T \in \Pi^n_{r;r_1,\ldots,r_k,q,\ldots,q}(X_1,...,X_n;Y).$$

□

Corollary 2.4. *Let* k, n *be natural numbers,* $n \geq k \geq 2$*,* $X_{k+1},...,X_n, Y$ *be arbitrary Banach spaces and* $q \in [1,\infty)$*. If* X_j *has finite cotype* $s_j \geq 2$*,* $j = 1,...,k$ *and* $1 \leq 1/s_1 + \cdots + 1/s_k$*, then*

$$\Pi^n_{p;p_1,\ldots,p_k,q,\ldots,q}(X_1,...,X_n;Y) \subsetneqq \Pi^n_{r;r_1,\ldots,r_k,q,\ldots,q}(X_1,...,X_n;Y),$$

where $p_j = p$ *and* $r_j = r$ *for all* $j = 1,...,k$*, for all* r *so that*

$$1 \leq r < p < \min s_j^* \text{ if } s_j \neq 2 \text{ for some } j = 1,...,k,$$
$$1 \leq r < p \leq 2 \text{ if } s_j = 2 \text{ for all } j = 1,...,k.$$

In particular

$$\Pi^n_{p;p}(X_1,...,X_n;Y) \subsetneqq \Pi^n_{r;r}(X_1,...,X_n;Y)$$

for all r *so that*

$$1 \leq r < p < \min s_j^* \text{ if } s_j \neq 2 \text{ for some } j = 1,...,k,$$
$$1 \leq r < p \leq 2 \text{ if } s_j = 2 \text{ for all } j = 1,...,k.$$

Proof. Since $1 \leq 1/s_1 + \cdots + 1/s_k$, we can use $s = 1$ in the previous theorem. Since $p = p_i$ and $r = r_i$ for all $i = 1,...,k$ and $s = 1$, we conclude that

$$\Pi^n_{p;p,\ldots,p,q,\ldots,q}(X_1,...,X_n;Y) \subsetneqq \Pi^n_{r;r,\ldots,r,q,\ldots,q}(X_1,...,X_n;Y).$$

In fact, for any $1 \leq r < p$ there is a $\theta \in (0,1)$ so that

$$\frac{1}{r} = \frac{1-\theta}{1} + \frac{\theta}{p}$$

and since $p = p_i$ and $r = r_i$, the same $\theta \in (0,1)$ satisfies

$$\frac{1}{r_i} = \frac{1-\theta}{1} + \frac{\theta}{p_i}.$$

Choosing $q = p$, since $r < p$ we have

$$\Pi^{n}_{p;p}(X_1, ..., X_n; Y) \subsetneqq \Pi^{n}_{r;r,...,r,p,...,p}(X_1, ..., X_n; Y) \subsetneqq \Pi^{n}_{r;r}(X_1, ..., X_n; Y).$$

□

Example 2.5. *Let* $X_4, ..., X_n, Y$ *be arbitrary Banach spaces. Then*

$$\Pi^{n}_{p;p,p,p,q,...,q}(\ell_3, \ell_3, \ell_3, X_4, ..., X_n; Y) \subsetneqq \Pi^{n}_{r;r,r,r,q,...,q}(\ell_3, \ell_3, \ell_3, X_4, , , ., X_n; Y)$$

for all $q \in [1, \infty)$ *and* $1 \leq r < p < 3^*$. *In particular*

$$\Pi^{n}_{p;p}(\ell_3, \ell_3, \ell_3, X_4, ..., X_n; Y) \subsetneqq \Pi^{n}_{r;r}(\ell_3, \ell_3, \ell_3, X_4, , , ., X_n; Y)$$

for all $1 \leq r < p < 3^*$.

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