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Weak Convergence Theorems for Maximal Monotone Operators with Nonspreading mappings in a Hilbert space

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ABSTRACT

Let C be a closed convex subset of a real Hilbert space H. Let T be a nonspreading mapping of C into itself, let A be an α-inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H such that the domain of B is included in C. We introduce an iterative sequence of finding a point of $F(T)\cap(A+B)^{-1}0$, where $F(T)$ is the set of fixed points of T and $(A+B)^{-1}0$ is the set of zero points of $A+B$. Then, we obtain the main result which is related to the weak convergence of the sequence. Using this result, we get a weak convergence theorem for finding a common fixed point of a nonspreading mapping and a nonexpansive mapping in a Hilbert space. Further, we consider the problem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonspreading mapping.

RESUMEN

Sea C un subconjunto convexo cerrado de un espacio real de Hilbert H. Sea T una asignación de C en sí mismo, sea A una asignación monótona α-inversa de C en H y sea B un operador monotono máximal en H tal que el dominio de B está incluido en C. Se introduce una secuencia iterativa para encontrar un punto de $F(T) \cap (A+B)^{-1}0$, donde $F(T)$ es el conjunto de puntos fijos de T y $(A+B)^{-1}0$ es el conjunto de los puntos cero de $A+B$. Entonces, se obtiene el resultado principal que se relaciona con la convergencia débil de la secuencia.

Utilizando este resultado, obtenemos un teorema de convergencia para encontrar un punto común de una asignación fija y una asignación en un espacio de Hilbert. Además, consideramos el problema para encontrar un elemento común del conjunto de soluciones de un problema de equilibrio y el conjunto de puntos fijos de una asignación.

Keywords: Nonspreading mapping, maximal monotone operator, inverse strongly-monotone mapping, fixed point, iteration procedure.

Mathematics Subject Classification: 46C05.

1 Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. For a constant $\alpha > 0$, the mapping $A : C \to H$ is said to be α-inverse strongly monotone if for any $x, y \in C$,

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2 .$$

It is well-known that an α-inverse strongly monotone mapping is also Lipschitz continuous with a Lipschitz constant $\frac{1}{\alpha}$. Let S be a mapping of C into itself. We denote by $F(S)$ the set of fixed points of S. A mapping S of C into itself is nonexpansive if

$$\|Su - Sv\| \leq \|u - v\|, \quad \forall u, v \in C.$$

If $S : C \to C$ is a nonexpansive mapping, then $I - S$ is $\frac{1}{2}$-inverse strongly monotone, where I is the identity mapping on H; see, for instance, [18]. A mapping S of C into itself is nonspreading if

$$2\|Su - Sv\|^2 \leq \|Su - v\|^2 + \|Sv - u\|^2, \quad \forall u, v \in C;$$

see [6, 7]. A multi-valued mapping $B \subset H \times H$ is said to be monotone if $\langle x - y, u - v \rangle \geq 0$ for all $x, y \in H$, $u \in Bx$ and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. Recently, in the case when $S : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an α-inverse strongly monotone mapping and $B \subset H \times H$ is a maximal monotone operator, Takahashi, Takahashi and Toyoda [15] proved a strong convergence theorem for finding a point of $F(S) \cap (A+B)^{-1}0$, where $F(S)$ is the set of fixed points of S and $(A+B)^{-1}0$ is the set of zero points of $A+B$.

In this paper, motivated by Takahashi, Takahashi and Toyoda [15], we introduce an iteration sequence of finding a common point of the set $F(S)$ of fixed points of a nonspreading mapping S and the set $(A+B)^{-1}0$ of zero points of $A+B$, where $A\colon C \to H$ is an α-inverse strongly monotone mapping and $B \subset H \times H$ is a maximal monotone operator. Then, we prove a weak convergence theorem. Using this result, we get a weak convergence theorem for finding a common fixed point of a nonspreading mapping and a nonexpansive mapping in a Hilbert space. Further, we obtain a weak convergence theorem for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonspreading mapping.

2 Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let H be a real Hilbert space with inner product $\langle \cdot\,,\,\cdot \rangle$ and norm $\|\cdot\|$. A Hilbert space satisfies Opial's condition [10], that is,

$$\liminf_{n\to\infty} \|x_n - u\| < \liminf_{n\to\infty} \|x_n - v\|$$

if $x_n \rightharpoonup u$ and $u \neq v$; see [10]. Let C be a nonempty closed convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x - P_C x\| \leq \|x - y\|$ for all $x \in H$ and $y \in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle$$

for all $x, y \in H$. Further $\langle x - P_C x, y - P_C x\rangle \leq 0$ holds for all $x \in H$ and $y \in C$; see, for instance, [16].

Let $\alpha > 0$ be a given constant. A mapping $A\colon C \to H$ is said to be α-inverse strongly monotone if $\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2$ for all $x, y \in C$. We have that $\|Ax - Ay\| \leq (1/\alpha)\|x - y\|$ for all $x, y \in C$ if A is α-inverse strongly monotone. Let B be a mapping of H into 2^H. The effective domain of B is denoted by $D(B)$, that is, $D(B) = \{x \in H : Bx \neq \emptyset\}$. A multi-valued mapping B is said to be a monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(B)$, $u \in Bx$, and $v \in By$. A monotone operator B on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator B on H and $r > 0$, we may define a single-valued operator $J_r = (I + rB)^{-1}\colon H \to D(B)$, which is called the resolvent of B for $r > 0$. Let B be a maximal monotone operator on H and let $B^{-1}0 = \{x \in H : 0 \in Bx\}$. It is known that the resolvent J_r is firmly nonexpansive and $B^{-1}0 = F(J_r)$ for all $r > 0$.

We give the crucial lemmas in order to prove the main theorem.

Lemma 2.1 ([12])**.** *Let H be a real Hilbert space, let $\{\alpha_n\}$ be a sequence of real numbers such that $0 < a \leq \alpha_n \leq b < 1$ for all $n \in \mathbb{N}$ and let $\{v_n\}$ and $\{w_n\}$ be sequences in H such that for some c, $\limsup_{n\to\infty} \|v_n\| \leq c$, $\limsup_{n\to\infty} \|w_n\| \leq c$ and $\limsup_{n\to\infty} \|\alpha_n v_n + (1-\alpha_n) w_n\| = c$. Then $\lim_{n\to\infty} \|v_n - w_n\| = 0$.*

Lemma 2.2 ([19]). *Let H be a Hilbert space and let S be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - x\| \leq \|x_n - x\|$ for all $n \in \mathbb{N}$ and $x \in S$, then $\{P_S(x_n)\}$ converges strongly to some $z \in S$, where P_S stands for the metric projection on H onto S.*

Using Opial's theorem [10], we can also prove the following lemma; see, for instance, [18].

Lemma 2.3. *Let H be a Hilbert space and let $\{x_n\}$ be a sequence in H such that there exists a nonempty subset $S \subset H$ satisfying (i) and (ii):*

(i) For every $x^* \in S$, $\lim_{n\to\infty} \|x_n - x^*\|$ exists:

(ii) if a subsequence $\{x_{n_j}\} \subset \{x_n\}$ converges weakly to x^*, then $x^* \in S$.

Then there exists $x_0 \in S$ such that $x_n \rightharpoonup x_0$.

Let C be a nonempty closed convex subset of a real Hilbert space H, let $f : C \times C \to \mathbb{R}$ be a bifunction and let $A : C \to H$ be a nonlinear mapping. Then, we consider the following equilibrium problem [8]: Find $z \in C$ such that

$$f(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall y \in C. \tag{2.1}$$

The set of such $z \in C$ is denoted by $EP(f, A)$, i.e.,

$$EP(f, A) = \{z \in C : f(z, y) + \langle Az, y - z\rangle \geq 0,\ \forall y \in C\}.$$

In the case of $A \equiv 0$, $EP(f, A)$ is denoted by $EP(f)$. In the case of $F \equiv 0$, $EP(f, A)$ is also denoted by $VI(C, A)$. For solving the equilibrium problem, let us assume that the bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;

(A3) for all $x, y, z \in C$, $\limsup_{t\downarrow 0} f(tz + (1 - t)x, y) \leq f(x, y)$;

(A4) $f(x, \cdot)$ is convex and lower semicontinuous for all $x \in C$.

We know the following lemmas; see, for instance, [1] and [2].

Lemma 2.4 ([1]). *Let C be a nonempty closed convex subset of H, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0$$

for all $y \in C$.

Lemma 2.5 ([2]). *For $r > 0$ and $x \in H$, define the resolvent $T_r : H \to C$ of f for $r > 0$ as follows:*

$$T_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}$$

for all $x \in H$. Then, the following hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive, i.e., for all $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(iii) $F(T_r) = EP(f)$;

(iv) $EP(f)$ is closed and convex.

3 Main result

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then, a mapping S of C into itself is nonspreading if

$$2\|Su - Sv\|^2 \leq \|Su - v\|^2 + \|Sv - u\|^2, \quad \forall u, v \in C;$$

see [6, 7]. We know from [6, 7, 3] that if the bifunction $f : C \times C \to \mathbb{R}$ satisfies the conditions (A1), (A2), (A3) and (A4), then for any $r > 0$, T_r is a nonspreading mapping of C into itself. Further, we can give the following example of nonspreading mappings in a Hilbert space. Let H be a real Hilbert space; see [4]. Set $E = \{x \in H : \|x\| \leq 1\}$, $D = \{x \in H : \|x\| \leq 2\}$ and $C = \{x \in H : \|x\| \leq 3\}$. Define a mapping $S : C \to C$ as follows:

$$Sx \begin{cases} 0, & x \in D, \\ P_E x, & x \notin D. \end{cases}$$

Then, this mapping S is not nonexpansive but nonspreading because it is not continuous. This implies that the class of nonexpansive mappings does not contain the class of nonspreading mappings. Now, we can prove a weak convergence theorem. Before proving it, we give the following lemma.

Lemma 3.1. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let $\alpha > 0$. Let A be an α-inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for any $\lambda > 0$. Then, the following hold:*

(i) If $u, v \in (A + B)^{-1}0$, then $Au = Av$;

(ii) for any $\lambda > 0$, $u \in (A+B)^{-1}(0)$ if and only if $u = J_\lambda(I - \lambda A)u$.

Proof. (i) If $u, v \in (A+B)^{-1}(0)$, then $0 \in Au + Bu$ and $0 \in Av + Bv$. Then, we have $-Au \in Bu$ and $-Av \in Bv$. Since B is monotone, we have $\langle u - v, -Au - (-Av)\rangle \geq 0$. On the other hand, since A is α-inverse strongly monotone, we have $\langle u - v, Au - Av\rangle \geq \|Au - Av\|^2$. So, we have $\langle u - v, -Au - (-Av)\rangle = 0$ and hence $Au = Av$.

(ii) For any $\lambda > 0$, we have that

$$\begin{aligned} & u = J_\lambda(I - \lambda A)u \\ & \Leftrightarrow u - \lambda Au \in u + \lambda Bu \\ & \Leftrightarrow 0 \in \lambda Au + \lambda Bu \\ & \Leftrightarrow 0 \in Au + Bu \\ & \Leftrightarrow u \in (A+B)^{-1}(0). \end{aligned}$$

This completes the proof. □

Now, we can prove the main theorem.

Theorem 3.1. *Let C be a nonempty convex closed subset of a real Hilbert space H, let $A : C \to H$ be α-inverse strongly monotone, let $B : D(B) \subset C \to 2^H$ be maximal monotone, let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for any $\lambda > 0$, and let $T : C \to C$ be a nonspreading mapping. Assume that $F(T) \cap (A+B)^{-1}(0) \neq \emptyset$. For any $x = x_1 \in C$, define*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)T(J_{\lambda_n}(I - \lambda_n A)x_n), \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions $(*)$:

$$0 < c \leq \beta_n \leq d < 1 \text{ and } 0 < a \leq \lambda_n \leq b < 2\alpha. \tag{$*$}$$

Then, $x_n \rightharpoonup z_0 \in F(T) \cap (A+B)^{-1}(0)$, where $z_0 = \lim_{n\to\infty} P_{F(T)\cap(A+B)^{-1}(0)}(x_n)$.

Proof. Set $E = F(T) \cap (A+B)^{-1}(0)$. Let $y_n = J_{\lambda_n}(I - \lambda_n A)x_n$ for all $n \in \mathbb{N}$ and let $z \in E$. Since $z = J_{\lambda_n}(I - \lambda_n A)z$ from Lemma 3.1 and A is α-inverse strongly monotone, we have that

$$\begin{aligned} \|y_n - z\|^2 &= \|J_{\lambda_n}(I - \lambda_n A)x_n - J_{\lambda_n}(I - \lambda_n A)z\|^2 \\ &\leq \|x_n - \lambda_n Ax_n - z + \lambda_n Az\|^2 \\ &= \|x_n - z\|^2 - 2\lambda_n\langle x_n - z, Ax_n - Az\rangle + {\lambda^2}_n \|Ax_n - Az\|^2 \\ &\leq \|x_n - z\|^2 - 2\lambda_n\alpha \|Ax_n - Az\|^2 + \lambda_n^2 \|Ax_n - Az\|^2 \\ &= \|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha) \|Ax_n - Az\|^2 . \end{aligned} \tag{3.1}$$

From $(*)$, we have that

$$\|y_n - z\|^2 \leq \|x_n - z\|^2, \quad \forall n \in \mathbb{N}$$

and hence

$$\begin{aligned}\|x_{n+1}-z\| &= \|\beta_n x_n + (1-\beta_n)Ty_n - z\| \\ &\le \beta_n \|x_n - z\| + (1-\beta_n)\|Ty_n - z\| \\ &\le \beta_n \|x_n - z\| + (1-\beta_n)\|y_n - z\| \\ &\le \|x_n - z\|.\end{aligned}$$

This means that the condition (i) of Lemma 2.3 holds for $S = E$. We also obtain that $\lim_{n\to\infty}\|x_n - z\|$ exists. Thus, $\{x_n\}, \{Ax_n\}, \{y_n\}$ and $\{Ty_n\}$ are bounded. By the inequality (2),

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \beta_n \|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 \\ &\le \beta_n \|x_n - z\|^2 + (1-\beta_n)\{\|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)\|Ax_n - Ax\|^2\} \\ &\le \|x_n - z\|^2 + \lambda_n(\lambda_n - 2\alpha)(1-\beta_n)\|Ax_n - Az\|^2.\end{aligned}$$

Thus we have

$$\begin{aligned}0 &\le (1-d)a(2\alpha - d)\|Ax_n - Az\|^2 \\ &\le \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \to 0,\end{aligned}$$

as $n \to \infty$. This means that

$$\lim_{n\to\infty}\|Ax_n - Az\| = 0. \tag{3.2}$$

On the other hand, since J_{λ_n} is firmly nonexpansive, we have that

$$\begin{aligned}\|y_n - z\|^2 &= \|J_{\lambda_n}(I-\lambda_n A)x_n - J_{\lambda_n}(I-\lambda_n A)z\|^2 \\ &\le \langle y_n - z, (I-\lambda_n A)x_n - (I-\lambda_n A)z\rangle \\ &= \frac{1}{2}\{\|y_n - z\|^2 + \|(I-\lambda_n A)x_n - (I-\lambda_n A)z\|^2 \\ &\quad - \|y_n - z - (I-\lambda_n A)x_n + (I-\lambda_n A)z\|^2\} \\ &= \frac{1}{2}\{\|y_n - z\|^2 + \|x_n - z\|^2 \\ &\quad - \|y_n - z - (I-\lambda_n A)x_n + (I-\lambda_n A)z\|^2\} \\ &= \frac{1}{2}\{\|y_n - z\|^2 + \|x_n - z\|^2 - \|y_n - x_n\|^2 \\ &\quad - 2\lambda_n\langle y_n - x_n, Ax_n - Az\rangle - {\lambda_n}^2\|Ax_n - Az\|^2\}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\|y_n - z\|^2 &\le \|x_n - z\|^2 - \|y_n - x_n\|^2 \\ &\quad - 2\lambda_n\langle y_n - x_n, Ax_n - Az\rangle - {\lambda_n}^2\|Ax_n - Az\|^2\end{aligned}$$

and hence

$$\begin{aligned}\|x_{n+1}-z\|^2 \leq & \beta_n \|x_n - z\|^2 + (1-\beta_n)\|Ty_n - z\|^2 \\ \leq & \beta_n \|x_n - z\|^2 + (1-\beta_n)\|y_n - z\|^2 \\ \leq & \beta_n \|x_n - z\|^2 + (1-\beta_n)\{\|x_n - z\|^2 - \|y_n - x_n\|^2 \\ & - 2\lambda_n\langle y_n - x_n, Ax_n - Az\rangle - {\lambda_n}^2\|Ax_n - Az\|^2\} \\ \leq & \|x_n - z\|^2 - (1-d)\|y_n - x_n\|^2 - {\lambda_n}^2(1-\beta_n)\|Ax_n - Az\|^2 \\ & - 2\lambda_n(1-\beta_n)\langle y_n - x_n, Ax_n - Az\rangle.\end{aligned}$$

This means that

$$\begin{aligned}(1-d)\|y_n - x_n\|^2 \leq & \|x_n - z\|^2 - \|x_{n+1} - z\|^2 \\ & + \|Ax_n - Az\|\{2b(1-c)\|y_n - x_n\| + b^2(1-c)\|Ax_n - Az\|\}.\end{aligned}$$

Since $\{y_n\}$ and $\{x_n\}$ are bounded, $\lim_{n\to\infty}\|Ax_n - Az\| = 0$ and $\lim_{n\to\infty}\|x_n - z\|$ exists, we have

$$\lim_{n\to\infty}\|y_n - x_n\| = 0.$$

Since A is Lipschitz continuous, we also have

$$\lim_{n\to\infty}\|Ay_n - Ax_n\| = 0.$$

Let x^* be a weak cluster point of $\{x_n\}$. First, we prove that $x^* \in (A+B)^{-1}(0)$. Since $y_n = J_{\lambda_n}(I - \lambda_n A)x_n$, we have that

$$\begin{aligned}& y_n = (I+\lambda_n B)^{-1}(I-\lambda_n A)x_n \\ & \Leftrightarrow (I-\lambda_n A)x_n \in (I+\lambda_n B)y_n = y_n + \lambda_n By_n \\ & \Leftrightarrow x_n - y_n - \lambda_n Ax_n \in \lambda_n By_n \\ & \Leftrightarrow \frac{1}{\lambda_n}(x_n - y_n - \lambda_n Ax_n) \in By_n.\end{aligned}$$

Since B is monotone, we have that for $(u,v) \in B$,

$$\left\langle y_n - u, \frac{1}{\lambda_n}(x_n - y_n - \lambda_n Ax_n) - v \right\rangle \geq 0$$

and hence

$$\langle y_n - u, x_n - y_n - \lambda_n(Ax_n + v)\rangle \geq 0.$$

Suppose that a subsequence $\{x_{n_j}\} \subset \{x_n\}$ satisfies $x_{n_j} \rightharpoonup x^*$. Then, since A is α-inverse strongly monotone and $Ax_n \to Az$ by (3),

$$\left\langle x_{n_j} - x^*, Ax_{n_j} - Ax^*\right\rangle \geq \alpha\left\|Ax_{n_j} - Ax^*\right\|^2$$

implies that $Ax_{n_j} \to Ax^*$ as $j \to \infty$. Moreover, since $\lim_{n\to\infty} \|y_n - x_n\| = 0$ implies $y_{n_j} \rightharpoonup x^*$, we have

$$\lim_{j\to\infty} \left\langle y_{n_j} - u, x_{n_j} - y_{n_j} - \lambda_{n_j}(Ax_{n_j} + v)\right\rangle \geq 0$$

and hence $\langle x^* - u, -Ax^* - v\rangle \geq 0$. Since B is maximal monotone, $(-Ax^*) \in Bx^*$. That is, $x^* \in (A+B)^{-1}(0)$.

Next, we show $x^* \in F(T)$. Putting $c = \lim_{n\to\infty} \|x_n - z\|$, we have

$$\begin{aligned}
\limsup_{n\to\infty} \|Ty_n - z\| =& \limsup_{n\to\infty} \|Ty_n - Tz\| \\
\leq& \limsup_{n\to\infty} \|y_n - z\| \\
\leq& \limsup_{n\to\infty} \|x_n - z\| \leq c.
\end{aligned}$$

On the other hand, we have

$$\lim_{n\to\infty} \|x_{n+1} - z\| = \lim_{n\to\infty} \|\beta_n x_n + (1-\beta_n)Ty_n - z\| = c.$$

From Lemma 2.1, we have

$$\lim_{n\to\infty} \|(x_n - z) - (Ty_n - z)\| = \lim_{n\to\infty} \|x_n - Ty_n\| = 0. \tag{3.3}$$

We have also

$$\|y_n - Ty_n\| \leq \|y_n - x_n\| + \|x_n - Ty_n\|.$$

Hence, we have

$$\lim_{n\to\infty} \|y_n - Ty_n\| = 0.$$

Since $x_{n_j} \rightharpoonup x^*$ and $x_n - y_n \to 0$, we have $y_{n_j} \rightharpoonup x^*$. Now we shall show that $Tx^* = x^*$. Since T is nonspreading, we have

$$\begin{aligned}
0 \leq& (\|Ty_n - x^*\|^2 - \|Ty_n - Tx^*\|^2) + (\|Tx^* - y_n\|^2 - \|Ty_n - Tx^*\|^2) \\
=& 2\langle Ty_n, Tx^* - x^*\rangle + \|x^*\|^2 - \|Tx^*\|^2 + 2\langle Ty_n - y_n, Tx^*\rangle + \|y_n\|^2 - \|Ty_n\|^2 \\
\leq& \quad 2\langle Ty_n - y_n, Tx^* - x^*\rangle + 2\langle y_n, Tx^* - x^*\rangle + \|x^*\|^2 - \|Tx^*\|^2 \\
&+ 2\langle Ty_n - y_n, Tx^*\rangle + (\|y_n\| + \|Ty_n\|)(\|y_n - Ty_n\|).
\end{aligned}$$

Thus, we have that for all $j \in \mathbb{N}$,

$$\begin{aligned}
0 \leq& 2\left\langle Ty_{n_j} - y_{n_j}, Tx^* - x^*\right\rangle + 2\left\langle y_{n_j}, Tx^* - x^*\right\rangle + \|x^*\|^2 - \|Tx^*\|^2 \\
&+ 2\left\langle Ty_{n_j} - y_{n_j}, Tx^*\right\rangle + (\|y_{n_j}\| + \|Ty_{n_j}\|)(\|y_{n_j} - Ty_{n_j}\|).
\end{aligned}$$

Since $\lim_{n\to\infty} \left\|Ty_{n_j} - y_{n_j}\right\| = 0$ and $y_{n_j} \rightharpoonup x^*$ as $j \to \infty$, the above inequality implies that

$$\begin{aligned} 0 \leq& 2\langle x^*, Tx^* - x^*\rangle + \|x^*\|^2 - \|Tx^*\|^2 \\ =& 2\langle x^*, Tx^*\rangle - \|x^*\|^2 - \|Tx^*\|^2 \\ =& -\|x^* - Tx^*\|^2 . \end{aligned}$$

So, we have $Tx^* = x^*, i.e., x^* \in F(T)$. Therefore we obtain that

$$x^* \in E = F(T) \cap (A+B)^{-1}(0).$$

This implies that the condition (ii) of Lemma 2.3 holds for $S = E$. We also know that $\lim_{n\to\infty} \|x_n - z\|$ exists for $z \in S = E$. So, we have from Lemma 2.3 that there exists $z^* \in E$ such that $x_n \rightharpoonup z^*$ as $n \to \infty$. Moreover, since for any $z \in S = E$,

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \quad \forall n \in \mathbb{N},$$

by Lemma 2.2 there exists some $z_0 \in S$ such that $P_S(x_n) \to z_0$. The property of metric projection implies that

$$\langle z^* - P_S(x_n), x_n - P_S(x_n)\rangle \leq 0.$$

Therefore, we have

$$\langle z^* - z_0, z^* - z_0\rangle = \|z^* - z_0\|^2 \leq 0.$$

This means that $z^* = z_0$, i.e., $x_n \rightharpoonup z^* = \lim_{n\to\infty} P_E(x_n)$. □

4 Applications

Let H be a Hilbert space and let f be a proper lower semicontinuous convex function of H into $(-\infty, \infty]$. Then the subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \leq f(y), \ \forall y \in H\}$$

for all $x \in H$. By Rockafellar [11], it is shown that ∂f is maximal monotone. Let C be a nonempty closed convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) \begin{cases} 0, & \text{if } x \in C, \\ \infty, & \text{if } x \notin C. \end{cases}$$

Further, for any $u \in C$, we also define the normal cone $N_C(u)$ of C at u as follows;

$$N_C(u) = \{z \in H : \langle z, y - u\rangle \leq 0, \ \forall y \in C\}.$$

Then $i_C : H \to (-\infty, \infty]$ is a proper lower semicontinuous convex function on H and ∂i_C is a maximal monotone operator. Let $J_\lambda x = (I + \lambda \partial i_C)^{-1} x$ for $\lambda > 0$ and $x \in H$. Since

$$\begin{aligned} \partial i_C(x) &= \{z \in H : i_C(x) + \langle z, y - x\rangle \leq i_C(y), \ \forall y \in H\} \\ &= \{z \in H : \langle z, y - x\rangle \leq 0, \ \forall y \in C\} \\ &= N_C(x) \end{aligned}$$

for $x \in C$, we have

$$\begin{aligned} u = J_\lambda x &\Leftrightarrow (I + \lambda \partial i_C)^{-1} x = u \\ &\Leftrightarrow x \in u + \lambda \partial i_C(u) \\ &\Leftrightarrow x \in u + \lambda N_C(u) \\ &\Leftrightarrow x - u \in \lambda N_C(u) \\ &\Leftrightarrow \langle x - u, y - u \rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow P_C(x) = u. \end{aligned}$$

Similarly, we have that for $x \in C$,

$$\begin{aligned} x \in (A + \partial i_C)^{-1}(0) &\Leftrightarrow \langle -Ax, y - x \rangle \leq 0, \quad \forall y \in C \\ &\Leftrightarrow x \in VI(A, C). \end{aligned}$$

Thus, putting $B = \partial i_C$, we have $J_{\lambda_n} = P_C$ for any $n \in \mathbb{N}$. Thus, we have the following theorem from Theorem 3.1.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H, let A be an α-inverse strongly monotone mapping of C into H and let $T : C \to C$ be a nonspreading mapping. Assume $F(T) \cap (A + \partial i_C)^{-1}(0) = F(T) \cap VI(A, C) \neq \emptyset$. Define a sequence $\{x_n\}$ in C as follows: $x = x_1 \in C$ and*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) T(P_C(I - \lambda_n A)x_n)$$

for all $n \in \mathbb{N}$, where the sequences $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the condition $(*)$:

$$0 < c \leq \beta_n \leq d < 1 \text{ and } 0 < a \leq \lambda_n \leq b < 2\alpha. \tag{$*$}$$

Then, $x_n \rightharpoonup z_0 \in F(T) \cap VI(A, C)$ and $z_0 = \lim_{n \to \infty} P_{F(T) \cap VI(A,C)}(x_n)$.

Let $S : C \to C$ be nonexpansive. Then, $I - S$ is $\frac{1}{2}$-inverse strongly monotone. So, we obtain the following result.

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $S : C \to C$ be a nonexpansive mapping and let $T : C \to C$ be a nonspreading mapping. Assume that $F(T) \cap F(S) \neq \emptyset$. Let $x = x_1 \in C$ and define*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) T((1 - \lambda_n) x_n + \lambda_n S x_n)$$

for all $n \in N$, where $\{\lambda_n\}$ and $\{\beta_n\}$ satisfy the condition $(*)$:

$$0 < c \leq \beta_n \leq d < 1 \text{ and } 0 < a \leq \lambda_n \leq b < 1. \tag{$*$}$$

Then, $x_n \rightharpoonup z_0 \in F(T) \cap F(S)$ and $z_0 = \lim_{n \to \infty} P_{F(T) \cap F(S)}(x_n)$.

Proof. Put $A = I - S$. Then we have

$$\begin{aligned} P_C(x_n - \lambda_n Ax_n) &= P_C(x_n - \lambda_n(I-S)x_n) \\ &= P_C((1-\lambda_n)x_n + \lambda_n Sx_n) \\ &= (1-\lambda_n)x_n + \lambda_n Sx_n. \end{aligned}$$

For $u \in C$, we have $Su \in C$ and

$$\begin{aligned} u \in (A + \partial i_C)^{-1}(0) &\Leftrightarrow 0 \in Au + N_C(u) \\ &\Leftrightarrow Su - u \in N_C(u) \\ &\Leftrightarrow \langle Su - u, v - u\rangle \leq 0, \ \forall v \in C \\ &\Leftrightarrow P_C(Su) = u \\ &\Leftrightarrow Su = u. \end{aligned}$$

Thus, we obtain $(A + \partial i_C)^{-1}(0) = VI(A, C) = F(S)$. So, by Theorem 4.1 we have the desired result. □

Next, we deal with the equilibrium problem with nonspreading mappings in a Hilbert space. Takahashi, Takahashi and Toyoda [15] showed the following.

Theorem 4.3 ([15])**.** *Let C be a nonempty closed convex subset of a Hibert space H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the conditions (A1)-(A4). Define A_f as follows:*

$$A_f(x) \begin{cases} \{z \in H : f(x,y) \geq \langle y - x, z\rangle, \ \forall y \in C\}, & \text{if } x \in C, \\ \emptyset, & \text{if } x \notin C. \end{cases}$$

Then, $EP(f) = A_f^{-1}(0)$ and A_f is maximal monotone with the domain of A_f in C. Furthermore,

$$T_r(x) = (I + rA_f)^{-1}(x), \quad \forall r > 0.$$

We obtain the following theorem from Theorem 3.1.

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H, let $f : C \times C \to \mathbb{R}$ satisfy the conditions (A1)-(A4) and let T_λ be the resolvent of f for $\lambda > 0$. Let $S : C \to C$ be a nonspreading mapping. Assume that $F(T) \cap EP(f) \neq \emptyset$. For $x = x_1 \in C$, define*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)ST_{\lambda_n}x_n, \quad \forall n \in \mathbb{N},$$

where $\{\beta_n\}$ and $\{\lambda_n\}$ satisfy the following conditions:

$$0 < c \leq \beta_n \leq d < 1, \quad 0 < a \leq \lambda_n \leq b < \infty.$$

Then, $x_n \rightharpoonup z_0 \in F(T) \cap EP(f)$ and $z_0 = \lim_{n\to\infty} P_{F(S)\cap EP(f)}(x_n)$.

Proof. Suppose $A = 0$. Then, we have that

$$\langle x - y, Ax - Ay \rangle \geq \alpha \|Ax - Ay\|^2 = 0, \quad \forall \alpha \in \mathbb{R}.$$

So, we can choose $\alpha = \infty$ in Theorem 3.1. Since $T_{\lambda_n} = (I + \lambda_n A_f)^{-1}$ is the resolvent of A_f and A_f is maximal monotone, Theorem 3.1 implies that $x_n \rightharpoonup z_0 \in F(T) \cap A_f^{-1}(0)$. Moreover, we know $A_f^{-1}(0) = EP(f)$. So, we have the desired result. $\square$

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