CUBO A Mathematical Journal
Vol.13, Nº03, (17–48). October 2011

Applications and Lipschitz results of Approximation by Smooth Picard and Gauss-Weierstrass Type Singular Integrals

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ABSTRACT

We continue our studies in higher order uniform convergence with rates and in L_p convergence with rates. Namely, in this article we establish some Lipschitz type results for the smooth Picard type singular integral operators and for the smooth Gauss-Weierstrass type singular integral operators.

RESUMEN

Continuamos nuestros estudios sobre convergencia uniforme de orden superior con radios y sobre convergencia L_p con radios. Concretamente, en este artículo establecemos algunos resultados de tipo Lipschitz para operadores integrales suves del tipo Picard singulares y para operadores integrales singulares de tipo Gauss-Weierstrass.

Keywords: Smooth Picard Type singular integral, Smooth Gauss-Weierstrass Type singular integral, modulus of smoothness, rate of convergence, Lp convergence, Higher Order Uniform Convergence with Rates, sharp inequality, Lipschitz functions.

Mathematics Subject Classification: 26A15, 26D15, 41A17, 41A35, 41A60, 41A80.

1. Introduction

We are motivated by [1], [2], [3] and [4].

We denote by L_p, $1 \leq p < \infty$, the classes of functions $f(x)$, integrable in $-\infty < x < \infty$ with the norm

$$\|f\|_p = \left[\int_{-\infty}^{\infty} |f(u)|^p \, du\right]^{\frac{1}{p}}. \tag{1.1}$$

The *Picard singular integral* $P_\xi(f;x)$ corresponding to the function $f(x)$, is defined as follows

$$P_\xi(f;x) = \frac{1}{2\xi}\int_{-\infty}^{\infty} f(x+y)e^{-|y|/\xi}dy, \quad \text{for all } x \in \mathbb{R},\ \xi > 0. \tag{1.2}$$

The *Gauss Weierstrass singular integral* $W_\xi(f;x)$ corresponding to the function $f(x)$, is defined as follows

$$W_\xi(f;x) = \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty} f(x+y)e^{-y^2/\xi}dy, \quad \text{for all } x \in \mathbb{R},\ \xi > 0. \tag{1.3}$$

2. Convergence with Rates of Smooth Picard Singular Integral Operators

In the next we deal with the following *smooth Picard singular integral operators* $P_{r,\xi}(f;x)$ defined as follows.

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set

$$\alpha_j = \begin{cases} (-1)^{r-j}\binom{r}{j} j^{-n}, & j = 1,\dots,r, \\ 1 - \sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j} j^{-n}, & j = 0, \end{cases} \tag{2.1}$$

that is $\sum_{j=0}^{r} \alpha_j = 1$. Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable, we define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$P_{r,\xi}(f;x) := \frac{1}{2\xi}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{r}\alpha_j f(x+jt)\right)e^{-|t|/\xi}dt. \tag{2.2}$$

We assume that $P_{r,\xi}(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$.

We mention the useful here formula

$$\int_0^{\infty} t^k e^{-t/\xi}dt = \Gamma(k+1)\,\xi^{k+1}, k > -1. \tag{2.3}$$

We need to introduce

$$\delta_k := \sum_{j=1}^{r} \alpha_j j^k, \quad k = 1, \ldots, n \in \mathbb{N}. \tag{2.4}$$

Denote by $\lfloor \cdot \rfloor$ the integral part.

We give a special related result.

Proposition 1. Let f be defined as above in this section. *It holds that*

$$|P_{2,\xi}(f;x) - f(x)| \leq \frac{1}{\xi}\int_0^{\infty}\left(\int_0^{|t|}\omega_2(f', w)dw\right)e^{-t/\xi}dt. \tag{2.5}$$

Proof. In Theorem 1 of [1] we use $n = 1$, $r = 2$. □

We also present the Lipschitz type result corresponding to the Theorem 1 of [1].

Theorem 2. Let f be defined as above in this section, with $n \in \mathbb{N}$. *Furthermore we assume the following Lipschitz condition:* $\omega_r\left(f^{(n)}, \delta\right) \leq K\delta^{r-1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then it holds that*

$$\left|P_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} f^{(2m)}(x)\delta_{2m}\xi^{2m}\right| \leq K\Gamma(\gamma + r)\,\xi^{n+r+\gamma-1}. \tag{2.6}$$

In L.H.S.(2.6) *the sum collapses when* $n = 1$.

Proof. As in the proof of Theorem 1, of [1], we get again that

$$P_{r,\xi}(f;x) - f(x) = \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k \frac{1}{2\xi}\left(\int_{-\infty}^{\infty} t^k e^{-|t|/\xi}dt\right) + \mathcal{R}_n^*, \tag{2.7}$$

where

$$\mathcal{R}_n^* := \frac{1}{2\xi}\int_{-\infty}^{\infty} \mathcal{R}_n(0,t)e^{-|t|/\xi}dt, \tag{2.8}$$

with

$$\mathcal{R}_n(0,t) := \int_0^t \frac{(t-w)^{n-1}}{(n-1)!}\tau(w)dw, \tag{2.9}$$

and

$$\tau(w) := \sum_{j=0}^{r} \alpha_j j^n f^{(n)}(x + jw) - \delta_n f^{(n)}(x).$$

Also we get

$$|\mathcal{R}_n(0,t)| \leq \int_0^{|t|} \frac{(|t| - w)^{n-1}}{(n-1)!}\omega_r(f^{(n)}, w)dw. \tag{2.10}$$

Using the Lipschitz type condition we obtain

$$\begin{aligned}
|\mathcal{R}_n(0,t)| &\le \int_0^{|t|} \frac{(|t|-w)^{n-1}}{(n-1)!} K w^{r-1+\gamma} dw \\
&= \frac{K|t|^{n+r+\gamma-2}}{(n-1)!} \int_0^{|t|} \left(1-\frac{w}{|t|}\right)^{n-1} \left(\frac{w}{|t|}\right)^{r-1+\gamma} dw \\
&= \frac{K|t|^{n+r+\gamma-1}}{(n-1)!} \int_0^1 (1-y)^{n-1} y^{r-1+\gamma} dy \\
&= \frac{K|t|^{n+r+\gamma-1}\Gamma(\gamma+r)}{\Gamma(n+\gamma+r)}. \qquad (2.11)
\end{aligned}$$

Then, by (2.3), we obtain

$$\begin{aligned}
|\mathcal{R}_n^*| &\le \frac{1}{2\xi}\int_{-\infty}^{\infty} \frac{K|t|^{n+r+\gamma-1}\Gamma(\gamma+r)}{\Gamma(n+\gamma+r)} e^{-|t|/\xi} dt \\
&= \frac{K}{2\xi}\frac{\Gamma(\gamma+r)}{\Gamma(n+\gamma+r)} \int_{-\infty}^{\infty} |t|^{n+r+\gamma-1} e^{-|t|/\xi} dt \\
&= \frac{K}{\xi}\frac{\Gamma(\gamma+r)}{\Gamma(n+\gamma+r)} \int_0^{\infty} t^{n+r+\gamma-1} e^{-t/\xi} dt \\
&\overset{(2.3)}{=} K\Gamma(\gamma+r)\,\xi^{n+r+\gamma-1}. \qquad (2.12)
\end{aligned}$$

We also notice that

$$\begin{aligned}
P_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k \frac{1}{2\xi}\left(\int_{-\infty}^{\infty} t^k e^{-|t|/\xi} dt\right) &= \\
P_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} f^{(2m)}(x)\delta_{2m}\xi^{2m} &= \mathcal{R}_n^*. \qquad (2.13)
\end{aligned}$$

By (2.12) and (2.13) we complete the proof of the theorem. □

Corollary 3. Let f be defined as above in this section. Furthermore we assume the following Lipschitz condition $\omega_2(f',\delta) \le K\delta^{1+\gamma}$, $K>0$, $0<\gamma\le 1$, for any $\delta>0$. Then

$$|P_{2,\xi}(f;x) - f(x)| \le K\Gamma(\gamma+2)\,\xi^{2+\gamma}. \qquad (2.14)$$

Proof. In Theorem 2 we use $n=1$, $r=2$. □

For the case $n=0$ we have

Theorem 4. Let f be defined as above in this section, with $n=0$. *Furthermore we assume the following Lipschitz condition:* $\omega_r(f,\delta) \le K\delta^{r-1+\gamma}$, $K>0$, $0<\gamma\le 1$, for any $\delta>0$. *It holds that*

$$|P_{r,\xi}(f;x) - f(x)| \le K\Gamma(r+\gamma)\,\xi^{r+\gamma-1}. \qquad (2.15)$$

Proof. As in the proof of Corollary 1, of [1], with $n=0$, using the Lipschitz type condition, we get that

$$\begin{aligned} |P_{r,\xi}(f;x)-f(x)| &\leq \frac{1}{\xi}\int_0^\infty \omega_r(f,t)e^{-t/\xi}dt \\ &\leq \frac{1}{\xi}\int_0^\infty Kt^{r-1+\gamma}e^{-t/\xi}dt \\ &\overset{(2.3)}{=} K\Gamma(r+\gamma)\,\xi^{r+\gamma-1} \end{aligned} \quad (2.16)$$

This completes the proof of Theorem 4. □

Corollary 5. Let f be defined as above in this section, with $n=0$. *Furthermore we assume the following Lipschitz condition:* $\omega_2(f,\delta)\leq K\delta^{1+\gamma}$, $K>0$, $0<\gamma\leq 1$, for any $\delta>0$. *Then*

$$|P_{2,\xi}(f;x)-f(x)|\leq K\Gamma(2+\gamma)\,\xi^{\gamma+1}. \quad (2.17)$$

Proof. In Theorem 4 we use $r=2$. □

In the next we consider $f\in C^n(\mathbb{R})$, $n\geq 2$ even and the simple *smooth singular operator of symmetric convolution type*

$$P_\xi(f,x_0):=\frac{1}{2\xi}\int_{-\infty}^{\infty} f(x_0+y)e^{-|y|/\xi}dy,\quad \text{for all } x_0\in\mathbb{R},\ \xi>0. \quad (2.18)$$

That is

$$P_\xi(f;x_0)=\frac{1}{2\xi}\int_0^\infty \big(f(x_0+y)+f(x_0-y)\big)e^{-y/\xi}dy,\quad \text{for all } x_0\in\mathbb{R},\ \xi>0. \quad (2.19)$$

We assume that f is such that

$$P_\xi(f;x_0)\in\mathbb{R},\quad \forall x_0\in\mathbb{R},\forall\xi>0 \ \text{ and }\ \omega_2(f^{(n)},h)<\infty,\ h>0.$$

Note that $P_{1,\xi}=P_\xi$ and if $P_\xi(f;x_0)\in\mathbb{R}$ then $P_{r,\xi}(f;x_0)\in\mathbb{R}$.

Proposition 6. *Assume* $\omega_2(f,h)<\infty$*,* $h>0$. *Furthermore we assume the following Lipschitz condition:* $\omega_2(f,\delta)\leq K\delta^{1+\gamma}$, $K>0$, $0<\gamma\leq 1$, for any $\delta>0$. *Then*

$$\|P_\xi(f)-f\|_\infty\leq \frac{K\Gamma(2+\gamma)}{2}\xi^{\gamma+1}. \quad (2.20)$$

Proof. Using Proposition 1 of [1] we obtain

$$\begin{aligned} |P_\xi(f;x_0)-f(x_0)| &\leq \frac{1}{2\xi}\int_0^\infty \omega_2(f,y)e^{-y/\xi}dy \\ &\leq \frac{1}{2\xi}\int_0^\infty Ky^{1+\gamma}e^{-y/\xi}dy \\ &\overset{(2.3)}{=} \frac{K\Gamma(2+\gamma)}{2}\xi^{\gamma+1}, \end{aligned} \quad (2.21)$$

proving the claim of the proposition. ☐

Let

$$K_2(x_0) := P_\xi(f; x_0) - f(x_0) - \sum_{\rho=1}^{n/2} f^{(2\rho)}(x_0)\xi^{2\rho}. \quad (2.22)$$

We give

Theorem 7. *Let* $f \in C^n(\mathbb{R})$, n *even,* $P_\xi(f)$ *real valued. Furthermore we assume the following Lipschitz condition:* $\omega_2\left(f^{(n)}, \delta\right) \le K\delta^{1+\gamma}$, $K > 0$, $0 < \gamma \le 1$, for any $\delta > 0$. *Then*

$$|K_2(x_0)| \le \frac{K\Gamma(n+\gamma+2)}{2n!}\xi^{n+\gamma+1}. \quad (2.23)$$

Proof. Using Theorem 6 of [1] we obtain

$$\begin{aligned} |K_2(x_0)| &\le \frac{1}{2\xi n!}\int_0^\infty \omega_2(f^{(n)}, y) y^n e^{-y/\xi} dy \\ &\le \frac{1}{2\xi n!}\int_0^\infty K y^{1+\gamma} y^n e^{-y/\xi} dy \\ &\overset{(2.3)}{=} \frac{K\Gamma(n+\gamma+2)}{2n!}\xi^{n+\gamma+1}, \end{aligned} \quad (2.24)$$

proving the claim of the theorem. ☐

In particular we have

Corollary 8. *Let* $f \in C^4(\mathbb{R})$ *such that* $P_\xi(f)$ *is real valued. Furthermore we assume the following Lipschitz condition:* $\omega_2\left(f^{(4)}, \delta\right) \le K\delta^{1+\gamma}$, $K > 0$, $0 < \gamma \le 1$, for any $\delta > 0$. *Then*

$$|K_2(x_0)| \le \frac{K\Gamma(\gamma+6)}{48}\xi^{\gamma+5}. \quad (2.25)$$

Proof. In Theorem 7 we use $n = 4$. ☐

We also give

Corollary 9. *Let* $f \in C^2(\mathbb{R})$, *such that*

$$\omega_2(f'', |y|) \le 2A|y|^\gamma, \quad 0 < \gamma \le 2, \quad A > 0.$$

Then for $x_0 \in \mathbb{R}$ *we have*

$$\left|P_\xi(f; x_0) - f(x_0) - f''(x_0)\xi^2\right| \le \Gamma(\alpha+1)A\xi^{\gamma+2}. \quad (2.26)$$

Inequality (2.16) is sharp, namely it is attained at $x_0 = 0$ *by*

$$f_*(y) = \frac{A|y|^{\gamma+2}}{(\gamma+1)(\gamma+2)}.$$

Proof. In Theorem 7 of [1] we use $n = 2$. □

We also give

Corollary 10. Assume that $\omega_2(f, \xi) < \infty$ and $n = 0$. Then

$$\|P_{2,\xi}(f) - f\|_\infty \le 5\omega_2(f, \xi), \tag{2.27}$$

and as $\xi \to 0$,

$$P_{2,\xi} \overset{u}{\to} I \text{ with rates.}$$

Proof. By formula (37) of [1] with $r = 2$. □

Next let

$$K_1 := \left\| P_{r,\xi}(f; x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} \left[f^{(2m)}(x)\delta_{2m}\xi^{2m} \right] \right\|_{\infty,x}. \tag{2.28}$$

We present

Corollary 11. *Assuming* $f \in C^2(\mathbb{R})$ *and* $\omega_2(f'', \xi) < \infty$, $\xi > 0$ we have

$$\begin{aligned} K_1 &= \left\| P_{2,\xi}(f; x) - f(x) - f''(x)\delta_2\xi^2 \right\|_{\infty,x} \\ &\le \frac{21}{4}\xi^2\omega_2(f'', \xi). \end{aligned} \tag{2.29}$$

That is as $\xi \to 0$ *we get* $P_{2,\xi} \to I$, *pointwise with rates, given that* $\|f''\|_\infty < \infty$.

Proof. In Theorem 11 of [1] we use $r = n = 2$. □

We also present

Corollary 12. *Assuming* $f \in C^2(\mathbb{R})$ *and* $\omega_2(f'', \xi) < \infty$, $\xi > 0$ we have

$$\begin{aligned} \|K_2(x)\|_{\infty,x} &= \left\| P_\xi(f; x_0) - f(x_0) - f''(x_0)\xi^2 \right\|_{\infty,x} \\ &\le \frac{21}{8}\xi^2\omega_2(f'', \xi). \end{aligned} \tag{2.30}$$

That is as $\xi \to 0$ *we get* $P_\xi \to I$, *pointwise with rates, given that* $\|f''\|_\infty < \infty$.

Proof. In Theorem 12 of [1] we use $n = 2$. □

3. L_p Convergence with Rates of Smooth Picard Singular Integral Operators

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we let α_j as in (2.1).

Let $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_p(\mathbb{R})$, $1 \le p < \infty$, we define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral $P_{r,\xi}(f;x)$ as in (2.2).

We need the rth L_p-modulus of smoothness

$$\omega_r(f^{(n)}, h)_p := \sup_{|t| \le h} \|\Delta_t^r f^{(n)}(x)\|_{p,x}, \quad h > 0, \tag{3.1}$$

where

$$\Delta_t^r f^{(n)}(x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f^{(n)}(x + jt), \tag{3.2}$$

Here we have that $\omega_r(f^{(n)}, h)_p < \infty$, $h > 0$.

We need to introduce δ_k's as in (2.4).

We define

$$\Delta(x) := P_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} f^{(2m)}(x)\delta_{2m}\xi^{2m}. \tag{3.3}$$

We have the following results.

Corollary 13. *Let $n \in \mathbb{N}$ and the rest as above in this section. Then*

$$\|\Delta(x)\|_2 \le \frac{\sqrt{2\tau}\xi^n}{\sqrt{(2r+1)(4n-2)}(n-1)!}\omega_r(f^{(n)}, \xi)_2, \tag{3.4}$$

where

$$0 < \tau := \left[\int_0^\infty (1+u)^{2r+1} u^{2n-1} e^{-u} du - (2n-1)!\right] < \infty. \tag{3.5}$$

Hence as $\xi \to 0$ we obtain $\|\Delta(x)\|_2 \to 0$.

If additionally $f^{(2m)} \in L_2(\mathbb{R}), m = 1, 2, \ldots, \lfloor \frac{n}{2} \rfloor$ then $\|P_{r,\xi}(f) - f\|_2 \to 0$, as $\xi \to 0$.

Proof. In Theorem 1 of [2], we place $p = q = 2$. □

Corollary 14. *Let* f be as above in this section. In particular, for $n = 1$, *we have*

$$\|P_{r,\xi}(f;\cdot) - f\|_2 \le \frac{\sqrt{\tau}\xi}{\sqrt{(2r+1)}}\omega_r(f', \xi)_2, \tag{3.6}$$

where

$$0 < \tau := \left[\int_0^{\infty} (1+u)^{2r+1} u e^{-u} du - 1\right] < \infty. \tag{3.7}$$

Hence as $\xi \to 0$ *we obtain* $\|P_{r,\xi}(f;\cdot) - f\|_2 \to 0$.

Proof. In Theorem 1 of [2], we place $p = q = 2$, $n = 1$. □

Corollary 15. *Let* f be as above in this section and $n = 2$. *Then*

$$\|P_{r,\xi}(f;x) - f(x) - f''(x)\delta_2\xi^2\|_2 \leq \frac{\sqrt{2\tau}\xi^2}{\sqrt{6(2r+1)}}\omega_r(f'',\xi)_2, \tag{3.8}$$

where

$$0 < \tau := \left[\int_0^{\infty} (1+u)^{2r+1} u^3 e^{-u} du - 6\right] < \infty. \tag{3.9}$$

Hence as $\xi \to 0$ *we obtain* $\|\Delta(x)\|_2 \to 0$.

If additionally $f'' \in L_2(\mathbb{R})$, then $\|P_{r,\xi}(f) - f\|_2 \to 0$, as $\xi \to 0$.

Proof. In Theorem 1 of [2], we place $p = q = n = 2$. □

Next we present the Lipschitz type result corresponding to Theorem 1 of [2].

Theorem 16. *Let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}$, *and the rest as above in this section. Furthermore we assume the following Lipschitz condition:* $\omega_r\left(f^{(n)},\delta\right)_p \leq K\delta^{r-1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$\|\Delta(x)\|_p \leq \frac{\left(\Gamma\left(p\left(r-1+\gamma+n\right)+1\right)\right)^{\frac{1}{p}} 2^{(r+\gamma+n)}K}{\left[(n-1)!q^{\frac{1}{q}}p^{r-\frac{1}{q}+\gamma+n}(q(n-1)+1)^{\frac{1}{q}}\left(p\left(r-1+\gamma\right)+1\right)^{\frac{1}{p}}\right]}\xi^{(r-1+\gamma+n)}. \tag{3.10}$$

Hence as $\xi \to 0$ *we obtain* $\|\Delta(x)\|_p \to 0$.

If additionally $f^{(2m)} \in L_p(\mathbb{R}), m = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$ then $\|P_{r,\xi}(f) - f\|_p \to 0$, as $\xi \to 0$.

Proof. As in the proof of Theorem 1, [2], we get again

$$\begin{aligned} I \quad : \quad &= \int_{-\infty}^{\infty} |\Delta(x)|^p dx \\ &\leq c_1 \left(\int_{-\infty}^{\infty} \left(\left(\int_0^{|t|} \omega_r(f^{(n)},w)_p^p dw\right) |t|^{np-1} e^{-|pt|/2\xi}\right) dt\right), \end{aligned} \tag{3.11}$$

where

$$c_1 := \frac{2^{p-2}}{\xi q^{p-1}((n-1)!)^p(q(n-1)+1)^{p/q}}. \tag{3.12}$$

Using the Lipschitz condition, we obtain

$$\begin{aligned}
I &\leq c_1 \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} \left(K w^{r-1+\gamma}\right)^p dw \right) |t|^{np-1} e^{-p|t|/2\xi} dt \right) \\
&= \frac{c_1 K^p}{(p(r-1+\gamma)+1)} \left(\int_{-\infty}^{\infty} |t|^{p(r-1+\gamma+n)} e^{-p|t|/2\xi} dt \right) \\
&= \frac{2c_1 K^p}{(p(r-1+\gamma)+1)} \left(\int_0^{\infty} t^{p(r-1+\gamma+n)} e^{-pt/2\xi} dt \right) \\
&= \frac{2c_1 K^p}{(p(r-1+\gamma)+1)} \left(\frac{2}{p}\right)^{p(r-1+\gamma+n)+1} \left(\int_0^{\infty} z^{p(r-1+\gamma+n)} e^{-z/\xi} dz \right) \\
&\overset{(2.3)}{=} \frac{2c_1 K^p \Gamma(p(r-1+\gamma+n)+1)}{(p(r-1+\gamma)+1)} \left(\frac{2}{p}\right)^{p(r-1+\gamma+n)+1} \xi^{p(r-1+\gamma+n)+1}. \qquad (3.13)
\end{aligned}$$

Thus we obtain

$$I \leq \frac{\Gamma(p(r-1+\gamma+n)+1)}{q^{p-1}((n-1)!)^p (q(n-1)+1)^{p/q} p^{p(r-1+\gamma+n)+1}} \frac{2^{p(r+\gamma+n)} K^p}{(p(r-1+\gamma)+1)} \xi^{p(r-1+\gamma+n)}. \qquad (3.14)$$

That is finishing the proof of the theorem. □

In particular we have

Corollary 17. *Let* f such that *the following Lipschitz condition holds:* $\omega_7\left(f^{(4)}, \delta\right)_2 \leq K\delta^{6+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$, *and the rest as above in this section. Then*

$$\|\Delta(x)\|_2 \leq \frac{K}{6} \sqrt{\frac{(\Gamma(2\gamma+21))}{7(2\gamma+13)}} \xi^{(\gamma+10)}. \qquad (3.15)$$

Hence as $\xi \to 0$ *we obtain* $\|\Delta(x)\|_2 \to 0$.

If additionally $f^{(2m)} \in L_2(\mathbb{R}), m = 1, 2$, then $\|P_{7,\xi}(f) - f\|_2 \to 0$, as $\xi \to 0$.

Proof. In Theorem 16 we place $p = q = 2$, $n = 4$, and $r = 7$. □

The counterpart of Theorem 16 follows, case of $p = 1$.

Theorem 18. *Let* $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_1(\mathbb{R}), n \in \mathbb{N}$. *Furthermore we assume the following Lipschitz condition:* $\omega_r\left(f^{(n)}, \delta\right)_1 \leq K\delta^{r-1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$\|\Delta(x)\|_1 \leq \frac{K}{(n-1)!\,(r+\gamma)} \Gamma(r+\gamma+n)\, \xi^{r+\gamma+n-1}. \qquad (3.16)$$

Hence as $\xi \to 0$ *we obtain* $\|\Delta(x)\|_1 \to 0$.

If additionally $f^{(2m)} \in L_1(\mathbb{R}), m = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$ then $\|P_{r,\xi}(f) - f\|_1 \to 0$, as $\xi \to 0$.

Proof. As in the proof of Theorem 2 of [2] we get

$$\|\Delta(x)\|_1 \leq \frac{1}{2\xi(n-1)!} \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} \omega_r(f^{(n)}, w)_1 dw \right) |t|^{n-1} e^{-|t|/\xi} dt \right). \tag{3.17}$$

Consequently we have

$$\begin{aligned}
\|\Delta(x)\|_1 &\leq \frac{1}{2\xi(n-1)!} \left(\int_{-\infty}^{\infty} \left(\int_0^{|t|} K w^{r-1+\gamma} dw \right) |t|^{n-1} e^{-|t|/\xi} dt \right) && (3.18)\\
&= \frac{K}{2\xi(n-1)!} \left(\int_{-\infty}^{\infty} \left(\frac{|t|^{r+\gamma}}{r+\gamma} \right) |t|^{n-1} e^{-|t|/\xi} dt \right) \\
&= \frac{K}{2\xi(n-1)!\,(r+\gamma)} \left(\int_{-\infty}^{\infty} |t|^{r+\gamma+n-1} e^{-|t|/\xi} dt \right) \\
&= \frac{K}{\xi(n-1)!\,(r+\gamma)} \left(\int_0^{\infty} t^{r+\gamma+n-1} e^{-t/\xi} dt \right) \\
&\overset{(2.3)}{=} \frac{K}{(n-1)!\,(r+\gamma)} \Gamma(r+\gamma+n)\, \xi^{r+\gamma+n-1}, && (3.19)
\end{aligned}$$

proving (3.16). □

Corollary 19. *Let* $f \in C^2(\mathbb{R})$ and $f'' \in L_1(\mathbb{R})$. *Furthermore we assume the following Lipschitz condition:* $\omega_2(f'', \delta)_1 \leq K\delta^{1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$\|\Delta(x)\|_1 \leq \frac{K}{(2+\gamma)} \Gamma(4+\gamma)\, \xi^{\gamma+3}. \tag{3.20}$$

Hence as $\xi \to 0$ *we obtain* $\|\Delta(x)\|_1 \to 0$.

If additionally $f'' \in L_1(\mathbb{R})$,then $\|P_{2,\xi}(f) - f\|_1 \to 0$, as $\xi \to 0$.

Proof. In Theorem 18 we place $n = r = 2$. □

Next, when $n = 0$ we get

Proposition 20. *Let* $r \in \mathbb{N}$ *and the rest as above. Then*

$$\|P_{r,\xi}(f) - f\|_2 \leq \theta^{1/2} \omega_r(f, \xi)_2, \tag{3.21}$$

where

$$0 < \theta := \int_0^{\infty} (1+x)^{2r} e^{-x} dx < \infty. \tag{3.22}$$

Hence as $\xi \to 0$ we obtain $P_{r,\xi} \to$ unit operator I in the L_2 norm.

Proof. In the proof of Proposition 1 of [2] we use $p = q = 2$. □

We continue with

Proposition 21. *Let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and the rest as above. Furthermore we assume the following Lipschitz condition:* $\omega_r(f, \delta)_p \leq K\delta^{r-1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$\|P_{r,\xi}(f) - f\|_p \leq \sqrt[p]{\Gamma(p(r-1+\gamma)+1)} \frac{K}{q^{1/q}} \frac{2^{(r+\gamma)}\xi^{(r+\gamma-1)}}{p^{\left(r-1+\gamma+\frac{1}{p}\right)}}. \tag{3.23}$$

Hence as $\xi \to 0$ we obtain $P_{r,\xi} \to$ unit operator I in the L_p norm, $p > 1$.

Proof. As in the proof of Proposition 1 of [2] we find

$$\int_{-\infty}^{\infty} |P_{r,\xi}(f;x) - f(x)|^p dx$$

$$\begin{aligned}
&\leq \frac{1}{2^{p-1}\xi^p}\left(\frac{4\xi}{q}\right)^{p/q}\left(\int_0^\infty \omega_r(f,t)_p^p e^{-pt/(2\xi)} dt\right) \\
&\leq \frac{1}{2^{p-1}\xi^p}\left(\frac{4\xi}{q}\right)^{p/q}\left(\int_0^\infty \left(Kt^{r-1+\gamma}\right)^p e^{-pt/(2\xi)} dt\right) \\
&\overset{(2.3)}{=} \frac{K^p}{q^{p-1}} \frac{\Gamma(p(r-1+\gamma)+1)\, 2^{p(r+\gamma)}\xi^{(r-1+\gamma)p}}{p^{(p(r+\gamma-1)+1)}}.
\end{aligned} \tag{3.24}$$

We have established the claim of the proposition. □

Corollary 22. *Let f such that the following Lipschitz condition holds:* $\omega_4(f, \delta)_2 \leq K\delta^{3+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$, *and the rest as above in this section. Then*

$$\|P_{4,\xi}(f) - f\|_2 \leq \sqrt{\Gamma(2\gamma+7)} K\xi^{(3+\gamma)}. \tag{3.25}$$

Hence as $\xi \to 0$ we obtain $P_{4,\xi} \to$ unit operator I in the L_2 norm.

Proof. In Proposition 21 we place $p = q = 2$ and $r = 4$. □

In general, for the L_1 case, $n = 0$ we have

Proposition 23. It holds

$$\|P_{2,\xi} f - f\|_1 \leq 5\omega_2(f, \xi)_1. \tag{3.26}$$

Hence as $\xi \to 0$ *we get* $P_{2,\xi} \to I$ *in the* L_1 *norm.*

Proof. In the proof of Proposition 2 of [2] we use $r = 2$. □

Proposition 24. *We assume the following Lipschitz condition:* $\omega_r(f, \delta)_1 \leq K\delta^{r-1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. Then

$$\|P_{r,\xi} f - f\|_1 \leq K\Gamma(r+\gamma)\, \xi^{r-1+\gamma}. \tag{3.27}$$

Hence as $\xi \to 0$ *we get* $P_{r,\xi} \to I$ *in the* L_1 *norm.*

Proof. As in the proof of Proposition 2 of [2] we get

$$\begin{aligned}\int_{-\infty}^{\infty} |P_{r,\xi}(f;x) - f(x)|\, dx &\leq \frac{1}{\xi}\int_0^{\infty} \omega_r(f,t)_1 e^{-t/\xi} dt \\ &\leq \frac{K}{\xi}\int_0^{\infty} t^{r-1+\gamma} e^{-t/\xi} dt \\ &= K\Gamma\,(r+\gamma)\,\xi^{r-1+\gamma}, \end{aligned} \tag{3.28}$$

proving the claim. □

Corollary 25. *Assume the following Lipschitz condition:* $\omega_2\,(f,\delta)_1 \leq K\delta^{1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. Then

$$\|P_{2,\xi} f - f\|_1 \leq K\Gamma\,(2+\gamma)\,\xi^{1+\gamma}. \tag{3.29}$$

Hence as $\xi \to 0$ *we get* $P_{2,\xi} \to I$ *in the* L_1 *norm.*

Proof. In Proposition 24 we place $r = 2$. □

In the next we consider $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_p(\mathbb{R}), n = 0$ or $n \geq 2$ even, $1 \leq p < \infty$ and the similar *smooth singular operator of symmetric convolution type*

$$P_\xi(f;x) = \frac{1}{2\xi}\int_{-\infty}^{\infty} f(x+y)e^{-|y|/\xi} dy, \quad \text{for all } x \in \mathbb{R},\ \xi > 0. \tag{3.30}$$

Denote

$$K(x) := P_\xi(f;x) - f(x) - \sum_{\rho=1}^{n/2} f^{(2\rho)}(x)\xi^{2\rho}. \tag{3.31}$$

We give

Theorem 26. *Let* $n \geq 2$ *even and the rest as above. Then*

$$\|K(x)\|_2 \leq \left(\sqrt{\frac{\tilde{\tau}}{20(2n-1)}}\right)\frac{\xi^n}{(n-1)!}\omega_2(f^{(n)},\xi)_2, \tag{3.32}$$

where

$$0 < \tilde{\tau} = \left(\int_0^{\infty} (1+x)^5 x^{2n-1} e^{-x} dx - (2n-1)!\right) < \infty. \tag{3.33}$$

Hence as $\xi \to 0$ *we get* $\|K(x)\|_2 \to 0$.

If additionally $f^{(2m)} \in L_2(\mathbb{R}), m = 1, 2, \ldots, \frac{n}{2}$ then $\|P_\xi(f) - f\|_2 \to 0$, as $\xi \to 0$.

Proof. In the proof of Theorem 3 of [2] we use $p = q = 2$. □

It follows a Lipschitz type approximation result.

Theorem 27. *Let* $p, q > 1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$, $n \geq 2$ *even and the rest as above. Furthermore we assume the following Lipschitz condition:* $\omega_2\left(f^{(n)}, \delta\right)_p \leq K\delta^{\gamma+1}$, $K>0$, $0<\gamma\leq 1$, for any $\delta > 0$. *Then*

$$\|K(x)\|_p \leq \left(\frac{2}{p}\right)^{(\gamma+n+1)} \frac{K\left[\Gamma\left(p\left(\gamma+n+1\right)+1\right)\right]^{1/p}}{(n-1)!q^{1/q}p^{1/p}(q(n-1)+1)^{1/q}\left[p\left(\gamma+1\right)+1\right]^{1/p}}\xi^{\gamma+n+1}. \tag{3.34}$$

Hence as $\xi \to 0$ *we get* $\|K(x)\|_p \to 0$.

If additionally $f^{(2m)} \in L_p(\mathbb{R}), m = 1, 2, \ldots, \frac{n}{2}$ then $\|P_\xi(f) - f\|_p \to 0$, as $\xi \to 0$.

Proof. As in the proof of Theorem 3, of [2] we find

$$\begin{aligned}
\int_{-\infty}^{\infty} |K(x)|^p dx &\leq c_2\left(\int_0^\infty\left(\int_0^y \omega_2(f^{(n)}, t)_p^p dt\right) y^{pn-1}e^{-py/(2\xi)}dy\right)\\
&\leq K^p c_2\left(\int_0^\infty\left(\frac{y^{p(\gamma+1)+1}}{p\left(\gamma+1\right)+1}\right)y^{pn-1}e^{-py/(2\xi)}dy\right)\\
&= \frac{K^p c_2}{p\left(\gamma+1\right)+1}\left(\frac{2}{p}\right)^{p(\gamma+n+1)+1}\left(\int_0^\infty z^{p(\gamma+n+1)}e^{-z/\xi}dz\right)
\end{aligned}$$

$$\overset{(2.3)}{=} \frac{K^p c_2 \Gamma\left(p\left(\gamma+n+1\right)+1\right)}{p\left(\gamma+1\right)+1}\left(\frac{2}{p}\right)^{p(\gamma+n+1)+1}\xi^{p(\gamma+n+1)+1}. \tag{3.35}$$

where here we denoted

$$c_2 := \frac{1}{2\xi q^{p/q}((n-1)!)^p(q(n-1)+1)^{p/q}}. \tag{3.36}$$

We have established the claim of the theorem. □

Corollary 28. *Assume the following Lipschitz condition:* $\omega_2\left(f'', \delta\right)_2 \leq K\delta^{\gamma+1}$, $K>0$, $0<\gamma\leq 1$, for any $\delta>0$, and the rest as above in this section. *Then*

$$\|K(x)\|_2 \leq \sqrt{\frac{\Gamma\left(2\gamma+7\right)}{6\gamma+9}}\frac{K}{2}\xi^{\gamma+3}. \tag{3.37}$$

Hence as $\xi \to 0$ *we get* $\|K(x)\|_2 \to 0$.

If additionally $f'' \in L_2(\mathbb{R})$, then $\|P_\xi(f) - f\|_2 \to 0$, as $\xi \to 0$.

Proof. In Theorem 27 we place $p = q = n = 2$. □

Theorem 29. *Let* $f \in C^2(\mathbb{R})$ and $f'' \in L_1(\mathbb{R})$. *Here* $K(x) = P_\xi(f;x) - f(x) - f''(x)\xi^2$. *Then*

$$\|K(x)\|_1 \leq 8\omega_2(f'', \xi)_1\xi^2. \tag{3.38}$$

Hence as $\xi \to 0$ *we obtain* $\|K(x)\|_1 \to 0$.

Also $\|P_\xi(f) - f\|_1 \to 0$, as $\xi \to 0$.

Proof. In the proof of Theorem 4 of [2] we use $n = 2$. □

The Lipschitz case of $p = 1$ follows.

Theorem 30. *Let* $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_1(\mathbb{R})$, $n \geq 2$ *even. Furthermore we assume the following Lipschitz condition:* $\omega_2\left(f^{(n)}, \delta\right)_1 \leq K\delta^{\gamma+1}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$\|K(x)\|_1 \leq \frac{\Gamma(\gamma+n+2)\,K}{2(n-1)!\,(\gamma+2)}\xi^{\gamma+n+1}. \tag{3.39}$$

Hence as $\xi \to 0$ *we obtain* $\|K(x)\|_1 \to 0$.

If additionally $f^{(2m)} \in L_1(\mathbb{R}), m = 1, 2, \ldots, \frac{n}{2}$ then $\|P_\xi(f) - f\|_1 \to 0$, as $\xi \to 0$.

Proof. As in the proof of Theorem 4 of [2] we have

$$\begin{aligned} \|K(x)\|_1 &\leq \frac{1}{2\xi}\left(\int_0^\infty \left(\int_0^y \omega_2(f^{(n)}, t)_1\, dt\right)\frac{y^{n-1}}{(n-1)!}e^{-y/\xi}dy\right) \\ &\leq \frac{1}{2\xi}\left(\int_0^\infty \left(\int_0^y Kt^{\gamma+1}dt\right)\frac{y^{n-1}}{(n-1)!}e^{-y/\xi}dy\right) \\ &= \frac{K}{2\xi(n-1)!\,(\gamma+2)}\left(\int_0^\infty y^{\gamma+n+1}e^{-y/\xi}dy\right) \\ &\overset{(2.3)}{=} \frac{\Gamma(\gamma+n+2)\,K}{2(n-1)!\,(\gamma+2)}\xi^{\gamma+n+1}. \end{aligned} \tag{3.40}$$

We have proved the claim of the theorem. □

Corollary 31. *Let* $f \in C^6(\mathbb{R})$ and $f^{(6)} \in L_1(\mathbb{R})$. *Furthermore we assume the following Lipschitz condition:* $\omega_2\left(f^{(6)}, \delta\right)_1 \leq K\delta^{\gamma+1}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$\|K(x)\|_1 \leq \frac{\Gamma(\gamma+8)\,K}{240\,(\gamma+2)}\xi^{\gamma+7}. \tag{3.41}$$

Hence as $\xi \to 0$ *we obtain* $\|K(x)\|_1 \to 0$.

If additionally $f^{(2m)} \in L_1(\mathbb{R}), m = 1, 2, 3$ then $\|P_\xi(f) - f\|_1 \to 0$, as $\xi \to 0$.

Proof. In Theorem 30 we place $n = 6$. □

The case of $n = 0$ follows.

Proposition 32. Let f *as above in this section. Then*

$$\|P_\xi(f) - f\|_2 \leq \frac{\sqrt{65}}{2}\omega_2(f, \xi)_2. \tag{3.42}$$

Hence as $\xi \to 0$ *we obtain* $P_\xi \to I$ *in the* L_2 *norm.*

Proof. In the proof of Proposition 3 of [2] we use $p = q = 2$. $\square$

The related Lipschitz case for $n = 0$ comes next.

Proposition 33. *Let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and the rest as above. Furthermore we assume the following Lipschitz condition:* $\omega_2(f, \delta)_p \le K\delta^{1+\gamma}$, $K > 0$, $0 < \gamma \le 1$, for any $\delta > 0$. *Then*

$$\|P_\xi(f) - f\|_p \le \left(\frac{2}{p}\right)^{1+\gamma} \frac{[\Gamma((1+\gamma)p+1)]^{1/p} K}{q^{1/q}p^{1/p}} \xi^{1+\gamma}. \tag{3.43}$$

Hence as $\xi \to 0$ *we obtain* $P_\xi \to I$ *in the* L_p *norm,* $p > 1$.

Proof. As in the proof of Proposition 3 of [2] we get

$$\begin{aligned} \int_{-\infty}^{\infty} |P_\xi(f;x) - f(x)|^p dx &\le \frac{1}{2\xi q^{p/q}} \left(\int_0^\infty \omega_2(f,y)_p^p e^{-py/(2\xi)} dy\right) \\ &\le \frac{K^p}{2\xi q^{p/q}} \left(\int_0^\infty y^{(1+\gamma)p} e^{-py/(2\xi)} dy\right) \\ &\overset{(2.3)}{=} \frac{K^p}{q^{p/q}p} \left(\frac{2}{p}\right)^{(1+\gamma)p} \Gamma((1+\gamma)p+1)\, \xi^{(1+\gamma)p}. \end{aligned} \tag{3.44}$$

The proof of the claim is now completed. $\square$

A particular example follows

Corollary 34. *Let* f *as above in this section. Furthermore we assume the following Lipschitz condition:* $\omega_2(f, \delta)_2 \le K\delta^{1+\gamma}$, $K > 0$, $0 < \gamma \le 1$, for any $\delta > 0$. *Then*

$$\|P_\xi(f) - f\|_2 \le \frac{K}{2}\sqrt{\Gamma(3+2\gamma)}\xi^{1+\gamma}. \tag{3.45}$$

Hence as $\xi \to 0$ *we obtain* $P_\xi \to I$ *in the* L_2 *norm.*

Proof. In Proposition 33 we place $p = q = 2$. $\square$

It follows the Lipschitz type result

Proposition 35. *Assume the following Lipschitz condition:* $\omega_2(f, \delta)_1 \le K\delta^{\gamma+1}$, $K > 0$, $0 < \gamma \le 1$, for any $\delta > 0$. *It holds,*

$$\|P_\xi f - f\|_1 \le \frac{K}{2}\Gamma(\gamma+2)\,\xi^{\gamma+1}. \tag{3.46}$$

Hence as $\xi \to 0$ *we get* $P_\xi \to I$ *in the* L_1 *norm.*

Proof. As in the proof of Proposition 4 of [2] we derive

$$\begin{aligned} \int_{-\infty}^{\infty} |P_\xi(f;x) - f(x)| dx &\le \frac{1}{2\xi}\int_0^\infty \omega_2(f,y)_1 e^{-y/\xi} dy \\ &\le \frac{1}{2\xi}\int_0^\infty K y^{\gamma+1} e^{-y/\xi} dy \\ &\overset{(2.3)}{=} \frac{K}{2}\Gamma(\gamma+2)\,\xi^{\gamma+1}, \end{aligned} \tag{3.47}$$

proving the claim. □

4. Convergence with Rates of Smooth Gauss Weierstrass Singular Integral Operators

In the next we deal with the following *smooth Gauss Weierstrass singular integral operators* $W_{r,\xi}(f;x)$ defined as follows.

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we set α_j's as in (2.1).

Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable, we define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral

$$W_{r,\xi}(f;x) := \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \left(\sum_{j=0}^{r} \alpha_j f(x+jt) \right) e^{-t^2/\xi} dt. \tag{4.1}$$

We assume that $W_{r,\xi}(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$.

We mention the useful here formula

$$\int_0^{\infty} t^k e^{-t^2/\xi} dt = \frac{1}{2}\Gamma\left(\frac{k+1}{2}\right) \xi^{\frac{k+1}{2}}, \text{ for any } k > -1. \tag{4.2}$$

We also need to introduce δ_k's as in (2.4).

Proposition 36. Let $f \in C^1(\mathbb{R})$ be defined as above in this section, and assume that $W_{2,\xi}(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. *Then*

$$|W_{2,\xi}(f;x) - f(x)| \leq \frac{2}{\sqrt{\pi\xi}} \int_0^{\infty} \left(\int_0^{|t|} \omega_2(f', w) dw \right) e^{-\frac{t^2}{\xi}} dt. \tag{4.3}$$

Proof. In Theorem 1 of [3] we use $n = 1$, $r = 2$. □

We present the Lipschitz type result corresponding to the Theorem 1 of [3].

Theorem 37. Let $f \in C^n(\mathbb{R})$, $n \in \mathbb{Z}^+$ and assume that $W_{r,\xi}(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. *Furthermore we assume the following Lipschitz condition:* $\omega_r\left(f^{(n)}, \delta\right) \leq K\delta^{r-1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then it holds that*

$$\left| W_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} f^{(2m)}(x) \delta_{2m} \frac{1}{m!} \left(\frac{\xi}{4}\right)^m \right|$$
$$\leq \frac{K}{\sqrt{\pi}} \frac{\Gamma(\gamma+r)}{\Gamma(n+\gamma+r)} \Gamma\left(\frac{n+r+\gamma}{2}\right) \xi^{\frac{n+r+\gamma-1}{2}}. \tag{4.4}$$

In L.H.S.(4.4) *the sum collapses when* $n = 1$.

Proof. As in the proof of Theorem 1, of [3], we get again that

$$W_{r,\xi}(f;x) - f(x) = \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k \frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty} t^k e^{-\frac{t^2}{\xi}}dt\right) + \mathcal{R}_n^*, \tag{4.5}$$

where

$$\mathcal{R}_n^* := \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty} \mathcal{R}_n(0,t)e^{-\frac{t^2}{\xi}}dt, \tag{4.6}$$

with

$$\mathcal{R}_n(0,t) := \int_0^t \frac{(t-w)^{n-1}}{(n-1)!}\tau(w)dw, \tag{4.7}$$

and

$$\tau(w) := \sum_{j=0}^{r} \alpha_j j^n f^{(n)}(x+jw) - \delta_n f^{(n)}(x).$$

Also we get

$$|\mathcal{R}_n(0,t)| \leq \int_0^{|t|} \frac{(|t|-w)^{n-1}}{(n-1)!}\omega_r(f^{(n)},w)dw. \tag{4.8}$$

Using the Lipschitz type condition we obtain again

$$|\mathcal{R}_n(0,t)| \leq \frac{K|t|^{n+r+\gamma-1}\Gamma(\gamma+r)}{\Gamma(n+\gamma+r)}, \tag{4.9}$$

and, using (4.2), we obtain

$$\begin{aligned} |\mathcal{R}_n^*| &\leq \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty} \frac{K|t|^{n+r+\gamma-1}\Gamma(\gamma+r)}{\Gamma(n+\gamma+r)}e^{-\frac{t^2}{\xi}}dt \\ &= \frac{K}{\sqrt{\pi\xi}}\frac{\Gamma(\gamma+r)}{\Gamma(n+\gamma+r)}\int_{-\infty}^{\infty} |t|^{n+r+\gamma-1}e^{-\frac{t^2}{\xi}}dt \\ &= \frac{2K}{\sqrt{\pi\xi}}\frac{\Gamma(\gamma+r)}{\Gamma(n+\gamma+r)}\int_0^{\infty} t^{n+r+\gamma-1}e^{-\frac{t^2}{\xi}}dt \\ &\overset{(4.2)}{=} \frac{K}{\sqrt{\pi}}\frac{\Gamma(\gamma+r)}{\Gamma(n+\gamma+r)}\Gamma\left(\frac{n+r+\gamma}{2}\right)\xi^{\frac{n+r+\gamma-1}{2}}. \end{aligned} \tag{4.10}$$

We notice also that

$$\begin{aligned} W_{r,\xi}(f;x) - f(x) - \sum_{k=1}^{n} \frac{f^{(k)}(x)}{k!}\delta_k \frac{1}{\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty} t^k e^{-\frac{t^2}{\xi}}dt\right) &= \\ W_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor}\left[\frac{f^{(2m)}(x)}{(2m)!\sqrt{\pi}}\delta_{2m}\Gamma\left(\frac{2m+1}{2}\right)\xi^m\right] &= \mathcal{R}_n^*. \end{aligned} \tag{4.11}$$

Furthermore we have that

$$\frac{1}{(2m)!\sqrt{\pi}}\Gamma\left(\frac{2m+1}{2}\right) =$$

$$\begin{aligned}
&= \frac{1}{(2m)\cdot(2m-1)\cdot\ldots\cdot 3\cdot 2\cdot 1}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{2m-1}{2}\cdot\frac{2m-3}{2}\cdot\ldots\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\\
&= \frac{1}{m!}\left(\frac{1}{4}\right)^m. \qquad (4.12)
\end{aligned}$$

By (4.10), (4.11) and (4.12) we complete the proof of the theorem. □

Corollary 38. Let $f \in C^1(\mathbb{R})$, and assume that $W_{2,\xi}(f;x) \in \mathbb{R}$ for all $x \in \mathbb{R}$. *Furthermore we assume the following Lipschitz condition:* $\omega_2(f',\delta) \le K\delta^{1+\gamma}$, $K>0$, $0<\gamma\le 1$, for any $\delta>0$. *Then*

$$|W_{2,\xi}(f;x) - f(x)| \le \frac{K}{(\gamma+2)\sqrt{\pi}}\Gamma\left(\frac{3+\gamma}{2}\right)\xi^{\frac{2+\gamma}{2}}. \qquad (4.13)$$

Proof. In Theorem 37 we use $n=1$, $r=2$. □

For the case $n=0$ we have

Theorem 39. Let f be defined as above in this section, with $n=0$. *Furthermore we assume the following Lipschitz condition:* $\omega_r(f,\delta) \le K\delta^{r-1+\gamma}$, $K>0$, $0<\gamma\le 1$, for any $\delta>0$. *It holds that*

$$|W_{r,\xi}(f;x) - f(x)| \le \frac{K}{\sqrt{\pi}}\Gamma\left(\frac{r+\gamma}{2}\right)\xi^{\frac{r+\gamma-1}{2}}. \qquad (4.14)$$

Proof. As in the proof of Corollary 1, of [3], with $n=0$, using the Lipschitz type condition, we get that

$$\begin{aligned}
|W_{r,\xi}(f;x) - f(x)| &\le \frac{2}{\sqrt{\pi\xi}}\int_0^\infty \omega_r(f,t)e^{-\frac{t^2}{\xi}}\,dt\\
&\le \frac{2}{\sqrt{\pi\xi}}\int_0^\infty Kt^{r-1+\gamma}e^{-\frac{t^2}{\xi}}\,dt\\
&\overset{(4.2)}{=} \frac{K}{\sqrt{\pi}}\Gamma\left(\frac{r+\gamma}{2}\right)\xi^{\frac{r+\gamma-1}{2}}. \qquad (4.15)
\end{aligned}$$

This completes the proof of Theorem 39. □

Corollary 40. Let f be defined as above in this section, with $n=0$. *Furthermore we assume the following Lipschitz condition:* $\omega_2(f,\delta) \le K\delta^{1+\gamma}$, $K>0$, $0<\gamma\le 1$, for any $\delta>0$. *Then*

$$|W_{2,\xi}(f;x) - f(x)| \le \frac{K}{\sqrt{\pi}}\Gamma\left(\frac{2+\gamma}{2}\right)\xi^{\frac{\gamma+1}{2}}. \qquad (4.16)$$

Proof. In Theorem 39 we use $r=2$. □

In the next we consider $f \in C^n(\mathbb{R})$, $n \geq 2$ even and the simple *smooth singular operator of symmetric convolution type*

$$W_\xi(f, x_0) := \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} f(x_0 + y) e^{-y^2/\xi} dy, \quad \text{for all } x_0 \in \mathbb{R},\ \xi > 0. \tag{4.17}$$

That is

$$W_\xi(f; x_0) = \frac{1}{\sqrt{\pi\xi}} \int_0^{\infty} (f(x_0 + y) + f(x_0 - y))\, e^{-y^2/\xi} dy, \quad \text{for all } x_0 \in \mathbb{R},\ \xi > 0. \tag{4.18}$$

We assume that f is such that

$$W_\xi(f; x_0) \in \mathbb{R}, \quad \forall x_0 \in \mathbb{R}, \forall \xi > 0 \ \text{ and } \ \omega_2(f^{(n)}, h) < \infty,\ h > 0.$$

Note that $W_{1,\xi} = W_\xi$ and if $W_\xi(f; x_0) \in \mathbb{R}$ then $W_{r,\xi}(f; x_0) \in \mathbb{R}$.

Proposition 41. *Assume* $f \in C^n(\mathbb{R}), \omega_2(f, h) < \infty$, $h > 0$. *Furthermore we assume the following Lipschitz condition:* $\omega_2(f, \delta) \leq K\delta^{1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$\|W_\xi(f) - f\|_\infty \leq \frac{K}{2\sqrt{\pi}} \Gamma\left(\frac{2+\gamma}{2}\right) \xi^{\frac{\gamma+1}{2}}. \tag{4.19}$$

Proof. Using Proposition 1 of [3] we obtain

$$\begin{aligned} |W_\xi(f; x_0) - f(x_0)| &\leq \frac{1}{\sqrt{\pi\xi}} \int_0^{\infty} \omega_2(f, y) e^{-y^2/\xi} dy \\ &\leq \frac{1}{\sqrt{\pi\xi}} \int_0^{\infty} K y^{1+\gamma} e^{-y^2/\xi} dy \\ &\overset{(4.2)}{=} \frac{K}{2\sqrt{\pi}} \Gamma\left(\frac{2+\gamma}{2}\right) \xi^{\frac{\gamma+1}{2}}, \end{aligned} \tag{4.20}$$

proving the claim of the proposition. □

Define the quantity

$$\overline{K}_2(x_0) := W_\xi(f; x_0) - f(x_0) - \sum_{\rho=1}^{n/2} f^{(2\rho)}(x_0) \frac{1}{\rho!} \left(\frac{\xi}{4}\right)^{\rho}. \tag{4.21}$$

We give

Theorem 42. *Let* $f \in C^n(\mathbb{R})$, n *even,* $W_\xi(f)$ *real valued. Furthermore we assume the following Lipschitz condition:* $\omega_2\left(f^{(n)}, \delta\right) \leq K\delta^{1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$|\overline{K}_2(x_0)| \leq \frac{K}{n! 2\sqrt{\pi}} \Gamma\left(\frac{n+\gamma+2}{2}\right) \xi^{\frac{n+\gamma+1}{2}}. \tag{4.22}$$

Proof. Using Theorem 6 of [3] we obtain

$$
\begin{aligned}
|\overline{K}_2(x_0)| &\leq \frac{1}{n!\sqrt{\pi\xi}}\int_0^\infty \omega_2(f^{(n)},y)y^n e^{-y^2/\xi}dy \\
&\leq \frac{1}{n!\sqrt{\pi\xi}}\int_0^\infty Ky^{1+\gamma}y^n e^{-y^2/\xi}dy \\
&\overset{(4.2)}{=} \frac{K}{n!2\sqrt{\pi}}\Gamma\left(\frac{n+\gamma+2}{2}\right)\xi^{\frac{n+\gamma+1}{2}},
\end{aligned}
\tag{4.23}
$$

proving the claim of the theorem. □

In particular we have

Corollary 43. *Let* $f \in C^4(\mathbb{R})$ *such that* $W_\xi(f)$ *is real valued. Furthermore we assume the following Lipschitz condition:* $\omega_2\left(f^{(4)},\delta\right) \leq K\delta^{1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$
|\overline{K}_2(x_0)| \leq \frac{K}{48\sqrt{\pi}}\Gamma\left(\frac{\gamma+6}{2}\right)\xi^{\frac{\gamma+5}{2}}. \tag{4.24}
$$

Proof. In Theorem 42 we use $n = 4$. □

We also give

Corollary 44. *Let* $f \in C^2(\mathbb{R})$, *such that*

$$
\omega_2(f'',|y|) \leq 2A|y|^\gamma, \quad 0 < \gamma \leq 2, \quad A > 0.
$$

Then for $x_0 \in \mathbb{R}$ *we have*

$$
\left|W_\xi(f;x_0) - f(x_0) - \frac{f''(x_0)\xi}{4}\right| \leq \frac{A}{(\gamma+1)(\gamma+2)\sqrt{\pi}}\Gamma\left(\frac{3+\gamma}{2}\right)\xi^{\frac{2+\gamma}{2}}. \tag{4.25}
$$

Inequality (4.25) is sharp, namely it is attained at $x_0 = 0$ *by*

$$
f_*(y) = \frac{A|y|^{\gamma+2}}{(\gamma+1)(\gamma+2)}.
$$

Proof. In Theorem 7 of [3] we use $n = 2$. □

We also give

Corollary 45. Assume that $\omega_2(f,\xi) < \infty$ and $n = 0$. Then

$$
\|W_{2,\xi}(f) - f\|_\infty \leq \left[\frac{2}{\sqrt{\pi}} + \frac{3}{2}\right]\omega_2(f,\sqrt{\xi}). \tag{4.26}
$$

and as $\xi \to 0$,

$$W_{2,\xi} \overset{u}{\to} I \text{ with rates.}$$

Proof. By formula (37) of [3] with $r = 2$. □

Define the quantity

$$\overline{K}_1 := \left\| W_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} f^{(2m)}(x)\delta_{2m}\frac{1}{m!}\left(\frac{\xi}{4}\right)^m \right\|_{\infty,x}. \tag{4.27}$$

We present

Corollary 46. *Assuming* $f \in C^2(\mathbb{R})$ *and* $\omega_2(f'', \xi) < \infty$, $\xi > 0$ we have

$$\begin{aligned} \overline{K}_1 &= \left\| W_{2,\xi}(f;x) - f(x) - f''(x)\delta_2\frac{\xi}{4} \right\|_{\infty,x} \\ &\leq \left\{ \frac{1}{3\sqrt{\pi}} + \frac{5}{16} \right\} \omega_2(f'', \sqrt{\xi})\xi. \end{aligned} \tag{4.28}$$

That is as $\xi \to 0$ *we get* $W_{2,\xi} \to I$, *pointwise with rates, given that* $\|f''\|_\infty < \infty$.

Proof. In Theorem 11 of [3] we use $r = n = 2$. □

We also present

Corollary 47. *Assuming* $f \in C^2(\mathbb{R})$ *and* $\omega_2(f'', \xi) < \infty$, $\xi > 0$ we have

$$\begin{aligned} \left\| \overline{K}_2(x) \right\|_{\infty,x} &= \left\| W_\xi(f;x_0) - f(x_0) - f''(x_0)\frac{\xi}{4} \right\|_{\infty,x} \\ &\leq \left\{ \frac{1}{6\sqrt{\pi}} + \frac{5}{32} \right\} \omega_2(f'', \sqrt{\xi})\xi. \end{aligned} \tag{4.29}$$

That is as $\xi \to 0$ *we get* $W_\xi \to I$, *pointwise with rates, given that* $\|f''\|_\infty < \infty$.

Proof. In Theorem 12 of [3] we use $n = 2$. □

5. L_p Convergence with Rates of Smooth Gauss Weierstrass Singular Integral Operators

For $r \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ we let α_j as in (2.1).

Let $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_p(\mathbb{R})$, $1 \le p < \infty$, we define for $x \in \mathbb{R}$, $\xi > 0$ the Lebesgue integral $W_{r,\xi}(f;x)$ as in (4.1).

The rth L_p-modulus of smoothness $\omega_r(f^{(n)}, h)_p$ was defined in (3.1). Here we have that $\omega_r(f^{(n)}, h)_p < \infty$, $h > 0$.

The δ_k's were introduced in (2.4).

We define

$$\Delta(x) := W_{r,\xi}(f;x) - f(x) - \sum_{m=1}^{\lfloor n/2 \rfloor} f^{(2m)}(x)\delta_{2m}\frac{1}{m!}\left(\frac{\xi}{4}\right)^m. \tag{5.1}$$

We have the following results.

Corollary 48. *Let* $n \in \mathbb{N}$ *and the rest as above in this section. Then*

$$\|\Delta(x)\|_2 \le \frac{\sqrt{2\tau}\xi^{\frac{n}{2}}}{(n-1)!\sqrt[4]{\pi}\sqrt{(2r+1)(2n-1)}}\omega_r(f^{(n)}, \sqrt{\xi})_2, \tag{5.2}$$

where

$$0 < \tau := \left[\int_0^\infty (1+u)^{2r+1}\ u^{2n-1}e^{-u^2}du - \int_0^\infty u^{2n-1}e^{-u^2}du\right] < \infty. \tag{5.3}$$

Hence as $\xi \to 0$ *we obtain* $\|\Delta(x)\|_2 \to 0$.

If additionally $f^{(2m)} \in L_2(\mathbb{R}), m = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$ then $\|W_{r,\xi}(f) - f\|_2 \to 0$, as $\xi \to 0$.

Proof. In Theorem 1 of [4], we place $p = q = 2$. □

Corollary 49. *Let* f be as above in this section. In particular, for $n = 1$, *we have*

$$\|W_{r,\xi}(f;\cdot) - f\|_2 \le \frac{\sqrt{2\tau}}{\sqrt[4]{\pi}\sqrt{(2r+1)}}\sqrt{\xi}\omega_r(f', \sqrt{\xi})_2, \tag{5.4}$$

where

$$0 < \tau := \left[\int_0^\infty (1+u)^{2r+1}\ ue^{-u^2}du - \frac{1}{2}\right] < \infty. \tag{5.5}$$

Hence as $\xi \to 0$ *we obtain* $\|W_{r,\xi}(f;\cdot) - f\|_2 \to 0$.

Proof. In Theorem 1 of [4], we place $p = q = 2$, $n = 1$. □

Corollary 50. *Let* f be as above in this section and $n = 2$. *Then*

$$\|W_{r,\xi}(f;x) - f(x) - \frac{f''(x)\delta_2}{4}\xi\|_2 \le \frac{\sqrt{2\tau}}{\sqrt[4]{\pi}\sqrt{3(2r+1)}}\xi\omega_r(f'', \sqrt{\xi})_2, \tag{5.6}$$

where

$$0<\tau:=\left[\int_0^\infty (1+u)^{2r+1}\ u^3 e^{-u^2}du-\frac{1}{2}\right]<\infty. \tag{5.7}$$

Hence as $\xi\to 0$ *we obtain* $\|\Delta(x)\|_2\to 0$.

If additionally $f''\in L_2(\mathbb{R})$, then $\|W_{r,\xi}(f)-f\|_2\to 0$, as $\xi\to 0$.

Proof. In Theorem 1 of [4], we place $p=q=n=2$. □

Next we present the Lipschitz type result corresponding to Theorem 1 of [4].

Theorem 51. *Let* $p,q>1$ *such that* $\frac{1}{p}+\frac{1}{q}=1$, $n\in\mathbb{N}$, *and the rest as above in this section. Furthermore we assume the following Lipschitz condition:* $\omega_r\left(f^{(n)},\delta\right)_p\le K\delta^{r-1+\gamma}$, $K>0$, $0<\gamma\le 1$, for any $\delta>0$.*Then*

$$\|\Delta(x)\|_p\le\frac{\left(\Gamma\left(\frac{p(r-1+\gamma+n)+1}{2}\right)\right)^{\frac{1}{p}}2^{\frac{(r+\gamma+n)}{2}}K\xi^{\frac{(r-1+\gamma+n)}{2}}}{\left[(n-1)!p^{\frac{r-\frac{1}{q}+\gamma+n}{2}}q^{\frac{1}{2q}}\pi^{\frac{1}{2p}}(q(n-1)+1)^{\frac{1}{q}}\left(p\left(r-1+\gamma\right)+1\right)^{\frac{1}{p}}\right]}. \tag{5.8}$$

Hence as $\xi\to 0$ *we obtain* $\|\Delta(x)\|_p\to 0$.

If additionally $f^{(2m)}\in L_p(\mathbb{R}), m=1,2,\dots,\left\lfloor\frac{n}{2}\right\rfloor$ then $\|W_{r,\xi}(f)-f\|_p\to 0$, as $\xi\to 0$.

Proof. As in the proof of Theorem 1, [4], we get again

$$I:=\int_{-\infty}^{\infty}|\Delta(x)|^p dx\le c_1\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\omega_r(f^{(n)},w)_p^p dw\right)|t|^{np-1}e^{-\frac{pt^2}{2\xi}}dt\right), \tag{5.9}$$

where

$$c_1:=\frac{2^{\frac{p-1}{2}}}{q^{\frac{p-1}{2}}\sqrt{\pi\xi}((n-1)!)^p(q(n-1)+1)^{p/q}}. \tag{5.10}$$

Using the Lipschitz condition, we obtain

$$\begin{aligned}
I &\le c_1\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\left(Kw^{r-1+\gamma}\right)^p dw\right)|t|^{np-1}e^{-\frac{pt^2}{2\xi}}dt\right)\\
&= \frac{c_1K^p}{(p\left(r-1+\gamma\right)+1)}\left(\int_{-\infty}^{\infty}|t|^{p(r-1+\gamma+n)}e^{-\frac{pt^2}{2\xi}}dt\right)\\
&= \frac{2c_1K^p}{(p\left(r-1+\gamma\right)+1)}\left(\int_0^{\infty}t^{p(r-1+\gamma+n)}e^{-\frac{pt^2}{2\xi}}dt\right)\\
&\overset{(4.2)}{=} \frac{c_1K^p\Gamma\left(\frac{p(r-1+\gamma+n)+1}{2}\right)}{(p\left(r-1+\gamma\right)+1)}\left(\frac{2}{p}\right)^{\frac{p(r-1+\gamma+n)+1}{2}}\xi^{\frac{p(r-1+\gamma+n)+1}{2}}.
\end{aligned} \tag{5.11}$$

Thus we obtain

$$I \leq \frac{K^p 2^{\frac{p(r+\gamma+n)}{2}} \Gamma\left(\frac{p(r-1+\gamma+n)+1}{2}\right) \xi^{\frac{p(r-1+\gamma+n)}{2}}}{q^{\frac{p-1}{2}} \sqrt{\pi}((n-1)!)^p (q(n-1)+1)^{p/q} (p(r-1+\gamma)+1) p^{\frac{p(r-1+\gamma+n)+1}{2}}}. \tag{5.12}$$

That is finishing the proof of the theorem. □

In particular we have

Corollary 52. *Let* f such that *the following Lipschitz condition holds:* $\omega_7\left(f^{(4)}, \delta\right)_2 \leq K\delta^{6+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$, *and the rest as above in this section. Then*

$$\|\Delta(x)\|_2 \leq \frac{K}{6}\sqrt{\frac{\Gamma\left(\frac{2\gamma+21}{2}\right)}{7\sqrt{\pi}(2\gamma+13)}}\,\xi^{\frac{(\gamma+10)}{2}}. \tag{5.13}$$

Hence as $\xi \to 0$ *we obtain* $\|\Delta(x)\|_2 \to 0$.

If additionally $f^{(2m)} \in L_2(\mathbb{R}), m = 1, 2$, then $\|W_{7,\xi}(f) - f\|_2 \to 0$, as $\xi \to 0$.

Proof. In Theorem 51 we place $p = q = 2$, $n = 4$, and $r = 7$. □

The counterpart of Theorem 51 follows, case of $p = 1$.

Theorem 53. *Let* $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_1(\mathbb{R}), n \in \mathbb{N}$. *Furthermore we assume the following Lipschitz condition:* $\omega_r\left(f^{(n)}, \delta\right)_1 \leq K\delta^{r-1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$.*Then*

$$\|\Delta(x)\|_1 \leq \frac{K}{(n-1)!(r+\gamma)\sqrt{\pi}}\Gamma\left(\frac{r+\gamma+n}{2}\right)\xi^{\frac{r+\gamma+n-1}{2}}. \tag{5.14}$$

Hence as $\xi \to 0$ *we obtain* $\|\Delta(x)\|_1 \to 0$.

If additionally $f^{(2m)} \in L_1(\mathbb{R}), m = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$ then $\|W_{r,\xi}(f) - f\|_1 \to 0$, as $\xi \to 0$.

Proof. As in the proof of Theorem 2, [4] we get

$$\|\Delta(x)\|_1 \leq \frac{1}{(n-1)!\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|}\omega_r(f^{(n)}, w)_1 dw\right)|t|^{n-1}e^{-t^2/\xi}dt\right). \tag{5.15}$$

Consequently we have

$$\begin{aligned}
\|\Delta(x)\|_1 &\leq \frac{1}{(n-1)!\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\left(\int_0^{|t|} Kw^{r-1+\gamma}dw\right)|t|^{n-1}e^{-t^2/\xi}dt\right)\\
&= \frac{K}{(n-1)!\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}\left(\frac{|t|^{r+\gamma}}{r+\gamma}\right)|t|^{n-1}e^{-t^2/\xi}dt\right)\\
&= \frac{K}{(n-1)!\,(r+\gamma)\sqrt{\pi\xi}}\left(\int_{-\infty}^{\infty}|t|^{r+\gamma+n-1}e^{-t^2/\xi}dt\right)\\
&= \frac{2K}{(n-1)!\,(r+\gamma)\sqrt{\pi\xi}}\left(\int_0^{\infty}t^{r+\gamma+n-1}e^{-t^2/\xi}dt\right)\\
&\overset{(4.2)}{=} \frac{K}{(n-1)!\,(r+\gamma)\sqrt{\pi\xi}}\Gamma\left(\frac{r+\gamma+n}{2}\right)\xi^{\frac{r+\gamma+n}{2}}. \qquad (5.16)
\end{aligned}$$

We have gotten that

$$\|\Delta(x)\|_1 \leq \frac{K}{(n-1)!\,(r+\gamma)\sqrt{\pi}}\Gamma\left(\frac{r+\gamma+n}{2}\right)\xi^{\frac{r+\gamma+n-1}{2}}. \qquad (5.17)$$

Hence the validity of (5.14). □

Corollary 54. *Let* $f \in C^2(\mathbb{R})$ and $f'' \in L_1(\mathbb{R})$. *Furthermore we assume the following Lipschitz condition:* $\omega_2(f'',\delta)_1 \leq K\delta^{1+\gamma}$, $K>0$, $0<\gamma\leq 1$, for any $\delta>0$. *Then*

$$\|\Delta(x)\|_1 \leq \frac{K}{(2+\gamma)\sqrt{\pi}}\Gamma\left(\frac{4+\gamma}{2}\right)\xi^{\frac{\gamma+3}{2}}. \qquad (5.18)$$

Hence as $\xi \to 0$ *we obtain* $\|\Delta(x)\|_1 \to 0$.

Also we get $\|W_{2,\xi}(f)-f\|_1 \to 0$, as $\xi \to 0$.

Proof. In Theorem 53 we place $n=r=2$. □

Next, when $n=0$ we get

Proposition 55. *Let* $r \in \mathbb{N}$ *and the rest as above. Then*

$$\|W_{r,\xi}(f)-f\|_2 \leq \frac{2^{\frac{3}{4}}\theta^{\frac{1}{2}}}{q^{\frac{1}{4}}\pi^{\frac{1}{4}}}\omega_r(f,\sqrt{\xi})_2, \qquad (5.19)$$

where

$$0<\theta := \int_0^{\infty}(1+t)^{2r}e^{-t^2}dt < \infty. \qquad (5.20)$$

Hence as $\xi \to 0$ we obtain $W_{r,\xi} \to$ unit operator I in the L_2 norm, $p>1$.

Proof. In the proof of Proposition 1 of [4] we use $p=q=2$. □

We continue with

Proposition 56. *Let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and the rest as above. Furthermore we assume the following Lipschitz condition:* $\omega_r(f, \delta)_p \leq K\delta^{r-1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$\|W_{r,\xi}(f) - f\|_p \leq \sqrt[p]{\Gamma\left(\frac{p(r-1+\gamma)+1}{2}\right)} \left(\frac{2}{p}\right)^{\frac{r+\gamma}{2}} \left(\frac{p}{q}\right)^{\frac{1}{2q}} \frac{K}{\sqrt[p]{\pi}} \xi^{\frac{(r-1+\gamma)}{2}}. \tag{5.21}$$

Hence as $\xi \to 0$ we obtain $W_{r,\xi} \to$ unit operator I in the L_p norm, $p > 1$.

Proof. As in the proof of Proposition 1 of [4] we find

$$\int_{-\infty}^{\infty} |W_{r,\xi}(f; x) - f(x)|^p dx$$

$$\begin{aligned}
&\leq \frac{2}{(\pi\xi)^{\frac{p}{2}}} \left(\frac{2\pi\xi}{q}\right)^{\frac{p}{2q}} \int_0^{\infty} \omega_r(f,t)_p^p e^{-\frac{pt^2}{2\xi}} dt \\
&\leq \frac{2K^p}{(\pi\xi)^{\frac{p}{2}}} \left(\frac{2\pi\xi}{q}\right)^{\frac{p}{2q}} \int_0^{\infty} t^{p(r-1+\gamma)} e^{-\frac{pt^2}{2\xi}} dt \\
&\overset{(4.2)}{=} \frac{K^p}{\pi^{\frac{p}{2}}} \left(\frac{2\pi}{q}\right)^{\frac{p}{2q}} \left(\frac{2}{p}\right)^{\frac{p(r-1+\gamma)+1}{2}} \Gamma\left(\frac{p(r-1+\gamma)+1}{2}\right) \xi^{\frac{p(r-1+\gamma)}{2}}.
\end{aligned} \tag{5.22}$$

We have established the claim of the proposition. □

Corollary 57. *Let f such that the following Lipschitz condition holds:* $\omega_4(f, \delta)_2 \leq K\delta^{3+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$, *and the rest as above in this section. Then*

$$\|W_{4,\xi}(f) - f\|_2 \leq \sqrt{\Gamma\left(\frac{2\gamma+7}{2}\right)} \frac{K}{\sqrt{\pi}} \xi^{\frac{(3+\gamma)}{2}}. \tag{5.23}$$

Hence as $\xi \to 0$ we obtain $W_{4,\xi} \to$ unit operator I in the L_2 norm.

Proof. In Proposition 56 we place $p = q = 2$ and $r = 4$. □

In the L_1 case, $n = 0$ we have

Proposition 58. It holds

$$\|W_{2,\xi}f - f\|_1 \leq \left(\frac{2}{\sqrt{\pi}} + \frac{3}{2}\right) \omega_2(f, \sqrt{\xi})_1. \tag{5.24}$$

Hence as $\xi \to 0$ *we get* $W_{2,\xi} \to I$ *in the* L_1 *norm.*

Proof. In the proof of Proposition 2 of [4] we use $r = 2$. □

Proposition 59. *We assume the following Lipschitz condition:* $\omega_r(f, \delta)_1 \leq K\delta^{r-1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. Then

$$\|W_{r,\xi}f - f\|_1 \leq \frac{K}{\sqrt{\pi}} \Gamma\left(\frac{r+\gamma}{2}\right) \xi^{\frac{r-1+\gamma}{2}}. \tag{5.25}$$

Hence as $\xi \to 0$ we get $W_{r,\xi} \to I$ in the L_1 norm.

Proof. As in the proof of Proposition 2 of [4] we get

$$\begin{aligned}\int_{-\infty}^{\infty} |W_{r,\xi}(f;x) - f(x)|\, dx &\leq \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} \omega_r(f,|t|)_1 e^{-t^2/\xi} dt \\ &\leq \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} K|t|^{r-1+\gamma} e^{-t^2/\xi} dt \\ &= \frac{2K}{\sqrt{\pi\xi}} \int_{0}^{\infty} t^{r-1+\gamma} e^{-t^2/\xi} dt \\ &\overset{(4.2)}{=} \frac{K}{\sqrt{\pi}} \Gamma\left(\frac{r+\gamma}{2}\right) \xi^{\frac{r-1+\gamma}{2}}. \end{aligned} \tag{5.26}$$

We have proved the claim of the proposition. □

Corollary 60. *Assume the following Lipschitz condition:* $\omega_2(f,\delta)_1 \leq K\delta^{1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. Then

$$\|W_{2,\xi} f - f\|_1 \leq \frac{K}{\sqrt{\pi}} \Gamma\left(\frac{2+\gamma}{2}\right) \xi^{\frac{1+\gamma}{2}}. \tag{5.27}$$

Hence as $\xi \to 0$ we get $W_{2,\xi} \to I$ in the L_1 norm.

Proof. In Proposition 59 we place $r = 2$. □

In the next we consider $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_p(\mathbb{R}), n = 0$ or $n \geq 2$ even, $1 \leq p < \infty$ and the similar *smooth singular operator of symmetric convolution type*

$$W_\xi(f;x) = \frac{1}{\sqrt{\pi\xi}} \int_{-\infty}^{\infty} f(x+y) e^{-y^2/\xi} dy, \quad \text{for all } x \in \mathbb{R},\ \xi > 0. \tag{5.28}$$

Denote

$$K(x) := W_\xi(f;x) - f(x) - \sum_{\rho=1}^{n/2} \frac{f^{(2\rho)}(x)}{\rho!} \cdot \left(\frac{\xi}{4}\right)^{\rho}. \tag{5.29}$$

We give

Theorem 61. *Let $n \geq 2$ even and the rest as above. Then*

$$\|K(x)\|_2 \leq \sqrt{\frac{\tilde{\tau}}{10\sqrt{\pi}(2n-1)}} \frac{\xi^{\frac{n}{2}}}{(n-1)!} \omega_2(f^{(n)}, \sqrt{\xi})_2, \tag{5.30}$$

where

$$0 < \tilde{\tau} = \int_0^{\infty} \left((1+u)^5 - 1\right) u^{2n-1} e^{-u^2} du < \infty. \tag{5.31}$$

Hence as $\xi \to 0$ we get $\|K(x)\|_2 \to 0$.

If additionally $f^{(2m)} \in L_2(\mathbb{R}), m = 1, 2, \ldots, \frac{n}{2}$ then $\|W_\xi(f) - f\|_2 \to 0$, as $\xi \to 0$.

Proof. In the proof of Theorem 3 of [4] we use $p = q = 2$. □

It follows a Lipschitz type approximation result.

Theorem 62. *Let $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $n \geq 2$ even and the rest as above. Furthermore we assume the following Lipschitz condition: $\omega_2\left(f^{(n)}, \delta\right)_p \leq K\delta^{\gamma+1}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. Then*

$$\|K(x)\|_p \leq \frac{K\left[\Gamma\left(\frac{p(\gamma+n+1)+1}{2}\right)\right]^{\frac{1}{p}}}{\sqrt{2}\pi^{\frac{1}{2p}}(n-1)!p^{\frac{1}{2p}}q^{\frac{1}{2q}}\left[q(n-1)+1\right]^{\frac{1}{q}}\left[p(\gamma+1)+1\right]^{\frac{1}{p}}}\left(\frac{2}{p}\right)^{\frac{(\gamma+n+1)}{2}}\xi^{\frac{(\gamma+n+1)}{2}}. \tag{5.32}$$

Hence as $\xi \to 0$ we get $\|K(x)\|_p \to 0$.

If additionally $f^{(2m)} \in L_p(\mathbb{R}), m = 1, 2, \ldots, \frac{n}{2}$ then $\|W_\xi(f) - f\|_p \to 0$, as $\xi \to 0$.

Proof. As in the proof of Theorem 3, of [4] we find

$$\begin{aligned}
\int_{-\infty}^{\infty}|K(x)|^p dx &\leq c_2\left(\int_0^{\infty}\left(\int_0^y \omega_2(f^{(n)}, t)_p^p dt\right)y^{pn-1}e^{-\frac{py^2}{2\xi}}dy\right)\\
&\leq K^p c_2\left(\int_0^{\infty}\left(\frac{y^{p(\gamma+1)+1}}{p(\gamma+1)+1}\right)y^{pn-1}e^{-\frac{py^2}{2\xi}}dy\right)\\
&= \frac{K^p c_2}{p(\gamma+1)+1}\left(\int_0^{\infty}y^{p(\gamma+n+1)}e^{-\frac{py^2}{2\xi}}dy\right)\\
&\overset{(4.2)}{=} \frac{K^p c_2}{p(\gamma+1)+1}\left(\frac{2}{p}\right)^{\frac{p(\gamma+n+1)+1}{2}}\\
&\quad \cdot\frac{1}{2}\Gamma\left(\frac{p(\gamma+n+1)+1}{2}\right)\xi^{\frac{p(\gamma+n+1)+1}{2}}.
\end{aligned} \tag{5.33}$$

where here we denoted

$$c_2 := \frac{1}{2^{\frac{p}{2q}}q^{\frac{p}{2q}}(q(n-1)+1)^{p/q}((n-1)!)^p\sqrt{\pi\xi}}. \tag{5.34}$$

We have established the claim of the theorem. □

Corollary 63. *Assume the following Lipschitz condition: $\omega_2(f'', \delta)_2 \leq K\delta^{\gamma+1}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$, and the rest as above in this section. Then*

$$\|K(x)\|_2 \leq \sqrt{\frac{\left[\Gamma\left(\frac{2\gamma+7}{2}\right)\right]}{\sqrt{\pi}\,[6\gamma+9]}}\frac{K}{2}\xi^{\frac{(\gamma+3)}{2}}. \tag{5.35}$$

Hence as $\xi \to 0$ we get $\|K(x)\|_2 \to 0$.

If additionally $f'' \in L_2(\mathbb{R})$, then $\|W_\xi(f) - f\|_2 \to 0$, as $\xi \to 0$.

Proof. In Theorem 62 we place $p = q = n = 2$. $\square$

Theorem 64. *Let* $f \in C^2(\mathbb{R})$ and $f'' \in L_1(\mathbb{R})$. *Here* $K(x) = W_\xi(f;x) - f(x) - \frac{f''(x)}{4}\xi$. *Then*

$$\|K(x)\|_1 \leq \left(\frac{1}{2\sqrt{\pi}} + \frac{3}{8}\right)\omega_2(f'', \sqrt{\xi})_1 \xi. \tag{5.36}$$

Hence as $\xi \to 0$ *we obtain* $\|K(x)\|_1 \to 0$.

Also $\|W_\xi(f) - f\|_1 \to 0$, as $\xi \to 0$.

Proof. In the proof of Theorem 4 of [4] we use $n = 2$. $\square$

The Lipschitz case of $p = 1$ follows.

Theorem 65. *Let* $f \in C^n(\mathbb{R})$ and $f^{(n)} \in L_1(\mathbb{R})$, $n \geq 2$ *even. Furthermore we assume the following Lipschitz condition:* $\omega_2\left(f^{(n)}, \delta\right)_1 \leq K\delta^{\gamma+1}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$\|K(x)\|_1 \leq \frac{\Gamma\left(\frac{\gamma+n+2}{2}\right)K}{(n-1)!\,(\gamma+2)\,2\sqrt{\pi}}\xi^{\frac{\gamma+n+1}{2}}. \tag{5.37}$$

Hence as $\xi \to 0$ *we obtain* $\|K(x)\|_1 \to 0$.

If additionally $f^{(2m)} \in L_1(\mathbb{R}), m = 1, 2, \ldots, \frac{n}{2}$ then $\|W_\xi(f) - f\|_1 \to 0$, as $\xi \to 0$.

Proof. As in the proof of Theorem 4 of [4] we have

$$\begin{aligned}
\|K(x)\|_1 &\leq \frac{1}{\sqrt{\pi\xi}}\int_0^\infty\left(\left(\int_0^y \omega_2(f^{(n)}, t)_1 dt\right)\frac{y^{n-1}}{(n-1)!}e^{-y^2/\xi}\right)dy \\
&\leq \frac{1}{\sqrt{\pi\xi}}\int_0^\infty\left(\left(\int_0^y Kt^{\gamma+1}dt\right)\frac{y^{n-1}}{(n-1)!}e^{-y^2/\xi}\right)dy \\
&= \frac{K}{(n-1)!\,(\gamma+2)\sqrt{\pi\xi}}\int_0^\infty\left(y^{\gamma+n+1}e^{-y^2/\xi}\right)dy \\
&\overset{(4.2)}{=} \frac{\Gamma\left(\frac{\gamma+n+2}{2}\right)K}{(n-1)!\,(\gamma+2)\,2\sqrt{\pi}}\xi^{\frac{\gamma+n+1}{2}}.
\end{aligned} \tag{5.38}$$

We have proved the claim of the theorem. $\square$

Corollary 66. *Let* $f \in C^6(\mathbb{R})$ and $f^{(6)} \in L_1(\mathbb{R})$. *Furthermore we assume the following Lipschitz condition:* $\omega_2\left(f^{(6)}, \delta\right)_1 \leq K\delta^{\gamma+1}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$\|K(x)\|_1 \leq \frac{\Gamma\left(\frac{\gamma+8}{2}\right)K}{240\,(\gamma+2)\sqrt{\pi}}\xi^{\frac{\gamma+7}{2}}. \tag{5.39}$$

Hence as $\xi \to 0$ *we obtain* $\|K(x)\|_1 \to 0$.

If additionally $f^{(2m)} \in L_1(\mathbb{R}), m = 1, 2, 3$ then $\|W_\xi(f) - f\|_1 \to 0$, as $\xi \to 0$.

Proof. In Theorem 65 we place $n = 6$. □

The case of $n = 0$ follows.

Proposition 67. Let f *as above in this section. Then*

$$\|W_\xi(f) - f\|_2 \leq \sqrt{\frac{2}{\sqrt{\pi}} + \frac{19}{16}}\,\omega_2(f, \sqrt{\xi})_2. \tag{5.40}$$

Hence as $\xi \to 0$ *we obtain* $W_\xi \to I$ *in the* L_2 *norm.*

Proof. In the proof of Proposition 3 of [4] we use $p = q = 2$. □

The related Lipschitz case for $n = 0$ comes next.

Proposition 68. *Let* $p, q > 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and the rest as above. Furthermore we assume the following Lipschitz condition:* $\omega_2(f,\delta)_p \leq K\delta^{1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$\|W_\xi(f) - f\|_p \leq \left(\frac{2}{p}\right)^{\frac{(1+\gamma)}{2}} \frac{\left[\Gamma\left(\frac{(1+\gamma)p+1}{2}\right)\right]^{\frac{1}{p}} K}{\pi^{\frac{1}{2p}} p^{\frac{1}{2p}} q^{\frac{1}{2q}} \sqrt{2}}\,\xi^{\frac{(1+\gamma)}{2}}. \tag{5.41}$$

Hence as $\xi \to 0$ *we obtain* $W_\xi \to I$ *in the* L_p *norm,* $p > 1$.

Proof. As in the proof of Proposition 3 of [4] we get

$$\begin{aligned}\int_{-\infty}^{\infty} |W_\xi(f;x) - f(x)|^p dx \;&\leq\; \frac{1}{\sqrt{\pi\xi}\,(2q)^{\frac{p}{2q}}} \int_0^\infty \omega_2(f,y)_p^p e^{\frac{-py^2}{2\xi}} dy \\ &\leq\; \frac{1}{\sqrt{\pi\xi}\,(2q)^{\frac{p}{2q}}} \int_0^\infty \left(Ky^{1+\gamma}\right)^p e^{\frac{-py^2}{2\xi}} dy\end{aligned}$$

$$\overset{(4.2)}{=} \frac{K^p}{\sqrt{\pi}\,(2q)^{\frac{p}{2q}}} \left(\frac{2}{p}\right)^{\frac{(1+\gamma)p+1}{2}} \frac{1}{2}\Gamma\left(\frac{(1+\gamma)p+1}{2}\right) \xi^{\frac{(1+\gamma)p}{2}}. \tag{5.42}$$

The proof of the claim is now completed. □

A particular example follows

Corollary 69. *Let* f *as above in this section. Furthermore we assume the following Lipschitz condition:* $\omega_2(f,\delta)_2 \leq K\delta^{1+\gamma}$, $K > 0$, $0 < \gamma \leq 1$, for any $\delta > 0$. *Then*

$$\|W_\xi(f) - f\|_2 \leq \frac{K}{2}\sqrt{\frac{\Gamma\left(\frac{3+2\gamma}{2}\right)}{\sqrt{\pi}}}\,\xi^{\frac{(1+\gamma)}{2}}. \tag{5.43}$$

Hence as $\xi \to 0$ *we obtain* $W_\xi \to I$ *in the* L_2 *norm.*

Proof. In Proposition 68 we place $p = q = 2$. □

We finish with the Lipschitz type result

Proposition 70. *Assume the following Lipschitz condition:* $\omega_2(f,\delta)_1 \le K\delta^{\gamma+1}$, $K > 0$, $0 < \gamma \le 1$, for any $\delta > 0$. *It holds,*

$$\|W_\xi f - f\|_1 \le \frac{K}{2\sqrt{\pi}}\Gamma\left(\frac{\gamma+2}{2}\right)\xi^{\frac{\gamma+1}{2}}. \tag{5.44}$$

Hence as $\xi \to 0$ *we get* $W_\xi \to I$ *in the* L_1 *norm.*

Proof. As in the proof of Proposition 4 of [4] we derive

$$\begin{aligned}\int_{-\infty}^{\infty} |W_\xi(f;x) - f(x)|dx &\le \frac{1}{\sqrt{\pi\xi}}\int_0^\infty \omega_2(f,y)_1 e^{-y^2/\xi}dy \\ &\le \frac{1}{\sqrt{\pi\xi}}\int_0^\infty Ky^{\gamma+1}e^{-y^2/\xi}dy \\ &\overset{(4.2)}{=} \frac{K}{2\sqrt{\pi}}\Gamma\left(\frac{\gamma+2}{2}\right)\xi^{\frac{\gamma+1}{2}}. \end{aligned} \tag{5.45}$$

We have established the claim. □

Received: September 2009. Revised: July 2010.

References

[1] George A. Anastassiou, "Basic Convergence with Rates of Smooth Picard Singular Operators", J. Comput. Anal. Appl., 8 (2006), 313–334.

[2] George A. Anastassiou, "Lp convergence with rates of smooth Picard singular operators", Differential & difference equations and applications, Hindawi Publ. Corp., New York, (2006), 31–45.

[3] George A. Anastassiou, Razvan A. Mezei, "Uniform Convergence with Rates of Smooth Gauss-Weierstrass Singular Integral Operators", Applicable Analysis, 88:7 (2009), 1015 — 1037.

[4] George A. Anastassiou, Razvan A. Mezei, "Lp Convergence with Rates of Smooth Gauss-Weierstrass Singular Operators", Nonlinear Studies, accepted 2009.