

The tree of primes in a field

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ABSTRACT

The product formula of algebraic number theory connects finite and infinite primes in a stringent way, a fact, while not hard to be checked, that has never ceased to be tantalizing. We propose a new concept of prime for any field and investigate some of its properties. There are *algebraic* primes, corresponding to valuations, such that every prime contains a largest algebraic one. For a number field, this algebraic part is zero just for the infinite primes. It is shown that the primes of any field form a tree with a kind of self-similar structure, and there is a binary operation on the primes, unexplored even for the rationals. Every prime defines a topology on the field, and each *compact* prime gives rise to a unique Haar measure, playing an essential part in the product formula.

RESUMEN

La fórmula producto de la Teoría de Números Algebraicos conecta primos finitos e infinitos de una fórmula estricta, un hecho no difícil de ser verificado, es que nunca cesa de ser estudiado. Nosotros proponemos un nuevo concepto de primos para cualquier cuerpo e investigamos algunas de sus propiedades. Hay primos algebraicos, correspondientes a valuaciones, tal que todo primo contiene un primo algebraico mayor. Para un número de cuerpos, esta parte algebraica es cero solamente para primos infinitos. Es demostrado que los primos de cualquier cuerpo forman un árbol con una clase de estructura auto-similar, y

hay una operación binaria sobre los primos inexplorada incluso para los racionales. Todo primo define una topología sobre el cuerpo, y todo primo compacto da origen a una única medida de Haar, jugando rol esencial en la fórmula producto.

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Let K be a field, and let $p \neq 0$ be a polynomial in $K[x]$. Assume first that p is normed and decomposes into powers of distinct linear factors:

$$p = (x - b_1)^{e_1} \cdots (x - b_r)^{e_r}. \quad (1)$$

Then p is completely determined by the function $e: K \rightarrow \mathbb{Z}$ with $e(b_i) := e_i$ and $e(b) := 0$ for all $b \notin \{b_1, \dots, b_r\}$. Furthermore, $e(b) \geq 0$ for all $b \in K$. The same holds for rational functions $p = \frac{g}{h} \in K(x)$ if $g, h \in K[x]$ are normed and decompose into linear factors. Here e may also attain negative values in \mathbb{Z} .

To complete the picture, replace K by the projective line $\mathbb{P}^1(K) := K \cup \{\infty\}$. If $K = \mathbb{R}$, we can calculate the limit of p for $x \rightarrow \infty$: In case that $\deg p := \deg g - \deg h < 0$, we have $\lim_{x \rightarrow \infty} p(x) = 0$, and $-\deg p$ can be regarded as the multiplicity of $\infty \in \mathbb{P}^1(K)$ as a zero of p . If $\deg p > 0$, then p has a pole in ∞ , i. e. a zero with negative multiplicity $-\deg p$. Therefore, we set

$$e(\infty) := -\deg p, \quad (2)$$

and this definition makes sense for an arbitrary field K . The function e vanishes almost everywhere, and

$$\sum_{b \in \mathbb{P}^1(K)} e(b) = 0. \quad (3)$$

In general, instead of (1), a polynomial p is of the form

$$p = a \cdot p_1^{e_1} \cdots p_r^{e_r} \quad (4)$$

with $a \in K^\times$ and different irreducible polynomials p_1, \dots, p_r . Then the p_i take the rôle of the points $b_i \in \mathbb{P}^1(K)$. However, K can be extended such that every p_i splits into $\deg p_i$ linear factors. To take this into account, a weight $f(p_i) := \deg p_i$ is attached to the point p_i , i. e. p_i counts $f(p_i)$ -fold. At infinity ∞ we still have $e(\infty) = -\deg p$, and $f(\infty) := 1$. So Eq. (3) turns into

$$\sum_{i=1}^r e(p_i) f(p_i) + e(\infty) f(\infty) = 0. \quad (5)$$

In other words, every point p_i has two invariants: the multiplicity $e(p_i)$, that is, the *ramification order*, and the weight $f(p_i)$, called the *inertial degree*.

Historically, the analogy between function fields and the field \mathbb{Q} of rationals became more and more apparent by the work of Gauß, Dedekind and Artin [11]. In this vein, it was natural to ask whether a formula like (5) holds for the field \mathbb{Q} .

In fact, every $n \in \mathbb{Q}^\times$ is of the form $n = \pm p_1^{e_1} \cdots p_r^{e_r}$ with different primes p_1, \dots, p_r and integers e_i . The unit ± 1 corresponds to the unit a in formula (4). To get a multiplicative analogue of (5), we

set $e(p_i) := e_i$ and $f(p_i) := p_i^{-1}$. For the infinite prime, we set $e(\infty) := 1$ and $f(\infty) := |n|$. So we get the product formula for \mathbb{Q} :

$$\prod_{i=1}^r f(p_i)^{e(p_i)} \cdot f(\infty)^{e(\infty)} = 1. \tag{6}$$

An analogous product formula holds for algebraic number fields (see, e. g., [8], III.1.3), or more generally, for global fields [10]. Artin and Whaples [1, 2] have shown that the product formula for algebraic number fields follows by formal arguments from its validity for \mathbb{Q} , and that a similar statement holds for function fields.

For a global field K , (finite or infinite) primes can be conceived as equivalence classes of absolute values $v: K \rightarrow \mathbb{R}$, normalized in a natural way using the modulus function of the locally compact completion \widehat{K}_v (see [4], VI.9, Proposition 1). In this way, the product formula admits a natural formulation, but we feel that it remains to be surprising: it just happens to be true - still a mystery! It deeply touches the relationship between the additive and the multiplicative structure of \mathbb{Q} .

To reveal this mystery, attempts have been made to elaborate on the concept of “prime” in a field [6, 7, 5, 13]. In the present paper, we give a new definition, partly anticipated by Krull [7], which might also shed some more light upon the distinction between finite and infinite primes of an algebraic number field. Namely, we define a *prime* in a field K to be a subset $P \subset K$ with the following properties:

- (P1) $1 \notin P$.
- (P2) $a, b \in P \Rightarrow ab \in P$.
- (P3) $ab \in P \Rightarrow a \in P$ or $b \in P$.
- (P4) $\exists d \in K^\times: P + P \subset dP$.

As a consequence, the subsets aP of K with $a \in K^\times$ form a chain. There are two types of primes - we call them *algebraic* and *transcendental* - such that the algebraic primes correspond to valuation rings in K . For every prime P , there is a largest algebraic prime $Q \subset P$ (Proposition 7.2); we denote it by P_\circ . If $P_\circ = 0$, we call P *purely transcendental*. For an algebraic number field K , the purely transcendental primes just correspond to the infinite primes.

There are unexpected properties of our concept of prime. As is well-known, the valuation rings of a field form a dual tree under inclusion. More generally, we show that all primes of a field K form a tree which is almost self-similar (section 2). Namely, we introduce a binary operation $P * Q$ on the primes which satisfies $P \subset P * Q$ (Proposition 2.4). Thus even for $K = \mathbb{Q}$, there are a lot more primes $P \subset \mathbb{Q}$ than rational prime numbers. A further investigation of this operation might be desirable.

Two primes define the same topology on K if and only if their intersection is non-zero (Proposition 3.3). Up to a constant, we associate a unique *pseudo-valuation* $v: K \rightarrow [0, \infty]$ to any prime of K (Proposition 3.4). This means that $v(a) = 0$ may also occur for non-zero $a \in K$. For a minimal (non-zero) prime P , the corresponding pseudo-valuation is of the form v^t with $t > 0$ for some absolute value v , and then P can be recovered from v . Every prime P makes K into a separated topological field such that the completion \widehat{K}_P is again a topological field. If P is open in its topology, then P is induced by a prime of \widehat{K}_P (Theorem 3.1).

We introduce *compact* primes, which are in a sense totally bounded (Definition 4.1). They always contain a minimal prime which is again compact. For example, all the primes of \mathbb{Q} - not only the minimal ones - are compact. To any compact prime in any field, we associate a Haar integral and

show that it admits a natural normalization (Theorem 6.1). For the minimal primes of \mathbb{Q} or $\mathbb{F}_p[x]$, this normalization corresponds to the standard normalization of absolute values. So the product formula becomes a natural statement on the minimal primes of certain fields.

A special case of our concept of prime already occurs in a paper of W. Krull [7]. Such primes are open in their topology, and we call them *Krull primes*. We show that every minimal prime is a Krull prime, and that every prime P of a field K contains a largest Krull prime P° (Proposition 3.1). There is a close relationship between algebraic primes and Krull primes. Namely, we show that $P_\circ \subset P^\circ$, so that there is no other prime between P_\circ and P° . Therefore, every compact Krull prime is either algebraic or purely transcendental (Proposition 7.5). Furthermore, a non-zero purely transcendental prime is a Krull prime if and only if it is minimal. The same holds for compact primes (Corollary of Proposition 7.6).

1 Primes

The two examples of the introduction show that up to units, the elements of certain fields are given by their values at minimal “primes”, where a relation between these values is given by a product formula (6). Let us first give a general definition of a prime.

For subsets P, Q of a field K and $c \in K$, define $P + Q := \{a + b \mid a \in P, b \in Q\}$ and $PQ := \{ab \mid a \in P, b \in Q\}$. If $P = \{a\}$, we simply write $a + Q := \{a\} + Q$ and $aQ := \{a\}Q$. The unit group of a ring R will be denoted by R^\times .

Definition 1.1. Let K be a field. We call a non-empty subset $P \subset K$ a *prime* of K if the following are satisfied ($\forall a, b \in K$):

- (P1) $1 \notin P$.
- (P2) $a, b \in P \Rightarrow ab \in P$.
- (P3) $ab \in P \Rightarrow a \in P$ or $b \in P$.
- (P4) $\exists d \in K^\times : P + P \subset dP$.

From (P1) and (P2), we get $-1 \notin P$, since $(-1)(-1) = 1$. Therefore, $(-1)(-a) = a$ gives

$$a \in P \Leftrightarrow -a \in P \tag{7}$$

by virtue of (P3). By (P4), every $a \in P$ satisfies $0 = a - a \in P + P \subset dP$ for some $d \in K^\times$. Hence

$$0 \in P. \tag{8}$$

It is easily checked that $P = 0 := \{0\}$ is a prime. We call it the *trivial* prime. By (P1) and (P2), a finite field has only the trivial prime. For a prime P , we call

$$\tilde{P} := \{a \in K \mid aP \subset P\} \tag{9}$$

the *hull* and

$$\partial P := \tilde{P} \setminus P \tag{10}$$

the *boundary* of P . The trivial prime 0 satisfies $\tilde{0} = K$ and $\partial 0 = K^\times$.

Proposition 1.1. *Let K be a field with a prime P . The $a\tilde{P}$ with $a \in K^\times$ form a chain.*

Proof. For $a, b \in K^\times$, assume that $a\tilde{P} \not\subset b\tilde{P}$. Then $ac \notin b\tilde{P}$ for some $c \in \tilde{P}$. For every $p \in P$, this gives $acb^{-1} \cdot ba^{-1}p = cp \in P$, hence $ba^{-1}p \in P$ by (P3). Therefore, $ba^{-1} \in \tilde{P}$, and thus $b\tilde{P} = a \cdot ba^{-1}\tilde{P} \subset a\tilde{P}$. \square

Proposition 1.2. *Let K be a field with a prime P . Then*

$$\partial P = \{a \in K^\times \mid aP = P\} = \{a \in K^\times \mid a, a^{-1} \notin P\}. \quad (11)$$

Proof. Assume that $a \in \partial P$. Then $a \neq 0$ and $aP \subset P$. For $b \in P$, we have $a \cdot a^{-1}b = b$. Thus (P3) gives $a^{-1}b \in P$. Therefore, $b \in aP$, which yields $P \subset aP \subset P$. Now let $a \in K^\times$ with $aP = P$ be given. Then $aa^{-1} = 1 \notin P$ implies $a^{-1} \notin P$. Similarly, $a^{-1}P = P$, hence $a \notin P$. Thus $\{a \in K \mid aP = P\} \subset \{a \in K^\times \mid a, a^{-1} \notin P\}$. Finally, assume that $a \in K^\times$ with $a, a^{-1} \notin P$. For $b \in P$, this gives $a^{-1} \cdot ab = b$, hence $ab \in P$ by (P3). Consequently, $a \in \tilde{P} \setminus P = \partial P$. \square

Note that (11) implies that ∂P is a subgroup of K^\times , and

$$a \notin \tilde{P} \Leftrightarrow a^{-1} \in P \quad (12)$$

for all $a \in K^\times$, that is, P and \tilde{P} determine each other. Furthermore, $-1 \in K \setminus P$, and thus $-1 \in \partial P$.

Definition 1.2. We call a prime P of a field K *algebraic* if $P + P \subset P$, that is, (P4) holds for $d = 1$, otherwise *transcendental*.

If P is algebraic, then $(\tilde{P} + \tilde{P})P \subset \tilde{P}P + \tilde{P}P \subset P + P \subset P$, which gives $\tilde{P} + \tilde{P} \subset \tilde{P}$. Thus \tilde{P} is a subring of K , and by (12), it follows that K is the quotient field of \tilde{P} . By Proposition 1.1, the principal ideals of \tilde{P} form a chain. Therefore, all ideals of \tilde{P} form a chain. Namely, if $I \not\subset J$ are ideals of \tilde{P} , there is some $a \in I \setminus J$. For all $b \in J$, this gives $a\tilde{P} \not\subset b\tilde{P}$, hence $b\tilde{P} \subset a\tilde{P} \subset I$. Furthermore, P is an ideal of \tilde{P} , and by (11) and (12), $\partial P = (\tilde{P})^\times$. Thus P is the unique maximal ideal of \tilde{P} . In other words, \tilde{P} is a valuation domain.

Example. For $K = \mathbb{Q}$, every rational prime p gives rise to an algebraic prime

$$P_p = \left\{ \frac{pa}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z}, p \nmid b \right\}. \quad (13)$$

By (12), we get $\tilde{P}_p = \mathbb{Z}_p$, whence

$$\tilde{P}_p/P_p = \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}. \quad (14)$$

Moreover, \mathbb{Q} has a transcendental prime

$$P_\infty := (-1, 1) \cap \mathbb{Q} = \{a \in \mathbb{Q} \mid |a| < 1\}. \quad (15)$$

Here (P4) holds for $d = 2$.

2 The primes of \mathbb{Q}

By (12), two primes $P \supset Q$ of a field K satisfy

$$Q \subset P \subset \tilde{P} \subset \tilde{Q}, \quad (16)$$

hence $cP \subset c\tilde{Q} \subset Q$ for all $c \in Q$. More precisely:

Proposition 2.1. *Let K be a field with a prime P . For any submonoid U of K^\times with $\tilde{P} \subset U$,*

$$Q := \{0\} \cup \{a \in K^\times \mid a^{-1} \notin U\} \quad (17)$$

is a prime with $Q \subset P$ and $\tilde{Q} = U$. Every prime $Q \subset P$ is of this form.

Proof. Let U be a submonoid of K^\times , and $\tilde{P} \subset U$. By (12), this implies that $Q \subset P$. Assume that $a, b \in Q \cap K^\times$. Then $a^{-1}, b^{-1} \notin U$. So $b^{-1} \notin \tilde{P}$, and therefore, $b \in P \subset U$. Now $ab \notin Q$ would yield $a^{-1}b^{-1} \in U$, whence $a^{-1} = a^{-1}b^{-1} \cdot b \in U$, which is impossible! Thus (P1) and (P2) hold for Q .

To prove (P3), we have to verify the implication $a, b \notin Q \Rightarrow ab \notin Q$, that is, $a^{-1}, b^{-1} \in U \Rightarrow a^{-1}b^{-1} \in U$. This follows since U is multiplicatively closed.

Finally, $Q + Q \subset P + P \subset dP$ holds for some $d \in K^\times$. To prove (P4), we can assume that $Q \neq 0$. For arbitrary $a \in K \setminus U$ and $b \in P \cap K^\times$, we have $a \notin \tilde{P}$. Thus $a^{-1} \in P$. So $b^{-1}a \in U$ would give $a \in bU \subset U$, a contradiction. Hence $b^{-1}a \notin U$, and thus $ba^{-1} \in Q$. So we get $P \subset aQ$, i. e. $Q + Q \subset daQ$, which completes the proof of (P4).

Conversely, let $Q \subset P$ be a prime. Then (12) shows that Q is of the form (17) with a submonoid $U := \tilde{Q}$ of K^\times . \square

Corollary. *For each pair P, Q of primes of a field K , there is an infimum $P \wedge Q$.*

Proof. By Proposition 2.1, the submonoid $\tilde{P}\tilde{Q}$ corresponds to a prime $P \wedge Q$ with the desired property. \square

Definition 2.1. We say that two primes P and Q of a field K are *dependent* if $P \wedge Q \neq 0$, otherwise *independent*. The minimal among the non-zero primes are called *minimal primes*.

Minimal primes can be characterized as follows.

Proposition 2.2. *A non-trivial prime P of a field K is minimal if and only if for every pair $a, b \in P \cap K^\times$, there exists some $n \in \mathbb{N}$ with $a^n \in bP$.*

Proof. Assume that $a, b \in P \cap K^\times$. Then $U := \bigcup_{n \in \mathbb{N}} a^{-n}P$ is a submonoid of K^\times , and $\tilde{P} \subset a^{-1}P \subsetneq U$. Suppose that $a^n \notin bP$ for all $n \in \mathbb{N}$. Then $b^{-1} \notin U$. By Proposition 2.1, there is a non-trivial prime $Q \subsetneq P$ with $\tilde{Q} = U$.

Conversely, let Q be a prime with $0 \neq Q \subsetneq P$. Choose $a \in P \setminus Q$ and $b \in Q \cap K^\times$. Then $a \notin Q$ implies that $a^{-1} \in \tilde{Q}$, hence $a^{-n} \in \tilde{Q}$ for all $n \in \mathbb{N}$. Furthermore, $bP \subset b\tilde{P} \subset b\tilde{Q} \subset Q$, and thus $a^{-n}bP \subset a^{-n}Q \subset Q \subset P$. Hence $a^{-n}b \in \tilde{P}$, that is, $a^n b^{-1} \notin P$ for all n . Thus $a^n \notin bP$ holds for $n \in \mathbb{N}$. \square

Now we are able to give an explicit description of the minimal primes of \mathbb{Q} . To this end, we need two auxiliary results.

Lemma 2.1. *Let P be a prime of \mathbb{Q} , such that $P + P \subset dP$ for some $d \in \mathbb{Q}^\times$. Then $nP \subset d^k P$ holds for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ with $|n| \leq 2^k$.*

Proof. For $k = 0$, this is trivial. If the assertion holds for some $k \in \mathbb{N}$, then every $n \in \mathbb{Z}$ with $|n| \leq 2^{k+1}$ can be written as a sum $n = r + s$ in \mathbb{Z} such that $|r|, |s| \leq 2^k$. Hence $nP \subset rP + sP \subset d^k P + d^k P \subset d^{k+1} P$. \square

Lemma 2.2. *Let P be a prime of \mathbb{Q} , and let $a > b$ be relatively prime in \mathbb{N} , such that $a(P+P) \subset bP$. Then*

$$na^a P \subset b^{a+k-1} P \tag{18}$$

holds for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}$ with $|n| < a^k$.

Proof. We proceed by induction. For $k = 0$, the statement (18) is trivial. Suppose that (18) is shown for some $k \in \mathbb{N}$. We prove (18) for $k + 1$. Assume that $|n| < a^{k+1}$. Since a and b are relatively prime, there are $m, r \in \mathbb{Z}$ with

$$n = ma + rb^k.$$

If r is replaced by $r + a$, the equation holds for $m - b^k$ instead of m . So we can assume that $|r| < a$. For $r \neq 0$, there are still two possibilities. We choose r such that the signs of n and r are not different. Then $|m| = \frac{1}{a}|n - rb^k| < a^k$. So the inductive hypothesis gives $ma^a P \subset b^{a+k-1} P$, hence $na^a P \subset ma \cdot a^a P + rb^k \cdot a^a P \subset b^{a+k-1} aP + rb^k a^a P$. Since $|r| \leq a - 1 < 2^{a-1}$, Lemma 2.1 implies that $rP \subset (\frac{b}{a})^{a-1} P$. Therefore, we get $na^a P \subset b^{a+k-1} aP + b^{a+k-1} aP \subset b^{a+k-1} bP = b^{a+k} P$. \square

As a consequence of Lemma 2.2, we have

Lemma 2.3. *Let P be a prime of \mathbb{Q} , and assume that $1 < c \in P$. Then there exists an integer $e > 1$ with $\mathbb{Z}e \subset P$ and $e(P+P) \subset P$.*

Proof. By (P4), there is some $d \in \mathbb{Q}^\times$ with $P+P \subset dP$. Choose $i \in \mathbb{N}$ such that $c^i > |d|$. Then $c^i \cdot |d|^{-1} = \frac{a}{b}$ with relatively prime $a, b \in \mathbb{N}$. Thus $a > b$ and $a(P+P) = bc^i d^{-1}(P+P) \subset bc^i P \subset bP$. For every $m \in \mathbb{Z}$, there exists some $k \in \mathbb{N}$ with $|m| \cdot b^a < (\frac{a}{b})^k$. We set $n := m \cdot b^{a+k}$. Then $|n| = |m| \cdot b^{a+k} < a^k$. By Lemma 2.2, this implies that $na^a P \subset b^{a+k-1} P$, that is, $ma^a bP \subset P$ for all $m \in \mathbb{Z}$, and thus $\mathbb{Z}a^{a+1} = \mathbb{Z}a^a b \cdot \frac{a}{b} \subset P$. Furthermore, $a^{a+1}(P+P) = a^a \cdot a(P+P) \subset a^a bP \subset P$. \square

Now we show that the list of minimal primes of \mathbb{Q} given in (13) and (15) is complete. The first part of the following theorem is essentially due to Ostrowski [9] (cf. [12]).

Theorem 2.1. *Every minimal prime P of \mathbb{Q} is of the form $P = P_p$ with a rational prime p or $p = \infty$, and every prime of \mathbb{Q} contains a minimal prime.*

Proof. By Proposition 2.2, the prime P_∞ is minimal. Therefore, if $|c| < 1$ for all $c \in P$, then $P \subset P_\infty$, and thus $P = P_\infty$. Otherwise, there exists some $c \in P$ with $c > 1$. By Lemma 2.3, there is an integer $e > 1$ with $\mathbb{Z}e \subset P$ and $e(P+P) \subset P$. Assume that $e = p_1 \cdots p_r$ with rational primes p_i . Suppose that $ep_i^{-r} \in P$ holds for all i . Then $1 = (ep_1^{-r}) \cdots (ep_r^{-r}) \in P$, a contradiction. Hence $ep^{-r} \notin P$ for some prime $p|e$. This implies that $e^{-1}p^r \in \tilde{P}$. For every $n \in \mathbb{Z}$, we thus get $(np)^r = n^r e \cdot e^{-1}p^r \in P\tilde{P} \subset P$. Therefore, $np \in P$, and thus $\mathbb{Z}p \subset P$.

Now let $p \nmid n$ for some $n \in \mathbb{Z}$. Then there are $u, v \in \mathbb{Z}$ with $un + vp^{r+1} = 1$. So the residue class $n + p^{r+1}\mathbb{Z}$ is invertible in $\mathbb{Z}/p^{r+1}\mathbb{Z}$. Therefore, we find some $k \in \mathbb{N} \setminus 0$ with $n^k \equiv 1 \pmod{p^{r+1}}$. With $s := (r+1)k$, we also have $n^s \equiv 1 \pmod{p^{r+1}}$, and thus $1 = n^s + mp^{r+1}$ for some $m \in \mathbb{Z}$. Suppose that $n^s \in p^r P$. Then $1 \in p^r P + p^r P = e^{-1}p^r \cdot e(P+P) \subset \tilde{P}P \subset P$, a contradiction. Hence $n^s p^{-r} \notin P$, and thus $n^{-s} p^r \in \tilde{P}$. So we get $(n^{-1}p)^s = n^{-s} p^{(r+1)k} \in n^{-s} p^r P \subset P$. This gives $n^{-1}p \in P$, whence $P_p \subset P$. \square

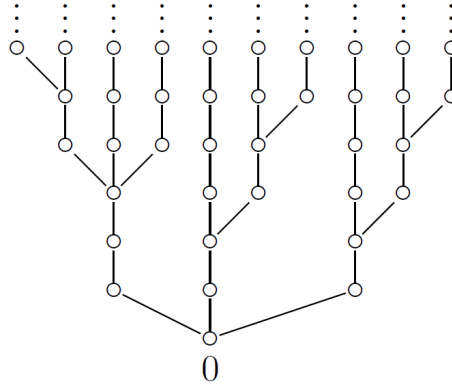
Remark. There exist non-minimal primes in \mathbb{Q} . For example,

$$P := 2\mathbb{Z}_2 \cup (\mathbb{Z}_2 \cap 3\mathbb{Z}_3)$$

is a prime. In fact, this is a general phenomenon. We show first that the primes form a tree with the trivial prime as a root:

Proposition 2.3. *Let K be a field with a prime P . Then the primes $Q \subset P$ form a chain.*

Proof. Let Q, Q' be primes with $Q, Q' \subset P$ and $Q' \not\subset Q$. Then there is an element $a \in Q' \setminus Q$. For $b \in Q$, the inclusions (16) give $b\tilde{P} \subset b\tilde{Q} \subset Q$. Hence $a\tilde{P} \not\subset b\tilde{P}$. Therefore, Proposition 1.1 yields $b \in b\tilde{P} \subset a\tilde{P} \subset a\tilde{Q}' \subset Q'$, and thus $Q \subset Q'$. \square



Moreover, the tree of primes has a strong symmetry:

Proposition 2.4. *Let K be a field with two primes P and Q . Then*

$$P * Q := P \cup (\tilde{P} \cap Q)$$

is a prime of K .

Proof. Obviously, $1 \notin P * Q$. Assume that $a, b \in P * Q$. If $a \in P$ or $b \in P$, then $ab \in P \subset P * Q$ since $a, b \in \tilde{P}$. For $a, b \in \tilde{P} \cap Q$, however, we have $ab \in \tilde{P} \cap Q$. This proves (P2). To verify (P3), let $a, b \in K$ with $ab \in P * Q$ be given. If $ab \in P$, then $a \in P$ or $b \in P$. Therefore, assume that $ab \in \tilde{P} \cap Q$ and $a, b \notin P$. Then $a^{-1}, b^{-1} \in \tilde{P}$. Hence $a = ab \cdot b^{-1} \in \tilde{P}$, and similarly, $b \in \tilde{P}$. Therefore, $a \in \tilde{P} \cap Q$ or $b \in \tilde{P} \cap Q$. Finally, assume that $d \in K^\times$ with $P + P \subset dP$. If $P = 0$, we have $P * Q = Q$. Otherwise, there exists some $a \in P \setminus 0$. Now $(\tilde{P} + \tilde{P})P \subset P + P \subset dP$, hence $(\tilde{P} + \tilde{P})d^{-1} \subset \tilde{P}$. Consequently, $(P * Q) + (P * Q) \subset \tilde{P} + \tilde{P} \subset d\tilde{P} = da^{-1}a\tilde{P} \subset da^{-1}(P * Q)$. \square

We call $P * Q$ the *lexicographic product* of P and Q . Thus for a fixed P , every prime Q yields a prime $P * Q \supset P$ which may coincide with P . Apart from this exception, the tree of primes is self-similar, i. e. the whole tree sits above each prime.

3 The topology of a prime

Let K be a field with a prime P . For $a, b \in K$, we define

$$a \leq b : \Leftrightarrow a\tilde{P} \subset b\tilde{P} \tag{19}$$

and $a < b$ if $a \not\leq b$. Then

$$\begin{aligned} a \leq b \leq c &\Rightarrow a \leq c \\ a \leq b &\Rightarrow ac \leq bc \end{aligned} \tag{20}$$

holds for $a, b, c \in K$, and Proposition 1.1 yields

$$a \leq b \text{ oder } b \leq a. \tag{21}$$

In particular, this implies the reflexivity $a \leq a$ for all $a \in K$. By (11), we have $aP = bP \Leftrightarrow b^{-1}aP = P \Leftrightarrow b^{-1}a \in \partial P$ for $a, b \in K^\times$. Therefore, (19) induces a linear order on $K^\times/\partial P$. From (19) and (12), we get

$$\tilde{P} = \{a \in K \mid a \leq 1\}; \quad P = \{a \in K \mid a < 1\}. \tag{22}$$

For $a \in K$ and $c \in K^\times$, we define the c -neighbourhood of a as follows:

$$U_c(a) = U_c^P(a) := a + cP. \tag{23}$$

A subset $U \subset K$ is *open* with respect to P if for every $a \in U$, there is an element $c \in K^\times$ with $U_c(a) \subset U$. The P -open sets define a topology on K , but it is not clear in advance that the c -neighbourhoods (23) are neighbourhoods in this topology. The following proposition shows that this is in fact the case.

Proposition 3.1. *Let K be a field with a prime P . The set*

$$P^\circ := \{a \in P \mid \exists n \in \mathbb{N} : a^n(P + P) \subset P\} \tag{24}$$

is open. If P is minimal, then $P^\circ = P$.

Proof. Assume that $a \in P^\circ$, say, $a^n(P + P) \subset P$. Then $a^{2n}(P + P + P + P) \subset a^n(P + P) \subset P$, and by induction, we get

$$a^{mn}(P + \dots + P) \subset P$$

for $m \in \mathbb{N}$, where $P + \dots + P$ may have at most 2^m summands. We set $b := a^{2n(n+1)}$. Now let $p \in P$ be arbitrary. Expanding $(a + bp)^{2n}$ gives 2^{2n} summands, namely, a^{2n} plus $2^{2n} - 1$ summands in bP . So we get

$$\begin{aligned} (a + bp)^{2n}(P + P) &\subset (a^{2n} + ba^{-2n^2}P)(P + P) \subset (a^{2n} + a^{2n}P)(P + P) \\ &\subset a^{2n}(P + P) + a^{2n}(P + P) \subset a^nP + a^nP \subset P. \end{aligned}$$

This yields $a + bP \subset P^\circ$. Finally, let P be minimal and $d \in K^\times$ with $P + P \subset dP$. Then $P \subset dP$, hence $d^{-1} \in \tilde{P}$. For $a \in P$, Proposition 2.2 yields an $n \in \mathbb{N}$ with $a^n \in ad^{-1}P$, whence $a^n(P + P) \subset ad^{-1}(P + P) \subset aP \subset P$. Thus $a \in P^\circ$. \square

In section 7, we will show that P° is again a prime.

Corollary. *Let K be a field with a minimal prime P . For every $a \in P$, there are $b, c \in P \cap K^\times$ such that $a\tilde{P} + bP \subset c\tilde{P}$.*

Proof. Assume that $P + P \subset dP$ with $d \in K^\times$ and $a \in P$. By Proposition 3.1, there is some $b \in d^{-1}P \cap K^\times$ with $a + bdP \subset P$. We show first that $a\tilde{P} + bP \subset P$. To this end, assume that $e \in \tilde{P}$.

Case I: $e \in d^{-1}P$. Then $ae + bP \subset d^{-1}P + d^{-1}P \subset P$.

Case II: $e \notin d^{-1}P$. Then $de \notin P$, hence $d^{-1}e^{-1} \in \tilde{P}$. This gives $ae + bP = e(a + bdd^{-1}e^{-1}P) \subset e(a + bdP) \subset P$.

If $a\tilde{P} = P$, we can set $c := a$. Otherwise, there exists some $c \in P \setminus a\tilde{P}$. Then $ca^{-1} \notin \tilde{P}$, hence $c^{-1}a \in P$. By the above, we find an element $b \in d^{-1}P \cap K^\times \subset P \cap K^\times$ with $c^{-1}a\tilde{P} + bP \subset P$. Whence $a\tilde{P} + cbP \subset cP \subset c\tilde{P}$. \square

Lemma 3.1. *Let K be a field with a prime P . Assume that $d \in K^\times$ with $P + P \subset dP$. Then every $a, b \in K^\times$ with $b \in ad^{-1}P$ satisfies*

$$\forall p \in P: (a + bp)^{-1} - a^{-1} \in da^{-2}bP.$$

Proof. Assume that $p \in P$. Then $bp \in aP$ implies that $a + bp \neq 0$. Suppose that $(1 + a^{-1}bp)d \in P$. Then $d = (1 + a^{-1}bp)d - a^{-1}bpd \in P + P \subset dP$, a contradiction. Hence $(1 + a^{-1}bp)^{-1}d^{-1} \in \tilde{P}$. From $(1 + a^{-1}bp)^{-1}(1 + a^{-1}bp) = 1$ we get

$$(1 + a^{-1}bp)^{-1} + (1 + a^{-1}bp)^{-1}a^{-1}bp = 1.$$

Multiplying by a^{-1} gives $(a + bp)^{-1} + (1 + a^{-1}bp)^{-1}a^{-2}bp = a^{-1}$, i. e. $(a + bp)^{-1} - a^{-1} = -(1 + a^{-1}bp)^{-1}a^{-2}bp \in d\tilde{P} \cdot a^{-2}bP \subset da^{-2}bP$. \square

Theorem 3.1. *Let K be a field with a prime P . The topology of P makes K into a separated topological field, and the completion \hat{K}_P of K with respect to P is a topological field. If P is open, then*

$$\hat{P} := \bigcup_{a \in P} \overline{a\tilde{P}} \quad (25)$$

is a prime in \hat{K}_P with $\hat{P} \cap K = P$, and \hat{P} defines the topology of \hat{K}_P .

Proof. By (P4), there is some $d \in K$ with $P + P \subset dP$. This implies that K is a topological ring. Hence K is a topological field by Lemma 3.1.

To show that the same holds for the completion \hat{K} of K , we have to verify that the image under the mapping $x \mapsto x^{-1}$ of every Cauchy filter \mathfrak{C} in K which does not have a cluster point at 0, is a Cauchy filter (see [3], III.6.8, Proposition 7). Thus let $c \in K^\times$ and $C \in \mathfrak{C}$ be such that $C \cap cP = \emptyset$. We can assume that $a - b \in cd^{-1}P$ for all $a, b \in C$. Choose any $a \in C$. Suppose that $b \in C \cap cd^{-1}P$. Then $a = (a - b) + b \in cd^{-1}P + cd^{-1}P \subset cP$, a contradiction. Hence $C \cap cd^{-1}P = \emptyset$. This gives $ac^{-1}d \notin P$, and thus $a^{-1}cd^{-1} \in \tilde{P}$, i. e. $a^{-1} \in c^{-1}d\tilde{P}$. For an arbitrary $e \in K^\times$, we find some $C' \subset C$ in \mathfrak{C} with $a - b \in c^2d^{-2}eP$ for all $a, b \in C'$. This gives $a^{-1} - b^{-1} = a^{-1}b^{-1}(b - a) \in c^{-2}d^2\tilde{P} \cdot c^2d^{-2}eP \subset eP$ for $a, b \in C'$.

Now let P be open. We show first that for $a \in P$,

$$\overline{a\tilde{P}} \subset P \quad (26)$$

holds in K . For $a = 0$, this is trivial. So we assume that $a \in P \cap K^\times$. Let $b \notin P$ be given. Then $b^{-1} \in \tilde{P}$, which gives $(a^{-1}b)^{-1} = b^{-1}a \in P$. Since P is open, the continuity of $x \mapsto x^{-1}$ implies that $(a^{-1}b + cP)^{-1} \subset P$ for some $c \in K^\times$. Hence $(a^{-1}b + cP) \cap \tilde{P} = \emptyset$, and thus $(b + acP) \cap a\tilde{P} = \emptyset$. This proves (26). Together with (25), we infer that $\hat{P} \cap K = P$.

Furthermore, (26) shows that \widehat{K}_P satisfies (P1). If $b, c \in \widehat{P}$, then $b, c \in \overline{a\widetilde{P}}$ for some $a \in P$. Hence $bc \in \overline{a\widetilde{P}} \cdot \overline{a\widetilde{P}} \subset \overline{a^2\widetilde{P}} \subset \widehat{P}$, which gives (P2). Similarly, (P4) is easily verified.

To prove (P3), assume that $b, c \in \widehat{K}_P$ satisfy $bc \in \widehat{P}$ and $b \notin \widehat{P}$, say, $bc \in \overline{a\widetilde{P}}$ for some $a \in P$. Then $b \notin \overline{p\widetilde{P}}$ for all $p \in P$. For any $p \in P$, we find a neighbourhood U of 0 in \widehat{K}_P such that $(b+U) \cap p\widetilde{P} = \emptyset$. Hence $(bc+Uc) \cap pc\widetilde{P} = \emptyset$. On the other hand, $(bc+Uc) \cap a\widetilde{P} \neq \emptyset$. Thus for any $p \in P$, we get $a\widetilde{P} \not\subset pc\widetilde{P}$. If $p \neq 0$, this implies that $p^{-1}c^{-1}a \notin \widetilde{P}$. Hence $pc a^{-1} \in P$ for all $p \in P$. Consequently, $ca^{-1} \in \widetilde{P}$, and thus $c \in a\widetilde{P} \subset \widehat{P}$.

It remains to verify that \widehat{P} defines the topology of \widehat{K}_P . For $a \in P$, we have $\overline{a\widetilde{P}} \subset \widehat{P}$. Conversely, (26) implies that $\widehat{P} \subset \overline{a\widetilde{P}}$. This completes the proof. \square

In particular, Theorem 3.1 implies that the limit of a sequence in K with respect to a prime P is unique. A minimal prime P is uniquely determined by its topology:

Proposition 3.2. *Let K be a field with a minimal prime P . Then*

$$P = \{a \in K \mid \lim_{n \rightarrow \infty} a^n = 0\}. \quad (27)$$

Proof. Let $a \in P$. By Proposition 2.2, every $c \in K^\times$ gives rise to some $N \in \mathbb{N}$ with $a^N \in cP$. Therefore, $a^n \in cP$ for all $n \geq N$, and thus $\lim_{n \rightarrow \infty} a^n = 0$. On the other hand, if $a \notin P$, then $a^{-1} \in \widetilde{P}$, which gives $a^{-n} \in \widetilde{P}$ for all $n \in \mathbb{N}$. But this implies that $a^n \notin P$ for all n . \square

If a prime contains a minimal prime P , then P can be determined topologically by means of Proposition 3.2. The following proposition decides whether two primes define the same topology.

Proposition 3.3. *For two primes $P, Q \neq 0$ of a field K , the following are equivalent.*

- (a) $PQ \neq K$.
- (b) $\widetilde{P}\widetilde{Q} \neq K$.
- (c) P and Q are dependent (see Definition 2.1).
- (d) There is some $a \in K^\times$ with $aP \subset Q$.
- (e) P and Q define the same topology.

Proof. (a) \Rightarrow (b): Suppose that $\widetilde{P}\widetilde{Q} = K$. For $a \in P \cap K^\times$ and $b \in Q \cap K^\times$, this implies that $K = ab\widetilde{P}\widetilde{Q} \subset PQ$, which is impossible. The implication (b) \Rightarrow (c) follows by Proposition 2.1.

(c) \Rightarrow (d): For every $a \in (P \wedge Q) \cap K^\times$, we have $aP \subset P \wedge Q \subset Q$.

(d) \Rightarrow (a): From $aP \subset Q$ with $a \in K^\times$, we get $aPQ \subset Q^2 \subset Q$, hence $PQ \subset a^{-1}Q \neq K$. Therefore, condition (d) is symmetric. It states that every Q -open set with respect to P is open. Hence (d) \Leftrightarrow (e). \square

Let K be a field with a prime P . A function $f: K \rightarrow [0, \infty]$ is said to be *monotonous* (with respect to P) if

$$a \leq b \Rightarrow f(a) \leq f(b) \quad (28)$$

holds for all $a, b \in K$. We call f *strictly monotonous* if (28) can be replaced by an equivalence. For the following definition, we set

$$\infty \cdot a = a \cdot \infty := \infty \quad (29)$$

for $0 < a \leq \infty$.

Definition 3.1. Let K be a field with a prime P . We call $v: K \rightarrow [0, \infty]$ a *pseudo-valuation* with respect to P if v is monotonous with $v(0) = 0$ and $v(1) = 1$, such that

$$v(ab) = v(a)v(b) \quad (30)$$

holds for all $a, b \in K$ with $v(a) \notin \{0, \infty\}$. We call v *finite* if $v(a) < \infty$ for all $a \in K$. If v is finite and strictly monotonous, we say that v is a P -*valuation*.

For $a \in K^\times$, the definition yields

$$v(a) = 0 \Leftrightarrow \overline{v(a^{-1})} = \infty. \quad (31)$$

Note that by (22), the prime P of a P -valuation v is given by $P = \{a \in K \mid v(a) < 1\}$. By [4], VI.6.1, a P -valuation is of the form v^t , where $t > 0$ and v is an absolute value on K .

Proposition 3.4. *Let K be a field with a prime P . For every $c \in K \setminus \tilde{P}$ and every real number $r > 1$, there is exactly one pseudo-valuation $v: K \rightarrow [0, \infty]$ with $v(c) = r$. If P is minimal, then v is a P -valuation.*

Proof. First, let v be a pseudo-valuation with $v(c) = r$, and let $a \in K$ be given. If $a \leq c^n$ for all $n \in \mathbb{Z}$, then $v(a) \leq r^n$ for all n , and thus $v(a) = 0$. If, however, $a \geq c^n$ for all $n \in \mathbb{Z}$, we get $v(a) = \infty$. Otherwise, there exists some $n \in \mathbb{N}$ with $c^{-n} \leq a \leq c^n$, and then $r^{-n} \leq v(a) \leq r^n$, hence $v(a) \notin \{0, \infty\}$. For such an a and an integer $n > 0$, there exists some $m \in \mathbb{Z}$ with $c^{m-1} \leq a^n \leq c^m$, which gives $r^{m-1} \leq v(a)^n \leq r^m$, i. e.

$$r^{\frac{m-1}{n}} \leq v(a) \leq r^{\frac{m}{n}}.$$

Thus v is unique:

$$v(a) = \inf\{r^{m/n} \mid m, n \in \mathbb{Z}, n > 0, a^n \leq c^m\}. \quad (32)$$

This formula extends to $v(a) \in \{0, \infty\}$. Conversely, we show that (32) defines a pseudo-valuation. Again, $v(a) = 0$ respectively $v(a) = \infty$ holds if $a \leq c^n$ resp. $a \geq c^n$ for all $n \in \mathbb{Z}$. Therefore, assume that $v(a) \notin \{0, \infty\}$. Then for every $n > 0$ in \mathbb{N} , there is an integer m with $c^{m-1} \leq a^n \leq c^m$. This implies (30) for $v(b) \in \{0, \infty\}$. Assume that $c^{k-1} \leq b^n \leq c^k$. Then we get $r^{\frac{m-1}{n}} \leq v(a) \leq r^{m/n}$ and $r^{\frac{k-1}{n}} \leq v(b) \leq r^{k/n}$. Hence $r^{\frac{m+k-2}{n}} \leq v(a)v(b) \leq r^{\frac{m+k}{n}}$. On the other hand, $c^{k+m-2} \leq (ab)^n \leq c^{m+k}$, and thus $r^{\frac{m+k-2}{n}} \leq v(ab) \leq r^{\frac{m+k}{n}}$. For $n \rightarrow \infty$, this gives (30). Furthermore, (32) implies that v is monotonous, and $v(0) = 0$, $v(1) = 1$ holds together with $v(c) = r$. Hence v is a pseudo-valuation. If P is minimal, Proposition 2.2 shows that for every $a \in P$, the inequality $v(a) < 1$ holds. Hence v is a P -valuation. \square

Examples. 1. For the transcendental prime $P_\infty = (-1, 1) \cap \mathbb{Q}$ of \mathbb{Q} , the unique P_∞ -valuation $v: \mathbb{Q} \rightarrow [0, \infty]$ with $r = c > 1$ (Proposition 3.4) is given by $v(a) := |a|$. So the completion is \mathbb{R} , and $\widehat{P}_\infty = (-1, 1)$. For a prime p and $P := p\mathbb{Z}_p$, every $a \in \mathbb{Q}^\times$ admits a unique representation $a = p^n \cdot \frac{b}{c}$ with $p \nmid b, c$ and $n \in \mathbb{Z}$. Then $v(a) = |a|_p := p^{-n}$ with $|0|_p := 0$ is a P -valuation $v: \mathbb{Q} \rightarrow [0, \infty]$, the p -*adic absolute value* of \mathbb{Q} . The c -neighbourhoods (23) of 0 are then of the form $p^n\mathbb{Z}_p$, $n \in \mathbb{Z}$. As usual, we write $\widehat{\mathbb{Q}}_p$ for the completion of \mathbb{Q} with respect to $P = p\mathbb{Z}_p$. Here we have $\widehat{P} = p\widehat{\mathbb{Z}}_p$, where $\widehat{\mathbb{Z}}_p$ denotes the ring of p -adic numbers.

2. For the field \mathbb{C} of complex numbers, $P := \{a \in \mathbb{C} \mid |a| < 1\}$ is a prime, and the absolute value is a P -valuation $v: \mathbb{C} \rightarrow [0, \infty]$. Here we have $\widehat{\mathbb{C}}_P = \mathbb{C}$.

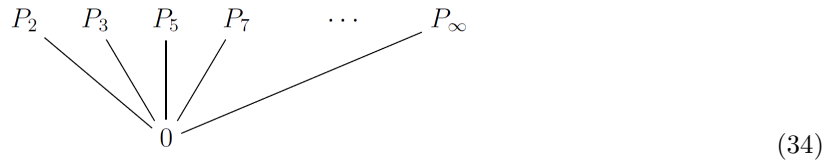
4 Compact primes and the Haar integral

In the sequel, we deal with primes like those occurring in algebraic number fields.

Definition 4.1. We call a prime P of a field K *compact* if for every $b \in K^\times$ there exist $a_1, \dots, a_n \in K$ with

$$P \subset (a_1 + bP) \cup \dots \cup (a_n + bP). \tag{33}$$

Example. The primes of \mathbb{Q} are compact: For the prime P_∞ this is trivial; for a rational prime p , we have $p\mathbb{Z}_p = \bigcup_{i=1}^{p^{n-1}} ip + p^n\mathbb{Z}_p$. So the minimal primes of \mathbb{Q} are compact.



By the following result, the other primes are compact, too:

Proposition 4.1. *Let K be a field. The primes Q which are dependent of a non-trivial compact prime P are compact.*

Proof. Assume that $b \in K^\times$. Since $P \wedge Q \neq 0$, there exists some $a \in (P \wedge Q) \cap K^\times$. As P is compact, there are $a_1, \dots, a_n \in K$ with $P \subset (a_1 + a^2bP) \cup \dots \cup (a_n + a^2bP)$. Hence $aQ \subset P \subset \bigcup_{i=1}^n (a_i + a^2bP) \subset \bigcup_{i=1}^n (a_i + abQ)$, and thus $Q \subset \bigcup_{i=1}^n (a^{-1}a_i + bQ)$. \square

Every compact prime P of a field K leads to a concept of integral. Let $\mathcal{C}_P(K)$ denote the set of real functions $f: K \rightarrow \mathbb{R}$ such that f vanishes on $K \setminus cP$ for some $c \in K^\times$, and is *uniformly continuous* with respect to P , i. e. for every $\varepsilon > 0$ there exists some $b \in K^\times$ such that $|f(a) - f(a')| < \varepsilon$ holds for all $a, a' \in K$ with $a - a' \in bP$. Clearly, $\mathcal{C}_P(K)$ is an \mathbb{R} -vector space. We write $f \leq g$ for $f, g \in \mathcal{C}_P(K)$ if $f(a) \leq g(a)$ holds for all $a \in K$. If, in addition, $f \neq g$, we write $f < g$. The subset of the $f > 0$ in $\mathcal{C}_P(K)$ will be denoted by $\mathcal{C}_P^+(K)$. We show first that $\mathcal{C}_P^+(K) \neq \emptyset$.

Proposition 4.2. *Let K be a field with a prime P . For every $b \in K^\times$, there exists a function $f \in \mathcal{C}_P^+(K)$ which satisfies $f(a) = 1$ for all $a \in bP$.*

Proof. Since $a \in bP \Leftrightarrow ab^{-1} \in P$, we can assume that $b = 1$. Furthermore, we can assume that $P \neq 0$. By (P4), there is an element $d \in K \setminus \tilde{P}$ with $P + P \subset dP$, and by Proposition 3.4, there exists a pseudo-valuation $v: K \rightarrow [0, \infty]$ with $v(d) = 2$. We show first that for every $n \in \mathbb{N}$ there exists some $c \in K^\times$ such that

$$d^{-1} \leq (1 + cp)^n \leq d \tag{35}$$

holds for all $p \in P$. From (P4) we get by induction $P + \dots + P \subset d^n P$, if the sum $P + \dots + P$ contains at most 2^n summands. For $c \in P$, we therefore have $(1 + cp)^n \in 1 + cP + \dots + cP \subset 1 + cd^n P$. Replacing c by cd^{-n} , the inequality (35) turns into $d^{-1} \leq 1 + cp \leq d$ for some $c \in K^\times$ and $p \in P$. By Lemma 3.1 (with $a = 1$), this reduces to the single inequality $1 + cp \leq d$. This is equivalent to $(1 + cp)d^{-1} \in \tilde{P}$, i. e. $(1 + cp)d^{-1}P \subset P$. By (P4), the latter is satisfied for $c = 1$.

Using v , we define $f: K \rightarrow \mathbb{R}$ by

$$f(a) := \begin{cases} 1 & \text{for } a \in P \\ 2 - v(a) & \text{for } a \in dP \setminus P \\ 0 & \text{for } a \notin dP \end{cases}$$

and prove the uniform continuity of f . Let $\varepsilon > 0$ be given. We have to find $c \in K^\times$ so that $|f(a) - f(a')| < \varepsilon$ holds for all $a, a' \in K$ with $a - a' \in cP$. By symmetry, we can assume that $a \in dP \setminus P$.

Case I: $a' \in P$. For $c \in d^{-1}P$, this implies that $a' \notin d^{-1}P$, since otherwise, $a = (a - a') + a' \in d^{-1}P + d^{-1}P \subset P$. Hence if $n \in \mathbb{N}$, the inequality (35) yields an element $c \in K^\times$ with $(1 + \frac{c}{a'}p)^n \leq d$ for all $p \in P$. For $a - a' = cp$, we thus have $a < \frac{a}{a'}$, hence $d^{-1} < 1 \leq a^n \leq (\frac{a}{a'})^n = (1 + \frac{c}{a'}p)^n \leq d$. So we get $2^{-1} \leq v(a^n) \leq 2$, and thus $2^{-\frac{1}{n}} \leq v(a) \leq 2^{\frac{1}{n}}$. Consequently, $|f(a) - f(a')| = |(2 - v(a)) - 1| = |v(a) - 1| < \varepsilon$ holds for sufficiently large n .

Case II: $a' \notin dP$. For any $n \in \mathbb{N}$, the inequality (35) shows that there exists some $c \in K^\times$ with $d^{-1} \leq (1 + \frac{c}{a'}p)^n$ for all $p \in P$. For $a - a' = cp$, this gives $d^{-1} \leq (1 + \frac{c}{a'}p)^n = (\frac{a}{a'})^n \leq (\frac{a}{a'})^n \leq 1 < d$. Therefore, $d^{n-1} \leq a^n \leq d^{n+1}$, hence $2^{n-1} \leq v(a)^n \leq 2^{n+1}$, and thus $2^{\frac{n-1}{n}} \leq v(a) \leq 2^{\frac{n+1}{n}}$. For sufficiently large n , we get $|f(a) - f(a')| = |v(a) - 2| < \varepsilon$.

Case III: $a' \in dP \setminus P$. Again by (35), for any $n \in \mathbb{N}$, there exists $c \in K^\times$ with $d^{-1} \leq (1 + \frac{c}{a'}p)^n \leq d$ for all $p \in P$. For $a - a' = cp$, we get $d^{-1} \leq (\frac{a}{a'})^n \leq d$ for all $p \in P$. This gives $2^{-1} \leq v(a)^n v(a')^{-n} \leq 2$, hence $2^{-\frac{1}{n}} \leq \frac{v(a)}{v(a')} \leq 2^{\frac{1}{n}}$. For sufficiently large n , it follows that $|f(a) - f(a')| = |v(a) - v(a')| < \varepsilon$. \square

Note that $f \in \mathcal{C}_P(K)$ implies that $|f| \in \mathcal{C}_P(K)$. Therefore, every $f \in \mathcal{C}_P(K)$ admits a decomposition

$$f = f_+ - f_- \tag{36}$$

with $f_+ := \frac{1}{2}(|f| + f)$ and $f_- := \frac{1}{2}(|f| - f)$, so that $|f| = f_+ + f_-$ and $f_+, f_- \geq 0$. For $f \in \mathcal{C}_P(K)$ and $a \in K$, we define $f^a \in \mathcal{C}_P(K)$ by

$$f^a(b) := f(b - a). \tag{37}$$

Definition 4.2. Let K be a field with a compact prime P . A linear form $I: \mathcal{C}_P(K) \rightarrow \mathbb{R}$ is said to be a *Haar integral* if $I(f) > 0$ holds for $f > 0$, and $I(f^a) = I(f)$ for all $a \in K$.

So the Haar integral is linear (like any integral) and monotonous ($f < g \Rightarrow I(f) < I(g)$), and invariant under translations. We will show first that a Haar integral exists and is unique up to a constant. Our proof is similar to [14], but does not assume that the topology of K given by the compact prime P is locally compact. In section 6, we will show that the constant can be chosen in a canonical way.

For $f, g \in \mathcal{C}_P^+(K)$, let $f : g$ be the infimum of all sums $r_1 + \dots + r_n$ with $r_1, \dots, r_n > 0$ in \mathbb{R} , such that $f \leq r_1 g^{a_1} + \dots + r_n g^{a_n}$ holds for suitable $a_1, \dots, a_n \in K$.

Proposition 4.3. *Let K be a field with a compact prime P . Every function $f \in \mathcal{C}_P(K)$ is bounded. For $f, g \in \mathcal{C}_P^+(K)$, we have $0 < f : g < \infty$.*

Proof. First let $f \in \mathcal{C}_P(K)$ and $b \in K^\times$ be given such that $f(a) = 0$ holds for $a \notin bP$. For every $\varepsilon > 0$, there exists some $c \in K^\times$ such that $|f(a) - f(a')| < \varepsilon$ holds for $a - a' \in cP$. Since P is compact, there are $a_1, \dots, a_n \in K$ with $bP \subset (a_1 + cP) \cup \dots \cup (a_n + cP)$. Hence f is bounded.

Next let $f, g \in \mathcal{C}_P^+(K)$ be given. Choose $a_0 \in K$ with $g(a_0) > 0$. Then we find $\varepsilon > 0$ in \mathbb{R} and $c \in K^\times$ with $g(a) \geq \varepsilon$ for $a - a_0 \in cP$. Assume that $f(a) = 0$ for $a \notin bP$ and $a_1, \dots, a_n \in K$ with $bP \subset (a_1 + cP) \cup \dots \cup (a_n + cP)$. Then there exists some $m \in \mathbb{N}$ with $f(a) \leq m\varepsilon$ for all $a \in bP$. Hence $f \leq mg^{a_1 - a_0} + \dots + mg^{a_n - a_0}$, which gives $f : g < \infty$. Furthermore, $g(a) \leq r$ for all $a \in K$ and a real $r > 0$. Now if $f(a) > 0$, we obtain $f : g \geq f(a)r^{-1} > 0$. \square

Lemma 4.1. *Let K be a field with a compact prime P . Let $b \in K^\times$ and $f, g \in \mathcal{C}_P^+(K)$ satisfy $f(a) = 0$ for $a \notin bP$ and $g(a) \geq \varepsilon$ for some $\varepsilon > 0$ in \mathbb{R} and $a \in b(P + P)$. Then $h := \frac{f}{g} \in \mathcal{C}_P^+(K)$, where $h(a) := 0$ for $a \notin bP$.*

Proof. For all $a, a' \in K$, we have:

$$\begin{aligned} |h(a) - h(a')| &= \frac{|f(a)g(a') - g(a)f(a')|}{g(a)g(a')} \\ &= \frac{|(f(a) - f(a'))g(a') + f(a')(g(a') - g(a))|}{g(a)g(a')} \\ &\leq \frac{|f(a) - f(a')|}{g(a)} + \frac{f(a')}{g(a')} \cdot \frac{|g(a) - g(a')|}{g(a)}. \end{aligned}$$

For $a, a' \notin bP$, this expression vanishes. Otherwise, if $a - a' \in bP$, then $a, a' \in bP + bP$. Hence

$$|h(a) - h(a')| \leq \varepsilon^{-1}|f(a) - f(a')| + \varepsilon^{-2}f(a') \cdot |g(a) - g(a')|$$

for all $a, a' \in K$. Since f is bounded by Proposition 4.3, this gives $h \in \mathcal{C}_P^+(K)$. \square

Proposition 4.4. *Let K be a field with a compact prime P . For every $f \in \mathcal{C}_P^+(K)$ and $c \in K^\times$, there are $f_1, \dots, f_n \in \mathcal{C}_P^+(K)$ and $a_1, \dots, a_n \in K$ such that $f_i(a) = 0$ holds for $a - a_i \notin cP$, so that $f = f_1 + \dots + f_n$.*

Proof. Assume that $f(a) = 0$ holds for $a \notin bP$, where $b \in K^\times$. By Proposition 4.2, there exists some $f' \in \mathcal{C}_P^+(K)$ which satisfies $f'(a) = 1$ for $a \in b(P + P)$. Furthermore, there exists some $g \in \mathcal{C}_P^+(K)$ that vanishes outside cP . By Proposition 4.3, we have $(f + f') : g < \infty$. Hence there are real numbers $r_1, \dots, r_n > 0$ and $a_1, \dots, a_n \in K$ with $f + f' \leq r_1g^{a_1} + \dots + r_n g^{a_n}$. By Lemma 4.1, $h := \frac{f}{r_1g^{a_1} + \dots + r_n g^{a_n}} \in \mathcal{C}_P^+(K)$, whence $f = r_1hg^{a_1} + \dots + r_nhg^{a_n}$. \square

5 Existence and Uniqueness

Now we are ready to prove existence and uniqueness of the Haar integral.

Proposition 5.1. *For every compact prime P of a field K , there exists a Haar integral.*

Proof. By (36), it is enough to find a map $I : \mathcal{C}_P^+(K) \rightarrow (0, \infty)$ which satisfies $I(rf + sg) = rI(f) + sI(g)$ for $r, s \geq 0$ and $I(f^a) = I(f)$ for $a \in K$. For $r > 0$ in \mathbb{R} and $f, g \in \mathcal{C}_P^+(K)$, we have $(rf : g) = r(f : g)$. Furthermore, for $f, g, h \in \mathcal{C}_P^+(K)$,

$$(f : g)(g : h) \geq f : h \tag{38}$$

In fact, assume that $f \leq r_1 g^{a_1} + \cdots + r_n g^{a_n}$ and $g \leq s_1 h^{b_1} + \cdots + s_m h^{b_m}$. Then $f \leq \sum_{i=1}^n r_i (\sum_{j=1}^m s_j h^{b_j})^{a_i} = \sum_{i=1}^n \sum_{j=1}^m r_i s_j h^{b_j + a_i}$, and thus $f : h \leq (\sum_{i=1}^n a_i) (\sum_{j=1}^m b_j)$.

To determine $I(f)$, let $f_0 \in \mathcal{C}_P^+(K)$ be fixed. Then $f : f_0$ can be regarded as a first approximation of $I(f) : I(f_0)$. An improvement is given by the function $I_g : \mathcal{C}_P^+(K) \rightarrow (0, \infty)$ with

$$I_g(f) := \frac{f : g}{f_0 : g} \quad (39)$$

for small $g \in \mathcal{C}_P^+(K)$. By (38), we have

$$(f_0 : f)^{-1} \leq I_g(f) \leq f : f_0. \quad (40)$$

For $b \in K^\times$, let V_b be the set of all I_g with $g(a) = 0$ for $a \notin bP$. By Proposition 4.2, $V_b \neq \emptyset$ for all $b \in K^\times$. By (40), we can regard I_g as a point of the compact space

$$X := \prod_{f \in \mathcal{C}_P^+(K)} [(f_0 : f)^{-1}, f : f_0]. \quad (41)$$

As the closed sets $\overline{V_b} \subset X$ form a chain, their intersection is non-empty. Hence there exists some $I \in \bigcap_{b \in K^\times} \overline{V_b}$. Consequently, for every $b \in K^\times$ and arbitrary $f_1, \dots, f_n \in \mathcal{C}_P^+(K)$ and $\varepsilon > 0$ in \mathbb{R} , there exists some $g \in \mathcal{C}_P^+(K)$ with $g(a) = 0$ for $a \notin bP$, so that for $i \in \{1, \dots, n\}$,

$$|I(f_i) - I_g(f_i)| < \varepsilon. \quad (42)$$

We will prove that I is a Haar integral. Firstly,

$$(f_1 + f_2) : g \leq (f_1 : g) + (f_2 : g)$$

implies that $I_g(f_1 + f_2) \leq I_g(f_1) + I_g(f_2)$ holds for $f_1, f_2, g \in \mathcal{C}_P^+(K)$. Hence (42) gives

$$I(f_1 + f_2) \leq I(f_1) + I(f_2) \quad (43)$$

for all $f_1, f_2 \in \mathcal{C}_P^+(K)$. For $0 < r < \infty$ and $f, g \in \mathcal{C}_P^+(K)$, we have $(rf : g) = r(f : g)$, i. e. $I_g(rf) = rI_g(f)$, and thus $I(rf) = rI(f)$, again by (42). Similarly, $I_g(f^a) = I_g(f)$ for all $a \in K$, hence $I(f^a) = I(f)$. From (41) and (42), it follows that I is monotonous. Thus it remains to show that the inequality (43) is an equation.

Let $f_1, f_2, g \in \mathcal{C}_P^+(K)$ be given. Then there exists some $b \in K^\times$ such that $f_1 + f_2$ vanishes outside bP . By Proposition 4.2, we find $f_3 \in \mathcal{C}_P^+(K)$ with $f_3(a) = 1$ for $a \in b(P + P)$. For an arbitrary $\varepsilon > 0$, define $f := f_1 + f_2 + \varepsilon f_3$ and $h_j := \frac{f_j}{f}$ for $j \in \{1, 2\}$. By Lemma 4.1, this gives $h_j \in \mathcal{C}_P^+(K)$. Hence there exists some $c \in K^\times$ such that $|h_j(a) - h_j(a')| < \varepsilon$ holds for $a - a' \in cP$ and $j \in \{1, 2\}$. Choose $g \in \mathcal{C}_P^+(K)$ with $g(a) = 0$ for $a \notin cP$, and

$$f \leq r_1 g^{a_1} + \cdots + r_n g^{a_n}.$$

Then $f_j(a) = f(a)h_j(a) \leq \sum_{i=1}^n r_i g^{a_i}(a)(h_j(a_i) + \varepsilon)$, hence

$$f_j : g \leq \sum_{i=1}^n r_i (h_j(a_i) + \varepsilon).$$

This implies $(f_1 : g) + (f_2 : g) \leq \sum_{i=1}^n r_i (1 + 2\varepsilon)$ since $h_1 + h_2 \leq 1$. So we get $(f_1 : g) + (f_2 : g) \leq (f : g)(1 + 2\varepsilon)$, and thus $I_g(f_1) + I_g(f_2) \leq I_g(f) \cdot (1 + 2\varepsilon)$. By (42), this gives $I(f_1) + I(f_2) \leq I(f)(1 + 2\varepsilon)$. Using (43), we get $I(f_1) + I(f_2) \leq (I(f_1 + f_2) + \varepsilon I(f_3))(1 + 2\varepsilon)$ for all ε , hence $I(f_1) + I(f_2) \leq I(f_1 + f_2)$. \square

Proposition 5.2. *Let K be a field with a compact prime P . For every $f_0 \in \mathcal{C}_P^+(K)$, there exists a unique Haar integral $I: \mathcal{C}_P(K) \rightarrow \mathbb{R}$ with $I(f_0) = 1$.*

Proof. Existence follows by Proposition 5.1. Therefore, let I be a Haar integral, and $f_0 \in \mathcal{C}_P^+(K)$. Then rI is a Haar integral for any $r > 0$ in \mathbb{R} . Assume first that $f, g \in \mathcal{C}_P^+(K)$ and $f \leq \sum_{i=1}^n r_i g^{a_i}$ with $r_i > 0$ and $a_i \in K$. Then $I(f) \leq \sum_{i=1}^n r_i I(g)$, hence

$$\frac{I(f)}{I(g)} \leq f : g. \tag{44}$$

For $f \in \mathcal{C}_P^+(K)$ and $\varepsilon > 0$, there exists some $b \in K^\times$ which satisfies $|f(a) - f(a')| < \varepsilon$ for $a - a' \in bP$. Choose $g \in \mathcal{C}_P^+(K)$ with $g(a) = 0$ for $a \notin bP$. Passing from $g(a)$ to $g(a) + g(-a)$, we can assume that $g(a) = g(-a)$ for all $a \in K$. Therefore, if $a, a' \in K$, we have $f(a')g^a(a') \geq (f(a) - \varepsilon)g^a(a')$, hence $I(fg^a) \geq (f(a) - \varepsilon)I(g)$, and thus

$$f(a) \leq \varepsilon + \frac{I(fg^a)}{I(g)}. \tag{45}$$

For every $\delta > 0$, there is some $c \in K^\times$ with $|g(a) - g(a')| < \delta$ for $a - a' \in cP$. By Proposition 4.4, there are elements $f_1, \dots, f_n \in \mathcal{C}_P^+(K)$ and $a_1, \dots, a_n \in K$ with $f = f_1 + \dots + f_n$ and $f_i(a') = 0$ for $a' - a_i \notin cP$. This gives $f_i(a')g(a' - a) \leq f_i(a')(g(a_i - a) + \delta)$ for all $a, a' \in K$, hence

$$I(fg^a) = \sum_{i=1}^n I(f_i g^a) \leq \sum_{i=1}^n I(f_i)(g^a(a_i) + \delta) = \sum_{i=1}^n I(f_i)(g^{a_i}(a) + \delta). \tag{46}$$

By Proposition 4.2, there exists some $h \in \mathcal{C}_P^+(K)$ such that $h(a') = 1$ holds for all a' with $f(a') + f_0(a') > 0$. Then (45) and (46) yield

$$f \leq \varepsilon h + \sum_{i=1}^n \frac{I(f_i)}{I(g)}(g^{a_i} + \delta h) = \left(\varepsilon + \frac{I(f)}{I(g)}\delta\right)h + \sum_{i=1}^n \frac{I(f_i)}{I(g)}g^{a_i}.$$

So we get $f : g \leq \varepsilon(h : g) + \frac{I(f)}{I(g)}$, and by virtue of (38) and (44),

$$\frac{f : g}{f_0 : g} \leq \varepsilon \cdot \frac{h : g}{f_0 : g} + \frac{I(f)}{I(g)(f_0 : g)} \leq \varepsilon(h : f_0) + \frac{I(f)}{I(f_0)}.$$

Similarly,

$$\frac{f_0 : g}{f : g} \leq \varepsilon(h : f) + \frac{I(f_0)}{I(f)} = \frac{I(f_0)}{I(f)} \left(1 + \varepsilon(h : f) \frac{I(f)}{I(f_0)}\right),$$

and thus

$$\frac{I(f)}{I(f_0)} \left(1 + \varepsilon(h : f) \frac{I(f)}{I(f_0)}\right)^{-1} \leq \frac{f : g}{f_0 : g} \leq \varepsilon(h : f_0) + \frac{I(f)}{I(f_0)}.$$

This implies that the quotient $\frac{f:g}{f_0:g}$ converges to $\frac{I(f)}{I(f_0)}$ for $b \rightarrow 0$, which shows that $I(f)$ is unique for $I(f_0) = 1$. \square

6 The normalized Haar integral

Let K be a field with a compact prime P . By Proposition 5.2, there exists a Haar integral $I: \mathcal{C}_P(K) \rightarrow \mathbb{R}$, unique up to a real constant. Furthermore, the proof of Proposition 5.2 yields an explicit determination of the Haar integrals. In fact, for any $b \in K^\times$, let $L(b)$ be the set of $g \in \mathcal{C}_P^+(K)$ which vanish

outside bP . Moreover, we set

$$L_s(b) := \{g \in L(b) \mid \forall a \in K: g(-a) = g(a)\}. \quad (47)$$

For $f, f_0 \in \mathcal{C}_P^+(K)$ and every $\varepsilon > 0$ in \mathbb{R} , There exists some $b \in K^\times$ such that

$$\left| \frac{I(f)}{I(f_0)} - \frac{f : g}{f_0 : g} \right| < \varepsilon \quad (48)$$

for all $g \in L_s(b)$. In this way, the quotient $\frac{I(f)}{I(f_0)}$ can be approximated by the expressions (39) with $g \in L_s(b)$.

For any $c \in K^\times$, consider the \mathbb{R} -linear map

$$\rho_c : \mathcal{C}_P(K) \rightarrow \mathcal{C}_P(K) \quad (49)$$

with $\rho_c(f)(a) := f(c^{-1}a)$ for all $f \in \mathcal{C}_P(K)$ and $a \in K$. Since $f(K \setminus bP) = 0 \Leftrightarrow \rho_c(f)(K \setminus cbP) = 0$, we have

$$\rho_c(L(b)) = L(cb). \quad (50)$$

Similarly, the uniform continuity is preserved under the map ρ_c . Furthermore, ρ_c is monotonous, i. e. for $f, g \in \mathcal{C}_P(K)$,

$$f \leq g \iff \rho_c(f) \leq \rho_c(g). \quad (51)$$

Since ρ_c is invertible, it induces a bijection

$$\rho_c : \mathcal{C}_P^+(K) \xrightarrow{\simeq} \mathcal{C}_P^+(K). \quad (52)$$

For $a, b \in K$ and $f \in \mathcal{C}_P(K)$, we finally have $\rho_c(f^a)(b) = f^a(c^{-1}b) = f(c^{-1}b - a) = f(c^{-1}(b - ca)) = \rho_c(f)(b - ca) = \rho_c(f)^{ca}(b)$, hence

$$\rho_c(f^a) = \rho_c(f)^{ca}. \quad (53)$$

Consequently, for any Haar integral $I : \mathcal{C}_P(K) \rightarrow \mathcal{C}_P(K)$ and $c \in K^\times$, it follows that $I\rho_c$ is again a Haar integral. So there is a real function $v_P : K \rightarrow [0, \infty]$ with $v_P(0) := 0$ and

$$I\rho_c = v_P(c)I \quad (54)$$

for $c \neq 0$, which does not depend on the choice of the Haar integrals I .

Now we prove that v_P is a pseudo-valuation. Using v_P , the Haar integral can be normalized in a natural way.

Proposition 6.1. *Let K be a field with a compact prime P . Then the uniquely defined function $v_P : K \rightarrow [0, \infty]$ in (54) is a pseudo-valuation with respect to P .*

Proof. From (54) we infer that v_P is multiplicative: $v_P(bc) = v_P(b)v_P(c)$ for all $b, c \in K$. In fact, for $b, c \in K^\times$ and a Haar integral I , $v_P(bc)I = I\rho_{bc} = I\rho_b\rho_c = v_P(b)I\rho_c = v_P(b)v_P(c)I$. Similarly, $v_P(1) = 1$.

To prove that v_P is monotonous, we use I to define a function $v_I : K \rightarrow [0, \infty)$ which is related to v_P . (In contrast to v_P , however, v_I depends on I .) For $b \in K$, we set $U(b) := \{f \in \mathcal{C}_P^+(K) \mid \forall a \in b\tilde{P}: f(a) = 1\}$, and

$$v_I(b) := \inf_{f \in U(b)} I(f). \quad (55)$$

Although the function (55) is not multiplicative, in general, it is always monotonous. In fact, if $b, c \in K$ with $b \leq c$, then $U(b) \supset U(c)$, and therefore, $v_I(b) \leq v_I(c)$.

Before we compare v_P with v_I , we prove, in analogy to (50), that

$$\rho_c(U(b)) = U(cb) \tag{56}$$

holds for $b, c \in K^\times$. This follows by the equivalence $\rho_c(f) \in U(cb) \Leftrightarrow f \in U(b)$ for all $f \in \mathcal{C}_P^+(K)$. For $b, c \in K^\times$, we have the equation $v_P(c) \cdot \inf_{f \in U(b)} I(f) = \inf_{f \in U(b)} I(\rho_c(f)) = \inf_{f \in U(cb)} I(f)$, which gives

$$v_I(cb) = v_P(c) \cdot v_I(b). \tag{57}$$

In particular, this gives $v_I(c) = v_P(c) \cdot v_I(1)$, i. e.

$$v_I = v_I(1) \cdot v_P. \tag{58}$$

By Proposition 4.2, we find a function $g \in L(1)$ with $g \leq 1$. For every $f \in U(1)$, we get $f \geq g$, hence $I(f) \geq I(g)$, and thus $v_I(1) \geq I(g) > 0$. By (58), the monotonous property of v_I carries over to v_P , i. e. v_P is a pseudo-valuation. \square

The fact that v_I depends on I enables us to normalize the Haar Integral:

Theorem 6.1. *Let K be a field with a compact prime P . There exists a unique Haar integral J with $v_J = v_P$.*

Proof. Let I be a Haar integral, and $J = rI$ with $r > 0$ in \mathbb{R} . Then $v_{rI}(1) = rv_I(1)$. By Eq. (58), the condition $v_J = v_P$ is equivalent to $v_J(1) = 1$, that is, $r = v_I(1)^{-1}$. \square

In the sequel, we simply call J the *integral* and $v_P = v_J$ the *normed* pseudo-valuation with respect to P . Then (55) shows that $v_P(b)$ measures the content of $b\tilde{P}$. Our normalization thus fixes the content of \tilde{P} to 1.

Next we determine the pseudo-valuations of a compact prime. First, let P be an arbitrary prime. We call a pseudo-valuation $v: K \rightarrow [0, \infty]$ with respect to P *trivial* if the values of v belong to $\{0, 1, \infty\}$.

Proposition 6.2. *Let K be a field with a prime P . There is a natural bijection between the primes $Q \subset P$ and the trivial pseudo-valuations $v: K \rightarrow [0, \infty]$ with respect to P .*

Proof. First, let $v: K \rightarrow [0, \infty]$ be a trivial pseudo-valuation with respect to P . By Proposition 2.1, the primes $Q \subset P$ correspond to the multiplicatively closed subsets $U \subset K$ with $\tilde{P} \subset U$, where $U = \tilde{Q}$. For such a Q , we define v by

$$v(a) := \begin{cases} 0 & \text{for } a \in Q \\ 1 & \text{for } a \in \tilde{\partial}Q \\ \infty & \text{for } a \in K \setminus \tilde{Q}. \end{cases} \tag{59}$$

Then v is monotonous with $v(0) = 0$ and $v(1) = 1$. For $a, b \in K$ with $v(a) = 1$, we have $b \in Q \Leftrightarrow ab \in Q$ and $b \in \tilde{Q} \Leftrightarrow ab \in \tilde{Q}$. Hence v is a pseudo-valuation by Definition 3.1. Conversely, let $v: K \rightarrow [0, \infty]$ be a trivial pseudo-valuation with respect to P . We set $U := \{a \in K \mid v(a) < \infty\}$. Then U is multiplicatively closed with $\tilde{P} \subset U$. The corresponding prime $Q \subset P$ satisfies $a \in \tilde{Q} \Leftrightarrow v(a) \leq 1$, and by (31), $a \in Q \Leftrightarrow a^{-1} \notin \tilde{Q} \Leftrightarrow v(a^{-1}) = \infty \Leftrightarrow v(a) = 0$ for $a \in K^\times$, in accordance with (59). \square

Proposition 6.3. *Let K be a field with a finite, non-trivial pseudo-valuation $v: K \rightarrow [0, \infty]$ with respect to a prime P . Then $Q := \{a \in K \mid v(a) < 1\} \subset P$ is a minimal prime.*

Proof. For $a, b \in Q \cap K^\times$, we have $v(ab) = v(a)v(b) < 1$, hence $ab \in Q$. If $a, b \notin Q$, then $v(a), v(b) \geq 1$, which gives $v(ab) = v(a)v(b) \geq 1$, i. e. $ab \notin Q$. This proves (P1), (P2), and (P3). Thus $Q \subset P$ implies that Q is a prime. Now assume that $a, b \in Q \cap K^\times$. Then there exists some $n \in \mathbb{N}$ with $v(a^n) = v(a)^n < v(b)$. Hence $a^n < b$, and thus $a^n \in bP$. So Q is minimal by Proposition 2.2. \square

Corollary. *Every compact prime $P \neq 0$ of a field K contains a minimal prime.*

Proof. Proposition 6.1 implies that v_P is a finite pseudo-valuation, and v_P is non-trivial since $P \neq 0$. Now Proposition 6.3 completes the proof. \square

7 Krull primes

By Proposition 3.4, every minimal prime P of a field K gives rise to a family of P -valuations. Conversely,

Proposition 7.1. *Let K be a field with a prime $P \neq 0$ and a P -valuation $v: K \rightarrow [0, \infty]$. Then P is minimal.*

Proof. Let $a, b \in P \cap K^\times$ be given. Then $v(a^{-1}) < \infty$ implies that $v(a) \neq 0$, and $a < 1$ yields $0 < v(a) < 1$. So there is an $n \in \mathbb{N}$ with $v(a^n) = v(a)^n \leq v(b)$. Hence $a^n \leq b$. By Proposition 2.2, we infer that P is minimal. \square

In algebraic number theory, the distinction between finite and infinite primes is crucial. For an arbitrary field, we have to distinguish between algebraic and transcendental primes. First, we prove

Lemma 7.1. *A prime P of a field K is algebraic if and only if \tilde{P} is a subring of K .*

Proof. If P is algebraic and $a, b \in \tilde{P}$, then $(a+b)P \subset P+P \subset P$, hence $a+b \in \tilde{P}$. Therefore, \tilde{P} is a subring. Conversely, assume that $\tilde{P} + \tilde{P} \subset \tilde{P}$ and $a, b \in P$, say, $a \leq b$. Then $ab^{-1} \in \tilde{P}$, which implies that $ab^{-1} + 1 \in \tilde{P}$. Hence $a+b = (ab^{-1} + 1)b \in P$. So we get $P+P \subset P$, and thus P is algebraic. \square

Proposition 7.2. *For every prime P of a field K , there is a largest algebraic prime $P_\circ \subset P$.*

Proof. The subring of K generated by \tilde{P} is

$$U := \{a_1 + \cdots + a_n \mid a_i \in \tilde{P}\}. \quad (60)$$

By Proposition 2.1, U corresponds to a prime $P_\circ \subset P$ with $\tilde{P}_\circ = U$. By Lemma 7.1, this prime meets the requirement. \square

There is a close relationship between P_\circ and the open subset P° of P given by Proposition 3.1. First, we prove

Proposition 7.3. *If P is a prime of a field K , then P° is a prime, and $P^{\circ\circ} = P^\circ$.*

Proof. Clearly, P° satisfies (P1) and (P2). Let $a, b \notin P^\circ$ be given. For any $n \in \mathbb{N}$, there are elements $p_n, q_n, r_n, s_n \in P$ with $a^n(p_n + q_n) \notin P$ and $b^n(r_n + s_n) \notin P$. Hence $a^n b^n (p_n + q_n)(r_n + s_n) \notin P$, and thus $a^n b^n (P + P + P + P) \not\subset P$ for all $n \in \mathbb{N}$. If $ab \in P^\circ$, there exists some $n \in \mathbb{N}$ with $(ab)^n (P + P) \subset P$, whence $(ab)^{2n} (P + P + P + P) \subset (ab)^n (P + P) \subset P$, a contradiction. This proves (P3) for P° . To verify (P4), choose $d \in K^\times$ with $P + P \subset dP$. We show that $P^\circ + P^\circ \subset dP^\circ$. To this end, assume that $a, b \in P^\circ$, say, $a^n (P + P) \subset P$ and $b^n (P + P) \subset P$. Then $(a + b)^{2n} (P + P) = \sum_{i=0}^{2n} \binom{2n}{i} a^i b^{2n-i} (P + P) \subset \sum_{i=0}^{2n} \binom{2n}{i} P \subset \sum_{i=1}^{2n} P \subset d^{2n} P$, which gives $(a + b)d^{-1} \in P^\circ$. Whence $P^\circ + P^\circ \subset dP^\circ$.

Finally, assume that $a \in P^\circ$, say, $a^n (P + P) \subset P$. Then $P + P \subset a^{-n} P$, and by the above, this implies that $P^\circ + P^\circ \subset a^{-n} P^\circ$. Hence $a^n (P^\circ + P^\circ) \subset P^\circ$, and thus $a \in P^{\circ\circ}$. \square

Primes P with $P = P^\circ$ were first considered by Krull [7]. We call them *Krull primes*. By Proposition 3.1, every minimal prime is a Krull prime. The connection to algebraic primes is given by

Proposition 7.4. *Let K be a field with a prime P . Then every prime $Q \subsetneq P^\circ$ of K is algebraic.*

Proof. Choose $a \in P^\circ \setminus Q$. So there is an $n \in \mathbb{N}$ with $a^n (P + P) \subset P$. Let $b \in Q$ be arbitrary. Then $a \notin Q$ implies that $a^{-1} \in \tilde{Q}$, hence $a^{-n-1} b \in Q$. This gives $b(\tilde{P} + \tilde{P}) \subset a^{n+1} Q(\tilde{P} + \tilde{P}) \subset a^n (a\tilde{P} + a\tilde{P})Q \subset a^n (P + P)Q \subset PQ \subset Q$. Assume that $a, b \in Q$, say $a\tilde{P} \subset b\tilde{P}$, and $b \neq 0$. Then $a + b = (ab^{-1} + 1)b \in (\tilde{P} + \tilde{P})b \subset Q$. So we get $Q + Q \subset Q$. \square

Corollary 1. *Let K be a field with a prime P . Then $P_\circ \subset P^\circ$, and there is no other prime between P_\circ and P° .*

Proof. Since $P_\circ(P + P) \subset P_\circ + P_\circ = P_\circ \subset P$, we have $P_\circ \subset P^\circ$. For every prime $Q \subsetneq P^\circ$, Proposition 7.4 gives $Q \subset P_\circ$. \square

Corollary 2. *Let K be a field with a prime P . A prime $Q \subset P$ is algebraic if and only if $Q \subset P_\circ$.*

Proof. Assume that $Q \subset P_\circ$. If $Q \neq P_\circ$, then $Q \subsetneq P_\circ \subset P^\circ$, hence Q is algebraic by Proposition 7.4. The converse is trivial. \square

We call a prime P *purely transcendental* if $P_\circ = 0$. If $P \neq 0$, then (P4) implies that $P^\circ \neq 0$. By Corollary 1, this shows that P° is minimal if $P \neq 0$ is purely transcendental. Thus Proposition 3.1 gives

Corollary 3. *A purely transcendental prime $P \neq 0$ is minimal if and only if P is a Krull prime.*

In the paragraph that follows Definition 1.2, we have shown that algebraic primes P of a field K correspond to the valuation rings \tilde{P} with quotient field K . If P is principal, say, $P = p\tilde{P} \neq 0$, and $\bigcap_{n=0}^\infty p^n \tilde{P} = 0$, then \tilde{P} is a discrete valuation domain. Now we turn our attention to compact primes.

Proposition 7.5. *A compact Krull prime P of a field K is either algebraic or purely transcendental.*

Proof. Assume that P is not purely transcendental. Then $P_\circ \neq 0$. Choose $a \in P_\circ \setminus 0$. Then $a\tilde{P}_\circ \subset P_\circ$. By Proposition 4.1, P_\circ is compact. So there are $a_1, \dots, a_n \in K$ with $P_\circ \subset \bigcup_{i=1}^n (a_i + aP_\circ)$, hence $\tilde{P}_\circ \subset a^{-1} P_\circ \subset \bigcup_{i=1}^n (a^{-1} a_i + P_\circ)$. Thus $|\tilde{P}_\circ / P_\circ| < \infty$. By (60), this gives $\tilde{P}_\circ \subset \tilde{P} + \dots + \tilde{P}$

with finitely many summands. Suppose that $P \neq P_\circ$. Then there exist $b \in P \setminus P_\circ$ and $n \in \mathbb{N}$ with $b^n(P + P) \subset P$. This gives $b^{n+1}(\tilde{P} + \tilde{P}) \subset b^n(b\tilde{P} + b\tilde{P}) \subset P$. Hence, for a suitably large $m \in \mathbb{N}$, we get $b^m\tilde{P}_\circ \subset b^m(\tilde{P} + \dots + \tilde{P}) \subset P$. If $b^{-m} \in \tilde{P}_\circ$, then $1 = b^m \cdot b^{-m} \in b^m\tilde{P}_\circ \subset P$, a contradiction. Hence $b^{-m} \notin \tilde{P}_\circ$, and thus $b^m \in P_\circ$. So we obtain $b \in P_\circ$, contrary to our assumption. Thus $P = P_\circ$. \square

Proposition 7.6. *For an algebraic prime $P \neq 0$ of a field K , the following are equivalent.*

- (a) P is compact and minimal.
- (b) P is a compact Krull prime.
- (c) \tilde{P} is a discrete valuation domain with finite residue field.

Proof. The implication (a) \Rightarrow (b) follows by Proposition 3.1.

(b) \Rightarrow (c): Assume that $a \in P \setminus 0$. Then there exist $a_1, \dots, a_n \in K$ with $P \subset \bigcup_{i=1}^n (a_i + a^2P)$, hence $\tilde{P} \subset a^{-1}P \subset \bigcup_{i=1}^n (a^{-1}a_i + aP)$. Thus $|\tilde{P}/aP| < \infty$. So we get $|\tilde{P}/P| < \infty$, and by Proposition 1.1, P is a principal ideal, say $P = p\tilde{P}$. Since $|\tilde{P}/aP| < \infty$ holds for all $a \in P \setminus 0$, it follows that $\bigcap_{n=0}^{\infty} P^n = 0$.

(c) \Rightarrow (a): Assume that $a, b \in P \cap K^\times$. Then there is some $n \in \mathbb{N}$ with $b \notin P^n$. Since $a^n \in P^n$, we have $b\tilde{P} \not\subset a^n\tilde{P}$, and thus $a^n\tilde{P} \subset b\tilde{P}$ by Proposition 1.1. Hence $a^{n+1} \in bP$. Thus, by Proposition 2.2, P is minimal. Assume that $P = p\tilde{P}$. Then $|\tilde{P}/P^n| = |\tilde{P}/P|^n$, which implies that P is compact. \square

Corollary. *A compact prime $P \neq 0$ of a field K is a Krull prime if and only if P is minimal.*

Proof. Assume that P is a compact Krull prime. If P is purely transcendental, Corollary 1 of Proposition 7.4 implies that P is minimal. Otherwise, P is algebraic by Proposition 7.5, hence minimal by Proposition 7.6. The converse follows by Proposition 3.1. \square

As already shown in section 6, Proposition 4.1 implies that every non-trivial compact prime P contains a minimal prime.

8 The product formula

In this section, we focus our attention upon the minimal compact primes P of a field K . If P is algebraic, then Proposition 7.6 implies that \tilde{P} is a discrete valuation domain with finite residue field. So we have a chain

$$\dots \subsetneq P^2 \subsetneq P \subsetneq P^0 \subsetneq P^{-1} \subsetneq P^{-2} \dots, \quad (61)$$

with $P^0 = \tilde{P}$ and $P^n := p^n\tilde{P}$. For $a \in K^\times$, let $e(a)$ denote the greatest integer n with $a \in P^n$. Thus

$$a\tilde{P} = P^{e(a)} \quad (62)$$

holds for any $a \in K^\times$. To get the normalized P -valuation, we define a function $\chi_n : K \rightarrow [0, \infty)$ for any $n \in \mathbb{Z}$ by

$$\chi_n(a) := \begin{cases} 1 & \text{for } a \in P^n \\ 0 & \text{for } a \notin P^n. \end{cases} \quad (63)$$

For $a, b \in K$ with $a - b \in P^n$, we have $|\chi_n(a) - \chi_n(b)| = 0$, which yields $\chi_n \in \mathcal{C}_P^+(K)$.

We set $r_P := |\tilde{P}/P|$. Then $P^n/P^{n+1} = p^n\tilde{P}/p^{n+1}\tilde{P} \cong \tilde{P}/p\tilde{P}$, hence $|P^n/P^{n+1}| = r_P$, and thus $|P^n/P^{n+m}| = r_P^m$ for $m \in \mathbb{N}$. This gives $\chi_n : \chi_{n+m} = r_P^m$, hence

$$\frac{\chi_n : \chi_{n+m}}{\chi_0 : \chi_{n+m}} = \frac{r_P^m}{r_P^{n+m}} = r_P^{-n}.$$

Therefore, by (48), there is a Haar integral I with $I(\chi_0) = 1$ and $I(\chi_n) = r_P^{-n}$. For $b \in K^\times$, Eq. (62) shows that $\chi_{e(b)}$ is the smallest function in $\mathcal{C}_P^+(K)$, which takes the value 1 on $b\tilde{P}$. Now (55) gives $v_I(b) = r_P^{-e(b)}$, hence $v_I(1) = 1$. By (58), we get a normalized P -valuation

$$v_P(a) = r_P^{-e(a)}. \tag{64}$$

Let us return to the product formula (6) of the introduction. The normalized P_p -valuations $v_p := v_{P_p}$ for the primes P_p of \mathbb{Q} are obtained by (64):

$$v_p(a) = p^{-e(a)}, \tag{65}$$

where $e(a)$ is the exponent of p in a prime factorization of a . Thus v_p coincides with the p -adic absolute value (cf. section 3, Example 1).

The topology of the prime $P_\infty = (-1, 1) \cap \mathbb{Q}$ of \mathbb{Q} is the usual one. Therefore, a function $f: \mathbb{Q} \rightarrow \mathbb{R}$ belongs to $\mathcal{C}_\infty(\mathbb{Q}) := \mathcal{C}_{P_\infty}(\mathbb{Q})$ if and only if it is uniformly continuous (in the usual sense) and vanishes outside a bounded interval. The Haar integral with respect to P_∞ is just the Riemann-integral. Therefore, the normalized P_∞ -valuation v_∞ can be obtained from (54). In fact, for $f \in \mathcal{C}_\infty(\mathbb{Q})$ and $c \in \mathbb{Q}^\times$, we get by the substitution $b = c^{-1}a$:

$$\int_{\mathbb{Q}} f \rho_c(a) da = \int_{\mathbb{Q}} f(c^{-1}a) da = |c| \int_{\mathbb{Q}} f(b) db.$$

So the normalized P_∞ -valuation is the absolute value:

$$v_\infty(a) = |a|. \tag{66}$$

In the sequel, we denote the set of minimal primes of a field K by $\mathcal{P}(K)$. From (65) and (66) we thus get for all $a \in \mathbb{Q}^\times$ the product formula:

$$\prod_{P \in \mathcal{P}(\mathbb{Q})} v_P(a) = 1. \tag{67}$$

Lemma 8.1. *Let K be a field with a minimal prime P , and let $a, b \in K^\times$ be such that $a^n \subset b\tilde{P}$ for all $n \in \mathbb{N}$. Then $a \in \tilde{P}$.*

Proof. Suppose that $a \notin \tilde{P}$. Then $a^{-1} \in P$. By Proposition 2.2, there is an $n \in \mathbb{N}$ with $a^{-n} \in b^{-1}P$. Hence $1 = a^n a^{-n} \in b\tilde{P}b^{-1}P \subset P$, a contradiction! \square

Proposition 8.1. *Every minimal prime of a field K of characteristic $p > 0$ is algebraic.*

Proof. Let F be the prime field of K . Suppose that $a \in F \setminus \tilde{P}$. Then $a^{-1} \in P$. Thus $|F^\times| = p - 1$ implies that $1 = (a^{-1})^{p-1} \in P$, a contradiction. Therefore, $F \subset \tilde{P}$. Choose $d \in K^\times$ with $\tilde{P} + \tilde{P} \subset d\tilde{P}$.

For $a, b \in \widetilde{P}$ and an arbitrary $n \in \mathbb{N}$ we find some $m \in \mathbb{N}$ with $mn < 2^m$. Then $(a+b)^{mn} = \sum_{i=0}^{mn} \binom{mn}{i} a^i b^{mn-i} \in \sum_{i=1}^{2^m} \widetilde{P} \subset d^m \widetilde{P}$. Hence $((a+b)^n d^{-1})^m \in \widetilde{P}$, and thus $(a+b)^n d^{-1} \in \widetilde{P}$. By Lemma 3.1, this gives $a+b \in \widetilde{P}$. So $\widetilde{P} + \widetilde{P} \subset \widetilde{P}$. Now Lemma 2.4 implies that P is algebraic. \square

For a rational prime p , let $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ denote the field with p elements. Consider the quotient field $\mathbb{F}_p(x)$ of $\mathbb{F}_p[x]$. By Proposition 8.1, every minimal prime of $\mathbb{F}_p(x)$ is algebraic. For any normed irreducible polynomial $q \in \mathbb{F}_p[x]$, consider the subring

$$\mathbb{F}_p[x]_q := \left\{ \frac{f}{g} \in \mathbb{F}_p(x) \mid f, g \in \mathbb{F}_p[x], q \nmid g \right\} \quad (68)$$

of $\mathbb{F}_p(x)$. Then $\mathbb{F}_p[x]_q$ is a discrete valuation ring with residue class field \mathbb{F}_p . Hence $P_q := q\widetilde{\mathbb{F}_p(x)}$ is a prime of $\mathbb{F}_p(x)$ with $\widetilde{P}_q = \mathbb{F}_p[x]_q$. For all these primes, $\widetilde{P}_q \supset \mathbb{F}_p[x]$. Furthermore,

$$P_\infty := x^{-1}\mathbb{F}_p[x^{-1}]_{x^{-1}} \quad (69)$$

is a prime with $x \notin \widetilde{P}_\infty$, hence $\widetilde{P}_\infty \not\supset \mathbb{F}_p[x]$. In analogy with (15), we write P_∞ , which does not mean, however, that P_∞ is transcendental (which is impossible by Proposition 8.1).

The primes of the form P_q with q irreducible, or $q = \infty$, are compact Krull primes by Proposition 7.6. In fact, if $m = \deg q$ is the degree of an irreducible polynomial $q \in \mathbb{F}_p[x]$, then $\dim_{\mathbb{F}_p}(\widetilde{P}_q/P_q) = \dim_{\mathbb{F}_p}(\mathbb{F}_p[x]/q\mathbb{F}_p[x]) = m$, hence $|\widetilde{P}_q/P_q| = p^m$. For P_∞ , we have $|\widetilde{P}_\infty/P_\infty| = p$. The normalized P_q -valuation $v_q := v_{P_q}$ results from (64):

$$v_q(f) = p^{-e_q(f)\deg q}, \quad (70)$$

where $e_q(f)$ denotes the multiplicity of q in f . For example, $e_{x-1}(x^3 - 2x^2 + x) = 2$. Moreover, we get

$$v_\infty(f) = p^{\deg f}. \quad (71)$$

Proposition 8.2. *Let p be a rational prime. Every minimal compact prime of $\mathbb{F}_p(x)$ is of the form P_q with q irreducible, or $q = \infty$.*

Proof. Let P be a minimal compact prime of $\mathbb{F}_p(x)$. As in the proof of Proposition 8.1, we have $\mathbb{F}_p \subset \widetilde{P}$. Assume first that $x \in \widetilde{P}$. Then $\mathbb{F}_p[x] \subset \widetilde{P}$. Since P is a prime ideal of \widetilde{P} , it follows that $P \cap \mathbb{F}_p[x]$ is a prime ideal of $\mathbb{F}_p[x]$. If $P \cap \mathbb{F}_p[x] = 0$, then $a \notin P$ for all $a \in \mathbb{F}_p[x] \setminus 0$. Hence $a^{-1} \in \widetilde{P}$, and thus $\mathbb{F}_p(x) = \widetilde{P}$, i. e. $P = 0$. Consequently, $P \cap \mathbb{F}_p[x]$ is a maximal ideal of $\mathbb{F}_p[x]$. As $\mathbb{F}_p[x]$ is a principal ideal domain, there is an irreducible polynomial $q \in \mathbb{F}_p[x]$ with $P \cap \mathbb{F}_p[x] = q\mathbb{F}_p[x]$. For every $f \in \mathbb{F}_p[x]$ with $q \nmid f$, we have $f \notin P$, hence $f^{-1} \in \widetilde{P}$. Therefore, $P_q = q\mathbb{F}_p[x]_q \subset P$. As P is minimal, we obtain $P = P_q$. Finally, assume that $x \notin \widetilde{P}$. Then $x^{-1} \in P$. Applying the preceding argument to $\mathbb{F}_p[x^{-1}]$ instead of $\mathbb{F}_p[x]$, we get $P_\infty = x^{-1}\mathbb{F}_p[x^{-1}]_{x^{-1}} \subset P$. Thus $P_\infty = P$. \square

From (70) and (71) we again obtain the product formula by Proposition 8.2. Note that in both cases, the product formula arises in a natural way: The minimal primes P are all compact, and their normalized P -valuations are unique. So the formula describes an intrinsic relation among the minimal primes.

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