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An Identity Related to Derivations of Standard Operator Algebras and Semisimple H*-Algebras[1]

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ABSTRACT

In this paper we prove the following result. Let X be a real or complex Banach space, let $L(X)$ be the algebra of all bounded linear operators on X, and let $A(X) \subset L(X)$ be a standard operator algebra. Suppose $D : A(X) \to L(X)$ is a linear mapping satisfying the relation $D(A^n) = \sum_{j=1}^{n} A^{n-j} D(A) A^{j-1}$ for all $A \in A(X)$. In this case D is of the form $D(A) = AB - BA$, for all $A \in A(X)$ and some $B \in L(X)$, which means that D is a linear derivation. In particular, D is continuous. We apply this result, which generalizes a classical result of Chernoff, to semisimple H^*-algebras.

This research has been motivated by the work of Herstein [4], Chernoff [2] and Molnár [5] and is a continuation of our recent work [8] and [9] .Throughout, R will represent an associative ring. Given an integer $n \geq 2$, a ring R is said to be $n-$torsion free, if for $x \in R$, $nx = 0$

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implies $x = 0$. Recall that a ring R is prime if for $a, b \in R$, $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is semiprime in case $aRa = (0)$ implies $a = 0$. Let A be an algebra over the real or complex field and let B be a subalgebra of A. A linear mapping $D : B \to A$ is called a linear derivation in case $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. In case we have a ring R an additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$, such that $D(x) = ax - xa$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein [4] asserts that any Jordan derivation on a prime ring of characteristic different from two is a derivation. Cusack [3] generalized Herstein's result to $2-$torsion free semiprime rings. Let us recall that a semisimple H^*-algebra is a semisimple Banach $^*-$algebra whose norm is a Hilbert space norm such that $(x, yz^*) = (xz, y) = (z, x^*y)$ is fulfilled for all $x, y, z \in A$ (see [1]). Let X be a real or complex Banach space and let $L(X)$ and $F(X)$ denote the algebra of all bounded linear operators on X and the ideal of all finite rank operators in $L(X)$, respectively. An algebra $A(X) \subset L(X)$ is said to be standard in case $F(X) \subset A(X)$. Let us point out that any standard algebra is prime, which is a consequence of Hahn-Banach theorem.

RESUMEN

En este artículo nosotros provamos el seguiente resultado. Sea X un espacio de Banach real o complejo, sea $L(X)$ a algebra de todos los operadores linares acotados sobre X, y sea $A(X) \subset L(X)$ una algebra de operadores estandar. Suponga $D : A(X) \longrightarrow L(X)$ una aplicación lineal verificando la relación $D(A^n) = \sum_{j=1}^{n} A^{n-j} D(A) A^{j-1}$ para todo $A \in A(X)$. En este caso D es de la forma $D(A) = AB - BA$, para todo $A \in A(X)$ y algún $B \in L(X)$, lo que significa que D es una deriviación lineal. En particual, D es continua. Nosotros aplicamos este resultado el cual generaliza un resultado clásico de Chernoff, para H^*-algebras semisimple. Este trabajo fué motivado por un trabajo de Herstein [4], Chernoff [2] y Molnár [5] y este una continuación de nuestro reciente trabajo [8] y [9].

Key words and phrases: *Prime ring, semiprime ring, Banach space, standard operator algebra, H^*–algebra, derivation, Jordan derivation.*

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Let us start with the following result proved by Chernoff [2] (see also [6] and [8]).

THEOREM A. Let X be a real or complex Banach space and let $A(X)$ be a standard operator algebra on X. Let $D : A(X) \to L(X)$ be a linear derivation. In this case D is of the form $D(A) = AB - BA$, for all $A \in A(X)$ and some $B \in L(X)$. In particular, D is continuous.

It is our aim in this paper to prove the following result which generalizes Theorem A.

THEOREM 1. Let X be a real or complex Banach space and let $A(X)$ be a standard operator algebra on X. Suppose $D : A(X) \to L(X)$ is a linear mapping satisfying the relation

$$D(A^n) = \sum_{j=1}^{n} A^{n-j} D(A) A^{j-1}.$$

for all $A \in A(X)$. In this case D is of the form $D(A) = AB - BA$, for all $A \in A(X)$ and some $B \in L(X)$, which means that D is a linear derivation. In particular, D is continuous.

Proof. We

have the relation

$$D(A^n) = \sum_{j=1}^{n} A^{n-j} D(A) A^{j-1}. \tag{1}$$

Let A be from $F(X)$ and let $P \in F(X)$, be a projection with $AP = PA = A$. From the above relation one obtains

$$D(P) = PD(P) + (n-2)\, PD(P)P + D(P)P. \tag{2}$$

Right multiplication of the relation (2) by P gives

$$PD(P)P = 0. \tag{3}$$

Putting $A + P$ for A in the relation (1), we obtain

$$\begin{gathered}
\sum_{i=0}^{n} \binom{n}{i} D\left(A^{n-i} P^i\right) = \left(\sum_{i=0}^{n-1} \binom{n-1}{i} A^{n-1-i} P^i \right) D(A+P) + \\
\left(\sum_{i=0}^{n-2} \binom{n-2}{i} A^{n-2-i} P^i \right) D(A+P)(A+P) + \\
\left(\sum_{i=0}^{n-3} \binom{n-3}{i} A^{n-3-i} P^i \right) D(A+P)(A+P)^2 + \cdots + \\
(A+P)^2 D(A+P) \left(\sum_{i=0}^{n-3} \binom{n-3}{i} A^{n-3-i} P^i \right) + \\
(A+P) D(A+P) \left(\sum_{i=0}^{n-2} \binom{n-2}{i} A^{n-2-i} P^i \right) + D(A+P) \left(\sum_{i=0}^{n-1} \binom{n-1}{i} A^{n-1-i} P^i \right).
\end{gathered} \tag{4}$$

Using (1) and rearranging the equation (4) in sense of collecting together terms involving equal number of factors of P we obtain:

$$\sum_{i=1}^{n-1} f_i(A,P) = 0,$$

where $f_i(A,P)$ stands for the expression of terms involving i factors of P.

Replacing A by $A+2P$, $A+3P$, ..., $A+(n-1)P$ in turn in the equation (1), and expressing the resulting system of $n-1$ homogeneous equations of variables $f_i(A,P)$, $i=1,2,...,n-1$, we see that the coefficient matrix of the system is a van der Monde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & \cdots & 2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ n-1 & (n-1)^2 & \cdots & (n-1)^{n-1} \end{bmatrix}.$$

Since the determinant of the matrix is different from zero, it follows that the system has only a trivial solution.

In particular,

$$\begin{gathered} f_{n-2}(A,P) = n(n-1)D\left(A^2\right) - (n-1)(n-2)\left(A^2D(P)+D(P)A^2\right) - \\ ((n-2)(n-3)+(n-3)(n-4)+\cdots+3\cdot 2+2\cdot 1)\left(A^2D(P)P+PD(P)A^2\right) - \\ 2((n-2)+(n-3)+(n-4)+\cdots+3+2+1)(AD(A)P+PD(A)A) - \\ 4(1\cdot(n-2)+2\cdot(n-3)+3\cdot(n-4)+\cdots+(n-3)\cdot 2+(n-2)\cdot 1)AD(P)A - \\ 2(n-1)(AD(A)+D(A)A) = 0, \end{gathered}$$

and

$$\begin{gathered} f_{n-1}(A,P) = nD(A) - (PD(A)+D(A)P) - (n-1)(AD(P)+D(P)A) - \\ ((n-2)+(n-3)+(n-4)+\cdots+2+1)(AD(P)P+PD(P)A) - \\ (n-2)PD(A)P = 0. \end{gathered}$$

The above equations reduce to

$$\begin{gathered} n(n-1)D\left(A^2\right) = (n-1)(n-2)\left(A^2D(P)+D(P)A^2\right) + \\ \frac{1}{3}(n-3)(n-2)(n-1)\left(A^2D(P)P+PD(P)A^2\right) + \\ (n-2)(n-1)(AD(A)P+PD(A)A) + \\ 4(1\cdot(n-2)+2\cdot(n-3)+3\cdot(n-4)+\cdots+(n-3)\cdot 2+(n-2)\cdot 1)AD(P)A + \\ 2(n-1)(AD(A)+D(A)A), \end{gathered} \tag{5}$$

and

$$2nD(A) = 2(PD(A) + D(A)P) + 2(n-1)(AD(P) + D(P)A) + (n-2)(n-1)(AD(P)P + PD(P)A) + 2(n-2)PD(A)P, \tag{6}$$

respectively. Multiplying the relation (3) from both sides by A we obtain

$$AD(P)A = 0, \tag{7}$$

which reduces the relation (5) to

$$n(n-1)D(A^2) = (n-1)(n-2)(A^2D(P) + D(P)A^2) + \frac{1}{3}(n-3)(n-2)(n-1)(A^2D(P)P + PD(P)A^2) + (n-2)(n-1)(AD(A)P + PD(A)A) + 2(n-1)(AD(A) + D(A)A). \tag{8}$$

Applying the relation (3) and the fact that $AP = PA = A$, we have $PD(P)A = (PD(P)P)A = 0$. Similarly one obtains that $AD(P)P = 0$. The relations (8) and (6) can now be written as

$$nD(A^2) = (n-2)(A^2D(P) + D(P)A^2) + (n-2)(AD(A)P + PD(A)A) + 2(AD(A) + D(A)A), \tag{9}$$

and

$$nD(A) = PD(A) + D(A)P + (n-1)(AD(P) + D(P)A) + (n-2)PD(A)P = 0, \tag{10}$$

respectively. Right multiplication of the relation (10) by P gives

$$D(A)P = D(P)A + PD(A)P. \tag{11}$$

Similarly one obtains

$$PD(A) = AD(P) + PD(A)P. \tag{12}$$

Multiplying the relation (11) from the right side and the relation (12) from the left side by A, we obtain

$$D(A)A = D(P)A^2 + PD(A)A, \tag{13}$$

and

$$AD(A) = A^2D(P) + AD(A)P. \tag{14}$$

Combining relations (9), (13) and (14) we obtain

$$nD(A^2) = (n-2)(D(P)A^2 + PD(A)A) + (n-2)(A^2D(P) + AD(A)P) + 2(AD(A) + D(A)A) = (n-2)(AD(A) + D(A)A) + 2(AD(A) + D(A)A).$$

We have therefore

$$D(A^2) = D(A)A + AD(A) \tag{15}$$

for any $A \in F(X)$. From the relation (10) one can conclude that $D(A) \in F(X)$ for any $A \in F(X)$. We have therefore a Jordan derivation on $F(X)$. Since $F(X)$ is prime it follows that D is a derivation by Herstein's theorem. Applying Theorem A one can conclude that D is of the form

$$D(A) = AB - BA, \tag{16}$$

for all $A \in A(X)$ and some $B \in L(X)$. It remains to prove that the relation (16) holds on $A(X)$ as well. Let us introduce $D_1 : A(X) \to L(X)$ by $D_1(A) = AB - BA$ and consider $D_0 = D - D_1$. The mapping D_0 is, obviously, linear and satisfies the relation (1). Besides, D_0 vanishes on $F(X)$. It is our aim to prove that D_0 vanishes on $A(X)$ as well. Let $A \in A(X)$, let P be an one-dimensional projection and $S = A + PAP - (AP + PA)$. We have $D_0(S) = D_0(A)$. and $SP = PS = 0$. We have

$$D_0(A^n) = \sum_{j=1}^{n} A^{n-j} D_0(A) A^{j-1} \tag{17}$$

for all $A \in A(X)$. Applying the above relation we obtain

$$\sum_{j=1}^{n} S^{n-j} D_0(S) S^{j-1} = D_0(S^n) = D_0(S^n + P) = D_0((S+P)^n) =$$

$$\sum_{j=1}^{n} (S+P)^{n-j} D_0(S+P)(S+P)^{j-1} = \sum_{j=1}^{n} (S+P)^{n-j} D_0(A)(S+P)^{j-1} =$$

$$\sum_{j=1}^{n} (S^{n-j} + P) D_0(S)(S^{j-1} + P) = \sum_{j=1}^{n} S^{n-j} D_0(A) S^{j-1} +$$

$$\sum_{j=1}^{n} P D_0(A) S^{j-1} + \sum_{j=1}^{n} S^{n-j} D_0(A) P + P D_0(A) P.$$

We have therefore

$$\sum_{j=1}^{n} P D_0(A) S^{j-1} + \sum_{j=1}^{n} S^{n-j} D_0(A) P + P D_0(A) P = 0. \tag{18}$$

Multiplying the above relation from both sides by P we obtain

$$P D_0(A) P = 0, \tag{19}$$

which reduces the relation (18) to

$$\sum_{j=1}^{n} P D_0(A) S^{j-1} + \sum_{j=1}^{n} S^{n-j} D_0(A) P = 0. \tag{20}$$

Right multiplication of the above relation by P gives

$$\sum_{j=1}^{n} S^{n-j} D_0(A) P = 0. \tag{21}$$

Let us prove that

$$\sum_{j=1}^{n-1} k_j S^{n-1-j} D_0(A) P = 0 \tag{22}$$

holds where $k_j = 2^{n-1-j} - 2^{n-1}$, $j = 1, 2, ..., n-1$. Putting in the relation (21) $2A$ for A we obtain

$$\sum_{j=1}^{n} 2^{n-j} S^{n-j} D_0(A) P = 0.$$

Multiplying the relation (21) by 2^{n-1} and subtracting the relation so obtained from the above relation we obtain the relation (22). Since the relation (21) implies the relation (22) one can conclude by induction that $D_0(A)P = 0$. Since P is an arbitrary one-dimensional projection, it follows that $D_0(A) = 0$, for any $A \in A(X)$, which completes the proof of the theorem.

Let us point out that in case $n = 3$ Theorem 1 reduces to Theorem in [9].

THEOREM 2. Let A be a semisimple H^*-algebra and let $D : R \to R$ be a linear mapping satisfying the relation

$$D(x^n) = \sum_{j=1}^{n} x^{n-j} D(x) x^{j-1}$$

for all $x \in R$. In this case D is a linear derivation.

Proof. The proof goes through using the same arguments as in the proof of Theorem in [5] with the exception that one has to use Theorem 1 instead of Lemma in [5].

Since in the formulation of the results presented in this paper we have used only algebraic concepts, it would be interesting to study the problem in a purely ring theoretical context. We conclude with the following conjecture.

CONJECTURE. Let R be a semiprime ring with suitable torsion restrictions and let $D : R \to R$ be an additive mapping satisfying the relation

$$D(x^n) = \sum_{j=1}^{n} x^{n-j} D(x) x^{j-1}$$

for all $x \in R$. In this case D is a derivation.

In case R has the identity element the conjecture above was proved in [8]. Since semisimple H^*-algebras are semiprime, Theorem 2 proves the conjecture above in a special case.

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