

Modulation Spaces
with
 A_{∞}^{loc} -Weights

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ABSTRACT

In this paper we describe the function space $M_{p,q}^{s,w}$ with $w \in A_{\infty}^{\text{loc}}$ together with some related results of weighted modulation spaces.

RESUMEN

En este artículo describimos el espacio de la funciones $M_{p,q}^{s,w}$ con $w \in A_{\infty}^{\text{loc}}$ junto con algunos resultados relacionados a espacios de modulación con peso.

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1 Modulation Spaces

Modulation spaces, which were initiated by Feichtinger in 1983 (see [5]), were investigated for the purpose of measuring smoothness of functions and distributions in a way other than Besov spaces. Besov spaces as well as Triebel-Lizorkin spaces are very close to Sobolev spaces and are used in partial differential equations. These spaces are defined by way of dilations. Feichtinger took full advantage of the group structure of \mathbb{R}^n . Recall that \mathbb{R}^n carries the structure of a Lie group not with dilation but with addition. Therefore, it seems natural that we consider the translation.

The goal of the present paper is to combine the results in [17, 21]. The main results of [21] can be summarized as follows: Quite a few of the results of usual modulation spaces $M_{p,q}^s$ carries over to the A_∞^{loc} -weighted cases with $0 < p, q \leq \infty$. In the present paper we shall establish the following results on modulation spaces. To describe the result, we make a setup.

Assume that $W : \mathbb{R}^n \rightarrow (0, \infty)$ is a measurable function with A_∞^{loc} condition: There exists $1 < P < \infty$ such that W satisfies the A_P^{loc} condition

$$\sup_{Q:\text{cube}} \left(\frac{1}{|Q|} \int_Q W(x) dx \right) \left(\frac{1}{|Q|} \int_Q W(x)^{-\frac{1}{P-1}} dx \right)^{\frac{1}{P-1}} < \infty. \tag{1.1}$$

Suppose that the parameters p, q, s satisfy

$$0 < p < \infty, 0 < q < \infty, s \in \mathbb{R}. \tag{1.2}$$

Fix a window function $\varphi \in C_c^\infty$ so that it satisfies the non-degenerate condition:

$$\int_{\mathbb{R}^n} \varphi(x) dx \neq 0, \text{supp}(\varphi) \subset [-1, 1]^n. \tag{1.3}$$

We write $\varphi_{m,x}(z) = \exp(2\pi i m \cdot z) \varphi(z - x)$ for $m \in \mathbb{Z}^n$ and $x \in \mathbb{R}^n$. We define

$$\|f : M_{p,q}^{s,W}\|_g > \left(\sum_{m \in \mathbb{Z}^n} \langle m \rangle^q \left(\int_{\mathbb{R}^n} |\langle f, \varphi_{m,x} \rangle|^p W(x) dx \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \tag{1.4}$$

for $f \in C_c^\infty$, where we write $\langle x \rangle = \sqrt{1 + |x|^2}$.

Theorem 1. *Assume (1.1) and (1.2). Then different choices of admissible φ satisfying (1.3) will yield equivalent norms. That is, if φ_1, φ_2 satisfy (1.3), then the norm equivalence*

$$\|f : M_{p,q}^{s,W}\|_{\varphi_1} \simeq \|f : M_{p,q}^{s,W}\|_{\varphi_2} \tag{1.5}$$

holds for $f \in C_c^\infty(\mathbb{R}^n)$.

In view of (1.5), we shall write $\|f : M_{p,q}^{s,W}\|$ instead of $\|f : M_{p,q}^{s,W}\|_g$.

As for this (new) modulation norm $\|f : M_{p,q}^{s,W}\|$, we have the following quantitative information.

Lemma 1. *There exist $C > 0$ and $N \in \mathbb{N}$ depending only on W and p, q, s such that*

$$|\langle f, \psi \rangle| \leq C \|f : M_{p,q}^{s,W}\| \sum_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} e^{N|x|} |\partial^\alpha \psi(x)| \tag{1.6}$$

holds for all $\psi \in C_c^\infty$.

Denote by $M_{p,q}^{s,W}$ the (abstract) completion of C_c^∞ with $\|f : M_{p,q}^{s,W}\|$. In view of (1.6), we see that $M_{p,q}^{s,W}$ is a subset of \mathcal{D}' satisfying

$$|\langle f, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} e^{N|x|} |\partial^\alpha \varphi(x)| \tag{1.7}$$

holds for all $\varphi \in C_c^\infty$.

In the present paper we shall prove the molecular decomposition suitable for $M_{p,q}^{s,W}$.

Definition 1 (Molecule, Atom). *Let $s \in \mathbb{R}$.*

1. *Suppose that $K, N \in \mathbb{N}$ are large enough and fixed. A C^K -function $\tau : \mathbb{R}^n \rightarrow \mathbb{C}$ is said to be an $(s; m, l)$ -molecule, if it satisfies*

$$|\partial^\alpha (e^{-im \cdot x} \tau(x))| \leq \langle m \rangle^{-s} e^{-N|x-l|}, \quad x \in \mathbb{R}^n$$

for $|\alpha| \leq K$.

2. *Suppose that $K, N \in \mathbb{N}$ are large enough and fixed. A C^K -function $\tau : \mathbb{R}^n \rightarrow \mathbb{C}$ is said to be an $(s; m, l)$ -atom, if it satisfies*

$$|\partial^\alpha (e^{-im \cdot x} \tau(x))| \leq \langle m \rangle^{-s} \chi_{l+[-2,2]^n}, \quad x \in \mathbb{R}^n$$

for $|\alpha| \leq K$.

3. *Also set*

$$\mathcal{M}^s := \{\{\Psi_{ml}^s\}_{m,l \in \mathbb{Z}^n} : \text{each } \Psi_{ml}^s \text{ is an } (s; m, l)\text{-molecule}\}$$

$$\mathcal{A}^s := \{\{a_{ml}^s\}_{m,l \in \mathbb{Z}^n} : \text{each } a_{ml}^s \text{ is an } (s; m, l)\text{-atom}\}.$$

Next, we introduce a sequence space $m_{p,q}$ to describe the condition of the coefficients of the molecular decomposition.

Definition 2 (Sequence space $m_{p,q}$). Let $0 < p, q \leq \infty$. Given a sequence $\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n}$, define

$$\|\lambda : m_{p,q}^W\| > \left[\sum_{m \in \mathbb{Z}^n} \left\{ \int_{\mathbb{R}^n} \left| \sum_{l \in \mathbb{Z}^n} \lambda_{ml} \chi_{l+[0,1]^n}(x) \right|^p W(x) dx \right\}^{\frac{q}{p}} \right]^{\frac{1}{q}}.$$

Here a natural modification is made when p and/or q is infinite. $m_{p,q}^W$ is the set of doubly indexed sequences $\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n}$ for which the quasi-norm $\|\lambda : m_{p,q}^W\|$ is finite.

With these definitions in mind, we present a typical result in [21].

Theorem 2. Assume (1.1) and (1.2).

1. If $\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n} \in m_{p,q}^{s,W}$ and $\{\Psi_{ml}^s\}_{m,l \in \mathbb{Z}^n} \in \mathcal{M}^s$, then

$$f := \sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot \Psi_{ml}^s \quad (1.8)$$

converges unconditionally in the topology of $M_{p,q}^{s,W}$.

2. There exists $\{\alpha_{ml}^s\}_{m,l \in \mathbb{Z}^n} \in \mathcal{A}^s$ such that any $f \in M_{p,q}^{s,W}$ admits the following decomposition:

$$f = \sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot \alpha_{ml}^s, \quad (1.9)$$

where $\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n}$ satisfies

$$\|\lambda : m_{p,q}^{s,W}\| \leq C \|f : M_{p,q}^{s,W}\| \quad (1.10)$$

for some $C > 0$ independent of f .

In the early 90's, more and more applications were found out. For example, time-frequency analysis, which is a branch of signal analysis, deals with the translation and the modulation, so that modulation spaces come into play naturally.

Also, it is remarkable that modulation spaces are applied effectively to the pseudo-differential operators by Sjöstrand, Tachizawa and many researchers [12, 14, 15, 19, 22, 23, 24, 25]. Modulation spaces are applicable to various partial differential equations. For example, Baoxiang and Chunyan used modulation spaces to investigate the KdV equation (see [3]). Recently modulation spaces can be applied even to the modeling of wireless channels and the quantum mechanics [2].

Now we describe the organization of this paper. In Section 2 we describe other weighted modulation spaces and compare them with ours. Section 3 is devoted to establishing Theorem 1 as well as Lemma 1. Section 4 is intended as the proof of Theorem 2. In Section 5 we consider the weighted modulation space $M_{p,\infty}^{s,W}$. Finally in Section 6 we present some examples.

2 Various Weighted Modulation Spaces

Based on the standard notation of signal analysis, we adopt the following notations.

$$T_a f(x) := f(x - a), \quad M_b f(x) := e^{ib \cdot x} f(x), \quad a, b \in \mathbb{R}^n, \quad f \in \mathcal{S}'.$$

Fix $\varphi \in C_c^\infty$ be a positive non-zero function. Then define

$$\|f : M_{p,q}^s\| > \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\langle f, M_y T_x \varphi \rangle|^p dx \right)^{\frac{q}{p}} \langle y \rangle^{sq} dy \right)^{\frac{1}{q}}$$

for $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Denote by $M_{p,q}^s$ the set of all tempered distributions $f \in \mathcal{S}'$ for which the norm is finite. An important observation is that the function space $M_{p,q}^s$ does not depend on the specific choices of φ . For more details we refer to [11, 18].

In general by weighted modulation norm we mean the following norm given by

$$\|f : M_{p,q}^v\| > \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\langle f, M_y T_x \varphi \rangle|^p v(x,y) dx \right)^{\frac{q}{p}} dy \right)^{\frac{1}{q}}.$$

Note that $M_{p,q}^s$ is recovered by setting $v(x,y) = \langle y \rangle^{sq}$. There are many important classes of weights.

Definition 3.

1. A weight $v : \mathbb{R}^{2n} \rightarrow [0, \infty)$ is said to be a submultiplicative, if there exists a constant $C > 0$ such that $v(x + y) \leq C v(x)v(y)$ for all $x, y \in \mathbb{R}^{2n}$.
2. A weight $v : \mathbb{R}^{2n} \rightarrow [0, \infty)$ is said to be subconvolutive, if $v^{-1} \in L^1(\mathbb{R}^{2n})$ and $v^{-1} * v^{-1} \leq c v^{-1}$ for some constant $c > 0$.
3. A weight $v : \mathbb{R}^{2n} \rightarrow [0, \infty)$ is said to satisfy the Gelfand-Raikov-Shilov condition, if

$$\lim_{n \rightarrow \infty} v(nx)^{\frac{1}{n}} = 1$$

for all $x \neq 0$.

4. A weight $v : \mathbb{R}^{2n} \rightarrow [0, \infty)$ is said to satisfy the Beurling-Domar condition, if

$$\sum_{j=1}^{\infty} \frac{\log v(nx)}{n} < \infty.$$

5. A weight $v : \mathbb{R}^{2n} \rightarrow [0, \infty)$ is said to satisfy the logarithmic integral condition, if

$$\int_{|x| \geq 1} \frac{\log v(x)}{|x|^{n+1}} dx < \infty.$$

Example 1.

1. The function $e^{\alpha|x|}$ with $\alpha \geq 0$ is a submultiplicative weight. Similarly $\langle x \rangle^\alpha$ with $\alpha \geq 0$ is a submultiplicative weight.
2. The function $\langle x \rangle^{n+\varepsilon}$ is a subconvolutive weight.

We refer to [7] for more details of the submultiplicative, moderate and subconvolutive weights not only on \mathbb{R}^n but also on locally compact abelian groups.

Proposition 1. [13] *The Bourling-Domar condition is stronger than the Gelfand-Raikov-Shilov condition.*

Proof. This is just an easy consequence of the fact that the limit of a positive summable sequence is zero. \square

In the present paper, we consider weights of the form

$$v(x, y) = W(x)\langle y \rangle^s,$$

where $s \in \mathbb{R}$ and W belongs to the class A_∞^{loc} described just below. As the example $W(x) = |x|^\alpha$, $\alpha > -n$ shows, it can happen that v fails the submultiplicative condition or subconvolutive condition. Another similar example shows that v does not necessarily satisfy the Bourling-Domar condition.

Before we go further, we recall the definition of A_p^{loc} -weights. In the sequel by a "weight", we mean a non-negative measurable function $W \in L_{\text{loc}}^1$ satisfying $0 < W < \infty$ for a.e. and we define the local maximal operator M^{loc} by

$$M^{\text{loc}}f(x) := \sup_{\substack{x \in Q \\ Q: \text{cube}, |Q| \leq 1}} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Let $1 \leq p < \infty$. Then we define

$$A_p^{\text{loc}}(W) = \begin{cases} \text{ess. sup}_{x \in \mathbb{R}^n} \frac{M^{\text{loc}}W(x)}{W(x)} & \text{if } p = 1 \\ \sup_{\substack{Q: \text{cube} \\ |Q| \leq 1}} \left(\int_Q W(x) \frac{dx}{|Q|} \right) \cdot \left(\int_Q W(x)^{\frac{1}{1-p}} \frac{dx}{|Q|} \right)^{p-1} & \text{if } 1 < p < \infty. \end{cases}$$

The quantity $A_p^{\text{loc}}(W)$ is called the A_p^{loc} -norm of W . Then it is easy to see that

$$A_p^{\text{loc}}(W) \leq A_q^{\text{loc}}(W), \quad 1 \leq q \leq p < \infty.$$

The class A_p^{loc} of weights is the set of all weights W for which the norm $A_p^{\text{loc}}(W)$ is finite. We also define

$$A_\infty^{\text{loc}} := \bigcup_{1 \leq p < \infty} A_p^{\text{loc}}.$$

We remark that $|x|^{-n+\varepsilon} \in A_1^{\text{loc}}$ for all $0 < \varepsilon < n$ and that $e^{\alpha|x|} \in A_1^{\text{loc}}$ for all $\alpha \geq 0$.

Let W be a weight. Then we define

$$\|f : L_p^W\| > \left(\int_{\mathbb{R}^n} |f(x)|^p W(x) dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.$$

Here and below we assume that $W \in A_P^{\text{loc}}$ with $1 \leq P < \infty$ for the sake of definiteness.

3 Proof of Theorem 1

Now we prove Theorem 1 and Lemma 1. Before we prove Theorem 1, we first establish an auxiliary result (Proposition 2) and then we prove Theorem 1. Proposition 2 will be strengthened after we prove Lemma 1.

3.1 An auxiliary result on maximal operators

We write

$$p_N(\psi) > \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| \leq N} \sup_{x \in \mathbb{R}^n} e^{N|x|} |\partial^\alpha \psi(x)|$$

for $\psi \in C_c^\infty$.

Proposition 2. *Let $k \in \mathbb{Z}$, $N > 0$ and $0 < \eta \leq 1$. Then we have*

$$\sup_{\substack{\psi \in C_c^\infty \\ p_N(\psi) \leq 1}} |M_k \psi * f(x)|^\eta \leq c \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} \frac{|M_l \varphi * f(x-y)|^\eta}{\langle k-l \rangle^{N\eta} e^{N\eta|y|}} dy \tag{3.11}$$

for all $f \in C_c^\infty$.

Proof. First let us consider the case when $\eta = 1$. Note that

$$\begin{aligned} & \sum_{l \in \mathbb{Z}^n} \mathcal{F} \varphi(x+l)^2 \\ &= (2\pi)^{-\frac{n}{2}} \sum_{l \in \mathbb{Z}^n} \mathcal{F}[\varphi * \varphi](x+l) \\ &= (2\pi)^{-\frac{n}{2}} \sum_{m \in \mathbb{Z}^n} \left(\int_{\mathbb{R}^n} \sum_{l \in \mathbb{Z}^n} \mathcal{F}[\varphi * \varphi](y+l) \exp(-2\pi i y \cdot m) dy \right) \exp(2\pi i x \cdot m) \end{aligned}$$

$$> \sum_{m \in \mathbb{Z}^n} \varphi * \varphi(-2\pi m) \exp(2\pi i x \cdot m) \equiv \varphi * \varphi(0)$$

from the Poisson summation formula. Consequently we obtain

$$M_k \psi * f = c_n \sum_{l \in \mathbb{Z}} M_k \psi * M_l \varphi * M_l \varphi * f. \quad (3.12)$$

Now we shall estimate each summand. First of all, a repeated integration by parts yields that for all $N > 0$ there exists $c = c_N > 0$ such that

$$|M_k \psi * M_l \varphi(y)| \leq c \langle k - l \rangle^{-N} e^{-N|y|}.$$

As a consequence we obtain

$$|M_k \psi * M_l \varphi * M_l \varphi * f(x)| \leq c \langle k - l \rangle^{-N} \int_{\mathbb{R}^n} e^{-N|y|} |M_l \varphi * f(x - y)| dy.$$

Inserting (3.12), we obtain the result when $\eta = 1$. Namely we have proved

$$|M_k \psi * f(x)| \leq c \sum_{l \in \mathbb{Z}} \langle k - l \rangle^{-N} \int_{\mathbb{R}^n} e^{-N|y|} |M_l \varphi * f(x - y)| dy \quad (3.13)$$

up to this point. Of course, the constant $c > 0$ does depend on N . Now we pass to the case when $0 < \eta < 1$. We define

$$\mathcal{M}_{N,k} f(x) := \sup_{\substack{\psi \in C_c^\infty, p_N(\psi) \leq 1 \\ y \in \mathbb{R}, l \in \mathbb{Z}}} \frac{|M_l \psi * f(x - y)|}{\langle k - l \rangle^N e^{N|y|}}.$$

Then from (3.13) we deduce

$$\begin{aligned} \mathcal{M}_{N,k} f(x) &\leq c \sup_{\substack{y \in \mathbb{R} \\ l \in \mathbb{Z}}} \left(\frac{1}{\langle k - l \rangle^N e^{N|y|}} \sum_{m \in \mathbb{Z}} \int \frac{|M_m \varphi * f(x - y - z)|}{\langle m - l \rangle^N e^{N|z|}} dy \right) \\ &\leq c \sup_{y \in \mathbb{R}} \left(\sum_{m \in \mathbb{Z}} \int \frac{|M_m \varphi * f(x - y - z)|}{\langle m - k \rangle^N e^{N|y+z|}} dz \right) \\ &\leq c \mathcal{M}_{N,k} f(x)^{1-\eta} \sum_{m \in \mathbb{Z}} \int \frac{|M_m \varphi * f(x - y)|^\eta}{\langle m - k \rangle^{N\eta} e^{N\eta|y|}} dy. \end{aligned}$$

Here we have used the Peetre inequality $\langle a + b \rangle \leq \sqrt{2} \langle a \rangle \cdot \langle b \rangle$. As a result, we obtain

$$|M_k \psi * f(x)|^\eta \leq \mathcal{M}_{N,k} f(x)^\eta \leq c \sum_{m \in \mathbb{Z}} \int \frac{|M_m \varphi * f(x - y)|^\eta}{\langle m - k \rangle^{N\eta} e^{N\eta|y|}} dy,$$

since $\mathcal{M}_{N,k} f(x) < \infty$. □

Proposition 3. *Let $W \in A_P^{\text{loc}}$ and $F : \mathbb{R}^n \rightarrow [0, \infty)$ a measurable function. Then we have*

$$\left\{ \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} F(x-y)^\eta \frac{dy}{e^{B\eta|y|}} \right)^{\frac{p}{\eta}} W(x) dx \right\}^{\frac{1}{p}} \leq C \left(\int_{\mathbb{R}^n} F(x)^p W(x) dx \right)^{\frac{1}{p}} \quad (3.14)$$

for all $p > P\eta$ and $B \gg 1$.

Proof. By replacing p/η with p , we can assume that $\eta = 1$ and $p > P$. Let $\ell \geq 1$. We denote $\chi_r = \frac{\chi_{(-r,r)^n}}{r^n}$. Then define $M_{\leq \ell}^{\text{loc}} f(x) > \sup_{r \leq \ell} \chi_r * |f|(x)$. Then there exists $\alpha > 0$ such that

$$\left(\int_{\mathbb{R}^n} M_{\leq \ell}^{\text{loc}} f(x)^p dx \right)^{\frac{1}{p}} \leq e^{\alpha \ell} \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}. \quad (3.15)$$

Indeed, this inequality is true for $\ell = 1$ by the definition of A_P^{loc} . Since $\chi_r * \chi_1 \geq \chi_{r+1}$ for $r \geq 1$, we have

$$M_{\leq k}^{\text{loc}} \leq (M_{\leq 1}^{\text{loc}})^k.$$

As a consequence, we obtain (3.15).

Once we establish (3.15), (3.14) is an easy consequence of inequality

$$\begin{aligned} \int_{\mathbb{R}^n} F(x-y)e^{-B|y|} dy &\leq \sum_{j=1}^{\infty} \int_{(-2^j, 2^j)^n} F(x-y)e^{-2^{j-1}B} dy \\ &\leq 2^n \sum_{j=1}^{\infty} e^{-2^{j-1}B} M_{\leq 2^j}^{\text{loc}} F(x). \end{aligned}$$

The proof is therefore complete. □

3.2 Proof of Theorem 1

Let $W \in A_\infty^{\text{loc}}$ throughout. Then define

$$\|f_m : l_q(L_p^W)\| > \left(\sum_{m \in \mathbb{Z}^n} \|f_m : L_p^W\|^q \right)^{\frac{1}{q}}$$

for a family of measurable functions $\{f_m\}_{m \in \mathbb{Z}^n}$. Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then the modulation norm (1.4) can be written as

$$\|f : M_{p,q}^{s,W}\| > \left(\sum_{m \in \mathbb{Z}^n} \langle m \rangle^{qs} \|M_m \varphi * f : L_p^W\|^q \right)^{\frac{1}{q}}. \quad (3.16)$$

We are now in the position of establishing Theorem 1.

By Proposition 2 we have

$$|M_k \varphi_2 * f(x)|^\eta \leq c \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} \frac{|M_l \varphi_1 * f(x-y)|^\eta}{\langle k-l \rangle^{N\eta} e^{B\eta|y|}} dy.$$

If we invoke Proposition 3, we obtain

$$\|M_k \varphi_2 * f\|_{L_p^W} \leq c \sum_{l \in \mathbb{Z}} \frac{1}{\langle k-l \rangle^{N\eta}} \|M_l \varphi_1 * f\|_{L_p^W}$$

if $\eta < P/p, N \gg 1$. Hence it follows that

$$\begin{aligned} \sum_{k \in \mathbb{Z}^n} \left(\langle k \rangle^s \|M_k \varphi_2 * f\|_{L_p^W} \right)^q &\leq c \sum_{k \in \mathbb{Z}^n} \left(\sum_{l \in \mathbb{Z}^n} \frac{\langle k \rangle^s}{\langle k-l \rangle^{N\eta}} \|M_l \varphi_1 * f\|_{L_p^W} \right)^q \\ &\leq c \sum_{l \in \mathbb{Z}^n} \left(\langle l \rangle^s \|M_l \varphi_2 * f\|_{L_p^W} \right)^q, \end{aligned}$$

which implies $\|f\|_{M_{p,q}^{s,W}} \leq c \|f\|_{M_{p,q}^{s,W} \varphi_1}$. By symmetry Theorem 1 was proved completely.

3.3 Proof of Lemma 1

Instead of dealing with $\langle f, \psi \rangle$ directly, we have only to deal with $\psi * f(0)$, which is justified by the isomorphism $\psi \rightarrow \psi(\cdot)$. Proposition 3 and a normalization yield

$$|\psi * f(0)|^\eta \leq c p_N(\psi)^\eta \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} \frac{|M_l \varphi * f(y)|^\eta}{\langle l \rangle^{N\eta} e^{N\eta|y|}} dy$$

with $0 < \eta \ll \frac{\min(p, P, 1)}{2}$.

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|M_l \varphi * f(y)|^\eta}{e^{N\eta|y|}} dy &> \int_{\mathbb{R}^n} \frac{|M_l \varphi * f(y)|^\eta W(y)^{\eta/p}}{e^{N\eta|y|} W(y)^{\eta/p}} dy \\ &\leq (\|M_l \varphi * f\|_{L_p^W})^\eta \cdot \left(\int_{\mathbb{R}^n} \left(\frac{W(y)^{-\eta/p}}{e^{N\eta|y|}} \right)^{-p/(p-\eta)} dy \right)^{\frac{p-\eta}{\eta}}. \end{aligned}$$

Since $W^{-\frac{1}{p-1}} \in A_\infty^{\text{loc}}$, we see that $W^{\frac{\eta(p-\eta)}{\epsilon}} \in A_\infty^{\text{loc}}$. Hence, if we choose $s \gg 1$, then we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} (e^{-N\eta|y|} W(y)^{-\eta/p})^{-p/(p-\eta)} dy \\ &\leq \sum_{j=1}^\infty \int_{[-2^j, 2^j]} e^{-2^{j-1} N p/(p-\eta)} W(y)^{\frac{\eta(p-\eta)}{\epsilon}} dy \\ &\leq C_s \sum_{j=1}^\infty 2^{jn} e^{-2^{j-1} N p/(p-\eta)} M_{\leq 2^j}[\chi_1](y)^s W(y)^{\frac{\eta(p-\eta)}{\epsilon}} dy \end{aligned}$$

$< \infty$.

As a consequence, Lemma 1 was proved.

We define \mathcal{S}_e as the set of all C^∞ -functions f for which the norm

$$p_N(\psi) > \sum_{\alpha \in \mathbb{Z}_+^n, |\alpha| \leq N} \sup_{x \in \mathbb{R}^n} \exp(N|x|) |\partial^\alpha \psi(x)|$$

is finite. \mathcal{S}'_e is defined as the topological dual of \mathcal{S}_e . We remark that \mathcal{S}'_e is a special case of Gelfand-Shilov spaces (see [16]).

Proposition 4. *Proposition 3 remains valid for $f \in \mathcal{S}'_e$.*

Proof. Keep to the same notation as Proposition 3. The proof does not undergo any major change until the end of the proof of Proposition 3. If $\mathcal{M}_{N,K}f(x)$ were finite, then we would obtain

$$|M_k \varphi * f(x)|^\eta \leq \mathcal{M}_{N,K}f(x)^\eta \leq c \sum_{m \in \mathbb{Z}} \int \frac{|M_m \gamma * f(x-y)|^\eta}{\langle m-k \rangle^{N\eta} e^{N\eta|y|}} dy. \tag{3.17}$$

However, this does not always work because $\mathcal{M}_{N,K}f(x)$ can be infinite. We shall show that (3.17) still holds for all $f \in \mathcal{S}'_e(\mathbb{R})$ even when $\mathcal{M}_{N,K}f(x) = \infty$. For this purpose let us assume the most right-hand side (3.17) is finite. Otherwise there is nothing to prove. Assuming that the most right-hand side (3.17) is finite, we shall establish $\mathcal{M}_{N,K}f(x) < \infty$. Since $f \in \mathcal{S}'_e(\mathbb{R})$, there exist $N_f > 0$ such that $\mathcal{M}_{N,K}f(x) < \infty$ for all $N \geq N_f$. As a consequence (3.17) holds for such N and N . From this we deduce

$$|M_k \varphi * f(x)|^\eta \leq c \sum_{m \in \mathbb{Z}} \int \frac{|M_m \gamma * f(x-y)|^\eta}{\langle m-k \rangle^{N_f \eta} e^{N_f \eta|y|}} dy. \tag{3.18}$$

The constant in (3.17) being dependent implicitly on N , c in (3.17) must be dependent on f . To emphasize this dependence, let us write this constant as c_f . Then we have

$$\begin{aligned} |M_k \varphi * f(x)|^\eta &\leq c_f \sum_{m \in \mathbb{Z}} \int \frac{|M_m \gamma * f(x-y)|^\eta}{\langle m-k \rangle^{N_f \eta} e^{N_f \eta|y|}} dy \\ &\leq c_f \sum_{m \in \mathbb{Z}} \frac{1}{\langle m-k \rangle^{N_f \eta}} \int \frac{|M_m \gamma * f(x-y)|^\eta}{e^{N_f \eta|y|}} dy \end{aligned}$$

for all N with $N \leq N_f$. As a consequence for all $N > 0$, there exists c_f such that

$$|M_k \varphi * f(x)|^\eta \leq c_f \sum_{m \in \mathbb{Z}} \int \frac{|M_m \gamma * f(x-y)|^\eta}{\langle m-k \rangle^{N\eta} e^{N\eta|y|}} dy.$$

From the definition of the maximal operator $\mathcal{M}_{N,K}f(x)$, we have

$$\mathcal{M}_{N,K}f(x) \leq c_f \sup_{y \in \mathbb{R}} \left(\sum_{m \in \mathbb{Z}} \int \frac{|M_m \gamma * f(x-y-z)|^\eta}{\langle k-l \rangle^{N\eta} \langle m-l \rangle^{N\eta} e^{N\eta(|y|+|z|)}} dz \right)$$

$$\begin{aligned} &\leq c_f \sum_{m \in \mathbb{Z}^n} \int \frac{|M_m \gamma * f(x-z)|^\eta}{\langle k-m \rangle^{N\eta} e^{N\eta|z|}} dz \\ &< \infty. \end{aligned}$$

As a consequence (3.17) holds for all $f \in \mathcal{S}'_e(\mathbb{R})$. □

4 Proof of Theorem 2

A fundamental technique in harmonic analysis is to represent functions or distributions as a linear combination of functions of an elementary form. We shall investigate the structure of weighted modulation spaces.

We refer to [1, 4, 6, 8, 9, 10, 15, 20] for the definition of the molecules and atoms for different modulation spaces.

Now we prove Theorem 2.

1. Let $N \in \mathbb{N}$ be fixed. An integration by parts yields

$$\begin{aligned} &\langle m \rangle^s \left| \sum_{l, m \in \mathbb{Z}^n} \lambda_{ml} M_k \varphi * \Psi_{ml}^s(x) \right| \\ &\leq c \sum_{l, m \in \mathbb{Z}^n} \frac{|\lambda_{ml}|}{\langle k-m \rangle^N} \exp(-N|x-l|) \\ &\leq c \sum_{j=1}^\infty \sum_{l \in \mathbb{Z}^n} \frac{e^{-Nj}}{\langle k-m \rangle^N} M_{\leq j}^{\text{loc}} \left(\sum_{m \in \mathbb{Z}^n} \lambda_{ml} \chi_{Q_m} \right) \end{aligned}$$

for some constant c depending only on N . As a result, we obtain the desired result by virtue of (3.15).

2. Note that $M_m * \varphi * M_m \varphi * \psi = c \psi$ for all $\psi \in \mathcal{S}_e$, since we have seen that $\sum_{m \in \mathbb{Z}^n} \mathcal{F} \varphi(\xi + m)^2 =: I \neq 0$. We set

$$a_{ml}(x) := \frac{1}{I} \int_{l+[0,1]^n} M_m \varphi(y) M_m \varphi * f(x-y) dy.$$

Then we have $f = \sum_{l, m \in \mathbb{Z}^n} a_{ml}$ in \mathcal{S}'_e . Since

$$M_{-m} a_{ml}(x) = \frac{1}{I} \int_{l+[0,1]^n} M_m \varphi(y) \langle f, \exp(-im \cdot (y + *)) \varphi(x-y-*) \rangle dy,$$

we have $M_m[\partial^\alpha(M_{-m} a_{ml})](x) = \frac{1}{I} \int_{l+[0,1]^n} M_m \varphi(y) M_m[\partial^\alpha \varphi] * f(x-y) dy$. Therefore, if we define

$$\lambda_{ml} > \sup_{x \in l+[-2,2]^n} \sup_{|\alpha| \leq M} |\partial^\alpha(M_{-m} a_{ml})(x)|,$$

then, by Proposition 3, we have

$$\| \{ \lambda_{ml} \}_{m,l \in \mathbb{Z}^n} : m_{p,q}^W \| \leq c \| f : M_{p,q}^{s,W} \|.$$

Hence, it follows that $f = \sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot \frac{\alpha_{ml}}{\lambda_{ml}}$ is an atomic decomposition of f . This is the desired result.

5 Weighted Modulation Space $M_{p,\infty}^{s,W}$

A minor modification of the results above will yield a theory of the function space $M_{p,\infty}^{s,W}$. We define the function space $M_{p,\infty}^{s,W}$ as follows:

Definition 4. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Assume that $W \in A_\infty^{\text{loc}}$. Then define

$$\| f : M_{p,q}^{s,W} \| > \left\{ \sum_{l \in \mathbb{Z}^n} \langle m \rangle^{qs} \left(\int_{\mathbb{R}^n} |M_m \varphi * f(x)|^p W(x) dx \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}}$$

for $f \in \mathcal{S}'_e$.

Lemma 2. Let $0 < p < \infty$, $s \in \mathbb{R}$, $W \in A_\infty^{\text{loc}}$. If ε and q satisfy

$$\varepsilon > 0, 0 < q < \infty, q\varepsilon > n.$$

then we have $M_{p,\infty}^{s,W} \hookrightarrow M_{p,q}^{s-\varepsilon,W}$.

Proof. This follows from a fundamental inequality

$$\left(\sum_{m \in \mathbb{Z}^n} \langle m \rangle^{-q\varepsilon} |a_m|^q \right)^{\frac{1}{q}} \leq \sup_{m \in \mathbb{Z}^n} |a_m| \left(\sum_{m \in \mathbb{Z}^n} \langle m \rangle^{-q\varepsilon} \right)^{\frac{1}{q}}$$

which holds for all complex sequences $\{a_m\}_{m \in \mathbb{Z}^n}$. □

The atomic decomposition theorem can be formulated as follows:

Theorem 3. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. Assume that $W \in A_\infty^{\text{loc}}$.

1. The function space $M_{p,q}^{s,W}$ does not depend on the choice of specific φ satisfying (1.3).
2. If $\lambda = \{ \lambda_{ml} \}_{m,l \in \mathbb{Z}^n} \in m_{p,q}^{s,W}$ and $\{ \Psi_{ml}^s \}_{m,l \in \mathbb{Z}^n} \in \mathcal{M}_0^s$, then

$$f := \sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot \Psi_{ml}^s \tag{5.19}$$

converges unconditionally in the topology of \mathcal{S}'_e .

3. There exists $\{a_{ml}^s\}_{m,l \in \mathbb{Z}^n} \in \mathcal{A}^s$ such that any $f \in M_{p,q}^{s,W}$ admits the following decomposition:

$$f = \sum_{m,l \in \mathbb{Z}^n} \lambda_{ml} \cdot a_{ml}^s, \quad (5.20)$$

where $\lambda = \{\lambda_{ml}\}_{m,l \in \mathbb{Z}^n}$ satisfies

$$\|\lambda : m_{p,q}^{s,W}\| \leq C \|f : M_{p,q}^{s,W}\| \quad (5.21)$$

for some $C > 0$ independent of f .

Proof. Almost all the proofs remains unchanged except for the convergence in (5.19). This will be established by Lemma 2. \square

6 Examples

Here we shall present some examples of weights.

Example 2. A weight $W_a(\xi) = \exp(a|\xi|)$, $a \in \mathbb{R}$ belongs to the class of our admissible weights. It is interesting that $M_{p,q}^{s,W_a}$ is much larger than $M_{p,q}^s = M_{p,q}^{s,W_0}$ for $a < 0$.

Example 3. If we define $W(x) = (1 + |x|^2)^{\frac{a}{2}}$, then $M_{2,2}^{s,W}$ is the weighted Sobolev space.

Proposition 5. Let $0 < p < \infty$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. If we define $W(x) = (1 + |x|^2)^{\frac{a}{2}}$, then $M_{p,q}^{s,W} \subset \mathcal{S}'$.

Proof. In analogy with Proposition 2, we can prove

$$\sup_{\substack{\psi \in C_c^\infty \\ q_N(\psi) \leq 1}} |M_k \psi * f(x)|^\eta \leq c \sum_{l \in \mathbb{Z}} \int_{\mathbb{R}^n} \frac{|M_l \varphi * f(x-y)|^\eta}{\langle k-l \rangle^{N\eta} \langle y \rangle^{N\eta}} dy \quad (6.22)$$

for all $f \in C_c^\infty$, where $q_N(\psi) = \sum_{|\alpha| \leq N} \sup_{x \in \mathbb{R}^n} \langle x \rangle^N |\partial^\alpha \psi(x)|$. Therefore, we can proceed as in the proof of Lemma 1. \square

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