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Majorization for certain classes of analytic functions defined by a new operator

E. A. Eljamal and M. Darus
School of Mathematical Sciences,
Faculty of Science and Technology,
Universiti Kebangsaan Malaysia,
Bangi 43600 Selangor D. Ehsan, Malaysia.
email: n-ebtisam@yahoo.com , maslina@ukm.my

ABSTRACT

In the present paper, we investigate the majorization properties for certain classes of multivalent analytic functions defined by a new operator. Moreover, we pointed out some new and known consequences of our main result.

RESUMEN

En el presente artículo, investigamos las propiedades de mayorización para ciertas clases de funciones analíticas multivalentes definidas por un nuevo operador. Además, resaltamos algunas consecuencias -nuevas y conocidas- de nuestro resultado princresultado.

Keywords and Phrases: Majorization properties, multivalent functions, Ruscheweyh derivative operator, Hadamard product.

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1 Introduction

Let f and g be analytic in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$. We say that f is majorized by g in U and write

$$f(z) \ll g(z) \qquad (z \in U) \tag{1.1}$$

if there exists a function φ, analytic in U such that

$$|\varphi(z)| \leq 1 \quad \text{and} \quad f(z) = \varphi(z)g(z) \quad (z \in U). \tag{1.2}$$

It maybe noted here that (1.1) is closely related to the concept of quasi-subordination between analytic functions. Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1.3}$$

which are analytic and multivalent in the open unit disk U. In particular, if $p = 1$, then$A_1 = A$. For functions $f_j \in A_p$ given by

$$f_j(z) = z^p + \sum_{k=p+1}^{\infty} a_{k,j} z^k, (j = 1, 2; p \in \mathbb{N}), \tag{1.4}$$

we define the Hadamard product or convolution of two functions f_1 and f_2 by

$$f_1 * f_2(z) = z^p + \sum_{k=p+1}^{\infty} a_{k_1}, a_{k_2} z^k = (f_2 * f_1)(z). \tag{1.5}$$

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Definition 1.1. *Let the function* f *be in the class* A_p. *Ruscheweyh derivative operator is given by*

$$R^n = z^p + \sum_{k=p+1}^{\infty} C(k, n) a_k z^k. \tag{1.6}$$

Next we define the following differential operator,

$$D^0 = f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$$

$$\begin{aligned} D^1_{n,\lambda_1,\lambda_2,p} &= D^0 f(z) \frac{p - p\lambda_1 + \lambda_2(k-p)}{p + \lambda_2(k-p)} + (D^0 f(z))' \frac{z\lambda_1}{p + \lambda_2(k-p)} \\ &= z^p + \Sigma_{k=p+1}^{\infty} \left[\frac{p + (\lambda_1 + \lambda_2)(k-p)}{p + \lambda_2(k-p)} \right] a_k z^k, \end{aligned}$$

and

$$D^2_{n,\lambda_1,\lambda_2,p} = D^1_{n,\lambda_1,\lambda_2,p} f(z) \frac{p - p\lambda_1 + \lambda_2(k-p)}{p + \lambda_2(k-p)} + (D^1_{n,\lambda_1,\lambda_2,p} f(z))' \frac{z\lambda_1}{p + \lambda_2(k-p)}$$

$$= z^p + \Sigma_{k=p+1}^{\infty} \left[\frac{p + (\lambda_1 + \lambda_2)(k-p)}{p + \lambda_2(k-p)} \right]^2 a^k z^k.$$

In general,

$$D^m_{n,\lambda_1,\lambda_2,p} f(z) = D(D^{n-1} f(z) = z^p + \Sigma_{k=p+1}^{\infty} \left[\frac{p + (\lambda_1 + \lambda_2)(k-p)}{p + \lambda_2(k-p)} \right]^m a_k z^k \tag{1.7}$$

where $(m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda_2 \geq \lambda_1 \geq 0)$*. By applying convolution product on (1.6) and (1.7) we have the following operator*

$$D^m_{n,\lambda_1,\lambda_2,p} f(z) = z^p + \Sigma_{k=p+1}^{\infty} \left[\frac{p + (\lambda_1 + \lambda_2)(k-p)}{p + \lambda_2(k-p)} \right]^m C(k,n) a_k z^k, \tag{1.8}$$

where $C(k,n) = \frac{\Gamma(k+n)}{\Gamma(k)}$.

Moreover, for $m, n \in N_0, \quad \lambda_2 \geq \lambda_1 \geq 0$

$$(p + \lambda_2(k-p)) D^{m,n}_{\lambda_1,\lambda_2,p} f(z) = (p + \lambda_2(k-p) - p\lambda_1) D^{m,n}_{\lambda_1,\lambda_2,p} f(z) + \lambda_1 z (D^{m,n}_{\lambda_1,\lambda_2,p} f(z))' \tag{1.9}$$

Special cases of this operator include:

- the Ruscheweyh derivative operator in the case $D^{0,n}_{0,0,1} f(z) \equiv R^n$ [6],
- the Salagean derivative operator in the case $D^{m,0}_{1,0,1} f(z) \equiv D^m \equiv S^n$ [2],
- the generalized Salagean derivative operator introduced by Al-Oboudi in the case $D^{m,0}_{\lambda_1,0,1} f(z) \equiv D^m_{\lambda_1}$ [1],
- the generalized Ruscheweyh derivative operator in the case $D^{1,n}_{\lambda_1,0,1} f(z) \equiv D^{\lambda_1}_n$ [3], and
- the generalized Al-Shaqsi and Darus derivative operator in the case $D^{m,n}_{\lambda_1,0,1} f(z) \equiv D^{m,\lambda_1}_n$ [4].

To further our work, we need to define a class of functions as follows:

Definition 1.2. *A function* $f \in A_p$ *is said to be in the class* $S^{m,p,j}_{\lambda_1,\lambda_2,n}[A,B,\gamma]$ *of p-valent functions of complex order* $\gamma \neq 0$ *in* U *if and only if*

$$\left\{ 1 + \frac{1}{\gamma} \left(\frac{z(D^{m,n}_{\lambda_1,\lambda_2,p} f(z))^{(j+1)}}{(D^{m,n}_{\lambda_1,\lambda_2,p} f(z))^{(j)}} - p + j \right) \right\} \prec \frac{1 + Az}{1 + Bz}. \tag{1.10}$$

$(z \in U, p \in \mathbb{N}, j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \gamma \in \mathbb{C} - \{0\}, \lambda_2 \geq \lambda_1 \geq 0)$.

Clearly, we have the following relationships:

(i) $S^{0,1,0}_{0,0,0}[1,-1,\gamma] = S(\gamma)$

(ii) $S^{0,1,1}_{0,0,0}[1,-1,\gamma] = K(\gamma)$
(iii) $S^{0,1,0}_{0,0,0}[1,-1,1-\alpha] = S^*$ for $0 < \alpha < 1$. *The classes* $S(\gamma)$ *and* $K(\gamma)$ *are said to be classes of starlike and convex of complex order* $\gamma \neq 0$ *in* U *and* $S^*(\alpha)$ *denote the class of starlike functions of order* α *in* U.

A majorization problem for the class $S(\gamma)$ has been investigated by Altintas e.tal [5] and for the class S^*=$S^*(0)$ has been investigated by MacGregor [7]. In the present paper, we investigate a majorization problem for the class $S^{m,p,j}_{\lambda_1,\lambda_2,\alpha}[A,B,\gamma]$.

2 Majorization problem for the class $S^{m,p,j}_{\lambda_1,\lambda_2,n}[A,B,\gamma]$

Theorem 2.1. *Let the function* $f \in A_p$ *and suppose that* $g \in S^{m,p,j}_{\lambda_1,\lambda_2,n}[A,B,\gamma]$. *If* $(D^{m,n}_{\lambda_1,\lambda_2,p}f(z))^{(j)}$ *is majorized by* $(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j)}$ *in* U, *then*

$$\left|(D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z))^{(j)}\right| \leq \left|(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j)}\right| \quad \text{for} \quad |z| \leq r_0, \tag{2.1}$$

where $r_0 = r_0(p,\gamma,\lambda_1,\lambda_2,A,B)$ *is the smallest positive root of the equation*

$$r^3\left|\gamma(A-B) - \left(\frac{p+\lambda_2(k-p)}{\lambda_1}\right)B\right| - \left[\frac{p+\lambda_2(k-p)}{\lambda_1} + 2|B|\right]r^2 -$$

$$\left[\left|\gamma(A-B) - (\frac{p+\lambda_2(k-p)}{\lambda_1})B\right| + 2\right]r + \left(\frac{p+\lambda_2(k-p)}{\lambda_1}\right) = 0, \tag{2.2}$$

$$(-1 \leq B < A \leq 1; P \in \mathbb{N}; \gamma \in \mathbb{C} - \{0\}).$$

Proof. Since $g \in S^{m,p,j}_{\lambda_1,\lambda_2,n}[A,B,\gamma]$ we find from (1.10) that

$$1 + \frac{1}{\gamma}\left(\frac{z(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j+1)}}{(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j)}} - p + j\right) = \frac{1+Aw(z)}{1+Bw(z)} \tag{2.3}$$

($\gamma \in \mathbb{C} - 0, j, p \in \mathbb{N} \quad \text{and} \quad p > j$), where w is analytic in U with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < z \quad (z \in U).$$

From (2.3) we get

$$\frac{z(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j+1)}}{(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j)}} = \frac{(p-j) + [\gamma(A-B) + (p-j)B]w(z)}{1+Bw(z)} \tag{2.4}$$

and

$$z(D^{m,n}_{\lambda_1,\lambda_2,p}f(z))^{(j+1)} = (p + \frac{\lambda_2(k-p)}{\lambda_1})(D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z))^{(j)} +$$

$$(p-j-\frac{\lambda_2(k-p)}{\lambda_1})(D^{m,n}_{\lambda_1,\lambda_2,p}f(z))^{(j)}. \tag{2.5}$$

By virtue of (2.4) and (2.5) we get

$$\left|(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j)}\right| \leq \frac{\frac{p+\lambda_2(k-p)}{\lambda_1}[1+|B|z|]}{(\frac{p+\lambda_2(k-p)}{\lambda_1})|\gamma(A-B)-(\frac{p+\lambda_2(k-p)}{\lambda_1})|B|z|}|(D^{m+1,n}_{\lambda_1,\lambda_2,p}g(z))^{(j)}|. \tag{2.6}$$

Next, since $(D^{m,n}_{\lambda_1,\lambda_2,p}f(z))^{(j)}$ is majorized by $(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j)}$ in the unit disk U, we have from (1.2) that

$$(D^{m,n}_{\lambda_1,\lambda_2,p}f(z))^{(j)} = \varphi(z)(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j)}.$$

Differentiating it with respect to z and multiplying by z we get

$$z(D^{m,n}_{\lambda_1,\lambda_2,p}f(z))^{(j+1)} = z\varphi'(z)(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j)} + z\varphi(z)(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j+1)}.$$

Now by using (2.5) in the above equation, it yields

$$(D^{m,n}_{\lambda_1,\lambda_2,p}f(z))^{(j)} = \frac{z\varphi'(z)(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j)}}{\frac{p+\lambda_2(k-p)}{\lambda_1}} + \varphi(z)(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j)} \tag{2.7}$$

Thus, by noting that $\varphi \in \Omega$ satisfies the inequality (see, e.g. Nehari [8])

$$|\varphi'(z)| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2} \qquad (z \in U) \tag{2.8}$$

and using (2.6) and (2.8) in (2.7), we get

$$\left|(D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z))^{(j)}\right| \leq$$

$$\left[|\varphi(z)| + \frac{1-|\varphi(z)|^2}{1-|z|^2}\frac{|z|(1+|B||z|)}{\frac{p+\lambda_2(k-p)}{\lambda_1}-|\gamma(A-B)-(\frac{p+\lambda_2(k-p)}{\lambda_1})|B||z}\right]\left|(D^{m,n}_{\lambda_1,\lambda_2,p}g(z))^{(j+1)}\right| \tag{2.9}$$

which upon setting

$$|z| = r \quad and \quad |\varphi(z)| = \rho \quad (0 \leq \rho \leq 1)$$

leads us to the inequality

$$\left|(D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z))^{(j)}\right| \leq$$

$$\frac{\phi(\rho)}{(1-r^2)(\frac{p+\lambda_2(k-p)}{\lambda_1})-|\gamma(A-B)-(\frac{p+\lambda_2(k-p)}{\lambda_1})B|r}\left|(D^{m+1,n}_{\lambda_1,\lambda_2,p}g(z))^{(j)}\right| \tag{2.10}$$

where

$$\phi(\rho) = -r(1+|B|)\rho^2 + (1-r^2)$$

$$\left[(\frac{p+\lambda_2(k-p)}{\lambda_1})-|\gamma(A-B)+(\frac{p+\lambda_2(k-p)}{\lambda_1})B|r)\right]\rho + r(1+|B|r) \tag{2.11}$$

takes its maximum value at $\rho = 1$ with $r_1 = r_1(p, \gamma, \lambda_1, \lambda_2, A, B)$ for $r_1(p, \gamma, \lambda_1, \lambda_2, A, B)$ is the smallest positive root of equation (2.2). Furthermore, if $0 \le \rho \le r_1(p, \gamma, \lambda_1, \lambda_2, A, B)$, then function $\psi(\rho)$ defined by

$$\psi(\rho) = -\sigma(1 + |B|\sigma)\rho^2 + (1 - \sigma^2)$$
$$\left[(\frac{p + \lambda_2(k-1)}{\lambda_1}) - |\gamma(A - B) + (\frac{p + \lambda_2(k-p)}{\lambda_1})B|\sigma) \right] \rho + \sigma(1 + |B|\sigma) \tag{2.12}$$

is seen to be an increasing function on the interval $0 \le \rho \le 1$ so that

$$\psi(\rho) \le \psi(1) = (1 - \sigma^2)(\frac{p + \lambda_2(k-p)}{\lambda_1}) - |\gamma(A - B) + (\frac{p + \lambda_2(k-p)}{\lambda_1})B|\sigma) \tag{2.13}$$
$$0 \le \rho \le 1; (0 \le \sigma \le r_1(p, \gamma, \lambda_1, \lambda_2, A, B)).$$

Hence upon setting $\rho = 1$ in (2.13) we conclude that (2.1) of Theorem 2.1 holds true for $|z| \le r_1(p, \gamma, \lambda_1, \lambda_2, A, B)$ where $r_1(p, \gamma, \lambda_1, \lambda_2, A, B)$ is the smallest positive root of equation (2.2). This completes the proof of the Theorem 2.1.

Setting $p = 1$, $m = 0$, $A = 1, B = -1$ and $j = 0$ in Theorem 2.1 we get

Corollary 2.1. *Let the function* $f \in A$ *be analytic in the open unit disk* U*and suppose that* $g \in S^{0,1,0}_{0,0,0}[1, -1, \gamma] = S(\gamma)$. *If* $f(z)$ *is majorized by* $g(z)$ *in* U, *then*

$$|f'(z)| \le |g'(z)| \quad (|z| < r_3)$$

where

$$r_3 = r_3(\gamma) = \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|}.$$

This is a known result obtained by Altintas[5].

For $\gamma = 1$, the above corollary reduces to the following result:

Corollary 2.2. *Let the function* $f(z) \in A$ *be analytic univalent in the open unit disk* U *and suppose that* $g \in S^* = S^*(0)$. *If* f *is majorized by* g *in* U, *then*

$$|f'(z)| \le |g'(z)| \quad (|z| \le 2 - \sqrt{3})$$

which is a known result obtained by MacGregor [7].

Some other work related to the class defined by (1.3) can be seen in [9] and of course elsewhere. In fact, recently Ibrahim [10] used the concept of majorization to find solutions of fractional differential equations in the unit disk.

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