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Structure of Resolvents of Elliptic Cone Differential Operators: A Brief Survey

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ABSTRACT

The resolvent of an elliptic cone differential operator is surveyed under the aspect of its pseudodifferential structure and its asymptotic behavior as the spectral parameter tends to infinity. The exposition is descriptive and focuses on the case when the domain of the given operator is stationary.

RESUMEN

Se examina la resolvente de un operador diferencial de tipo cónico, elíptico, bajo el aspecto de su estructura pseudodiferencial y su comportamiento asintótico cuando el parámetro espectral tiende a infinito. La exposición es descriptiva y se enfoca en el caso cuando el dominio del operador dado es estacionario.

Key words and phrases: *Resolvents, trace asymptotics, manifolds with conical singularities.*

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1 Introduction

The purpose of this paper is to give a brief descriptive account of joint work with Thomas Krainer and Gerardo Mendoza on resolvents of general cone differential operators whose symbols satisfy natural ellipticity conditions. Cone operators arise particularly in the study of differential equations on a manifold with conical singularities – basic case of an incomplete Riemannian manifold.

The results presented here rely on the analytic and geometric approach developed in the series of papers [3]–[7]. There the reader can find details and further information, including complete proofs, examples, applications, as well as an extensive discussion on the existing literature in the subject.

We start our survey by reviewing the necessary functional analytic framework.

Let H be a Hilbert space and let $\mathcal{D}_0 \subset H$ be a dense subspace. Let A be a linear operator, initially defined as an unbounded operator $A : \mathcal{D}_0 \subset H \to H$.

We are interested in the closed extensions of A in H. In other words, we are looking for domains $\mathcal{D} \subset H$ with $\mathcal{D}_0 \subset \mathcal{D}$ to which A can be extended as a closed operator. There are two canonical such domains:

$$\mathcal{D}_{\min}(A) = \text{closure of } \mathcal{D}_0 \text{ in } H \text{ with respect to } \|\cdot\|_A,$$
$$\mathcal{D}_{\max}(A) = \{u \in H : Au \in H\},$$

where $\|u\|_A = \|u\|_H + \|Au\|_H$. Both domains are dense in H and the extension

$$A_{\mathcal{D}} : \mathcal{D} \subset H \to H$$

is closed if and only if $\mathcal{D}$ is a closed subspace of $\mathcal{D}_{\max}(A)$ that contains $\mathcal{D}_{\min}(A)$. Thus, there is a one-to-one correspondence between the closed extensions of A and the closed subspaces of $\mathcal{D}_{\max}(A)/\mathcal{D}_{\min}(A)$. If the operator A is fixed and there is no possible ambiguity, we will write $\mathcal{D}_{\min}$ and $\mathcal{D}_{\max}$ instead of $\mathcal{D}_{\min}(A)$ and $\mathcal{D}_{\max}(A)$.

If $A_{\mathcal{D}}$ is closed in H, so is $A_{\mathcal{D}} - \lambda = A_{\mathcal{D}} - \lambda I$ for every $\lambda \in \mathbb{C}$. If $A_{\mathcal{D}} - \lambda$ is invertible and $(A_{\mathcal{D}} - \lambda)^{-1}$ is bounded in H, λ is said to be an element of $\operatorname{res}(A_{\mathcal{D}})$, the resolvent set of $A_{\mathcal{D}}$. The family $(A_{\mathcal{D}} - \lambda)^{-1}$ is called the resolvent of $A_{\mathcal{D}}$, and the set $\operatorname{spec}(A_{\mathcal{D}}) = \mathbb{C} \setminus \operatorname{res}(A_{\mathcal{D}})$ is the spectrum of $A_{\mathcal{D}}$.

A closed sector (or ray) $\Lambda \subset \mathbb{C}$ is called a sector (or ray) of minimal growth for $A_{\mathcal{D}} : \mathcal{D} \to H$ if there exists $R > 0$ such that $A_{\mathcal{D}} - \lambda$ is invertible for every λ in

$$\Lambda_R = \{\lambda \in \Lambda : |\lambda| \geq R\},$$

and the resolvent satisfies either of the equivalent estimates

$$\left\|(A_{\mathcal{D}} - \lambda)^{-1}\right\|_{\mathscr{L}(H)} \leq C/|\lambda|, \quad \left\|(A_{\mathcal{D}} - \lambda)^{-1}\right\|_{\mathscr{L}(H,\mathcal{D})} \leq C,$$

for some $C > 0$ and all $\lambda \in \Lambda_R$.

Our research on elliptic cone operators has been guided by two basic goals: One is to find verifiable conditions on A and $\mathcal{D}$ for the resolvent of $A_{\mathcal{D}}$ to exist and for a sector $\Lambda \subset \mathbb{C}$ to be a sector of minimal growth for $A_{\mathcal{D}}$. This information is particularly relevant for nonselfadjoint operators. Secondly, we are interested in describing the pseudodifferential structure and asymptotic properties of the resolvent as the spectral parameter λ tends to infinity. In this paper, we will discuss our progress and main difficulties around these goals.

We finish this introduction by mentioning that the asymptotic information obtained for the resolvent can be directly applied, for instance, in the short-time asymptotic analysis of heat traces, and in the study of the meromorphic structure of zeta functions. This follows from the standard functional calculus, cf. [10], [15].

2 Cone Operators

Let M be a smooth compact n-dimensional manifold with boundary $Y = \partial M$. We fix a defining function x for Y and choose a collar neighborhood $[0, \varepsilon) \times Y$ of the boundary of M. Let E be a smooth vector bundle over M.

A cone differential operator of order m on sections of E is an element $A = x^{-m}P$ with P in $\operatorname{Diff}_b^m(M; E)$; the space of totally characteristic differential operators of order m, see [13]. Thus A is a linear differential operator on $C^\infty(\mathring{M}; E)$ of order m, which near Y, in local coordinates $(x, y) \in (0, \varepsilon) \times Y$, takes the form

$$A = x^{-m} \sum_{k+|\alpha| \le m} a_{k\alpha}(x, y)(xD_x)^k D_y^\alpha \tag{2.1}$$

with coefficients $a_{k\alpha}$ smooth up to $x = 0$.

These operators occur, for example, when introducing polar coordinates around a point or as Laplace-Beltrami operators corresponding to cone metrics, cf. [1].

Every cone operator $A \in x^{-m} \operatorname{Diff}_b^m(M; E)$ has a principal c-symbol ${}^c\boldsymbol{\sigma}(A)$ defined on the c-cotangent ${}^cT^*M$ of M. Over the interior of M, ${}^c\boldsymbol{\sigma}(A)$ is essentially the usual principal symbol of A. Near the boundary Y, ${}^c\boldsymbol{\sigma}(A)$ is of the form

$$\sum_{k+|\alpha|=m} a_{k\alpha}(x, y)\xi^k \eta^\alpha,$$

see (2.1). The operator A is said to be *c-elliptic* if ${}^c\boldsymbol{\sigma}(A)$ is invertible on ${}^cT^*M \backslash 0$, and the family $A - \lambda$ is *c-elliptic with parameter* $\lambda \in \Lambda \subset \mathbb{C}$ if ${}^c\boldsymbol{\sigma}(A) - \lambda$ is invertible on $({}^cT^*M \times \Lambda)\backslash 0$. With $A = x^{-m}P$ one associates the (indicial) family

$$\hat{P}(\sigma) = \sum_{k+|\alpha| \le m} a_{k\alpha}(0, y)\sigma^k D_y^\alpha,$$

also called the conormal symbol of A. If A is c-elliptic, then $\hat{P}(\sigma)$ is invertible for all $\sigma \in \mathbb{C}$ except a discrete set, $\operatorname{spec}_b(A)$, called the boundary spectrum of A.

Fix a positive b-density $\mathfrak{m}$ on M and let $L^2_b(M;E)$ denote the L^2 space with respect to a Hermitian form on E and the density $\mathfrak{m}$. For $s \in \mathbb{N}$ let

$$H^s_b(M;E) = \{u \in L^2_b(M;E) : Pu \in L^2_b(M;E)\ \forall P \in \operatorname{Diff}^s_b(M;E)\}.$$

Throughout this paper we will assume that A is a c-elliptic cone operator of order $m > 0$, and as reference Hilbert space we choose, for instance, $x^{-m/2}L^2_b(M;E)$. Consider A as a densely defined unbounded operator

$$A : C^\infty_c(\overset{\circ}{M};E) \subset x^{-m/2}L^2_b(M;E) \to x^{-m/2}L^2_b(M;E).$$

In [12] Lesch showed that, in the situation at hand, $\mathcal{D}_{\max}/\mathcal{D}_{\min}$ is finite dimensional and every closed extension of A,

$$A_{\mathcal{D}} : \mathcal{D} \subset x^{-m/2}L^2_b(M;E) \to x^{-m/2}L^2_b(M;E),$$

is Fredholm. Modulo $\mathcal{D}_{\min}$ the elements of $\mathcal{D}_{\max}$ are determined by their asymptotic behavior near the boundary of M. The structure of these asymptotics depends on the elements σ in the boundary spectrum of A with $|\Im\sigma| < m/2$.

In [9] it was shown that

$$\mathcal{D}_{\min} = \mathcal{D}_{\max} \cap \Big(\bigcap_{\varepsilon>0} x^{m/2-\varepsilon}H^m_b(M;E)\Big),$$

and $\mathcal{D}_{\min} = x^{m/2}H^m_b(M;E)$ if and only if $\operatorname{spec}_b(A) \cap \{\sigma \in \mathbb{C} : \Im\sigma = -\frac{m}{2}\} = \varnothing$. Moreover, there exists $\varepsilon > 0$ such that

$$\mathcal{D}_{\max} \hookrightarrow x^{-m/2+\varepsilon}H^m_b(M;E).$$

The embedding $(\mathcal{D}_{\max}, \|\cdot\|_A) \hookrightarrow x^{-m/2}L^2_b(M;E)$ is therefore compact. Thus, for every domain $\mathcal{D}$ with $\mathcal{D}_{\min} \subset \mathcal{D} \subset \mathcal{D}_{\max}$ and all $\lambda \in \mathbb{C}$,

$$A_{\mathcal{D}} - \lambda : \mathcal{D} \to x^{-m/2}L^2_b(M;E)$$

is Fredholm with $\operatorname{ind}(A_{\mathcal{D}} - \lambda) = \operatorname{ind} A_{\mathcal{D}}$. Consequently,

$$\operatorname{spec}(A_{\mathcal{D}}) \neq \mathbb{C} \Rightarrow \operatorname{ind} A_{\mathcal{D}} = 0.$$

Conversely, if $\operatorname{ind} A_{\mathcal{D}} = 0$, then $\operatorname{spec}(A_{\mathcal{D}})$ is either discrete or all of $\mathbb{C}$.

Remark 2.2. The complexity of the spectrum of a cone operator can already be observed in the simple case of the Laplacian on the interval $[0,1]$, see [6]. In that case, the following situations are possible:

- Closed extensions with index zero whose spectrum is empty.
- Closed extensions with index zero whose spectrum is $\mathbb{C}$.
- A family of domains $\mathcal{D}_\beta$ with $\mathcal{D}_\beta \to \mathcal{D}_0$ (in a suitable sense) such that $\operatorname{spec}(\Delta_{\mathcal{D}_\beta})$ is discrete and independent of β, but $\operatorname{spec}(\Delta_{\mathcal{D}_0}) = \mathbb{C}$.

3 The Model Operator and Rays of Minimal Growth

By means of a Taylor expansion at $x = 0$, a cone operator $A \in x^{-m}\operatorname{Diff}_b^m(M;E)$ induces a decomposition

$$x^m A = P_0 + x\tilde{P}_1,$$

where $\tilde{P}_1 \in \operatorname{Diff}_b^m(M;E)$ and P_0 is an operator with coefficients independent of x near Y. We let $Y^\wedge = [0,\infty) \times Y$ and consider P_0 as an element of $\operatorname{Diff}_b^m(Y^\wedge;E)$.

We call the operator $x^{-m}P_0 \in x^{-m}\operatorname{Diff}_b^m(Y^\wedge;E)$ the model operator of A and denote it by $A_\wedge$. If A is written as in (2.1) near the boundary, then

$$A_\wedge = x^{-m} \sum_{k+|\alpha|\le m} a_{k\alpha}(0,y)(xD_x)^k D_y^\alpha.$$

This operator acts on $C_c^\infty(\mathring{Y}^\wedge;E)$ and can be extended as a densely defined closed operator in $x^{-m/2}L_b^2(Y^\wedge;E)$. The domains of the minimal and maximal closed extensions of $A_\wedge$ are denoted by $\mathcal{D}_{\wedge,\min}$ and $\mathcal{D}_{\wedge,\max}$, and like for A, the space $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$ is finite dimensional. In fact, there is a natural isomorphism

$$\theta : \mathcal{D}_{\max}/\mathcal{D}_{\min} \to \mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$$

that allows passage from domains over M to domains over $Y^\wedge$. With a domain $\mathcal{D}$ for A we associate a domain $\mathcal{D}_\wedge$ for $A_\wedge$ defined via

$$\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} = \theta(\mathcal{D}/\mathcal{D}_{\min}). \tag{3.1}$$

The model operator and its canonical domains $\mathcal{D}_{\wedge,\min}$ and $\mathcal{D}_{\wedge,\max}$ exhibit an important invariance property with respect to the natural $\mathbb{R}_+$-action on $Y^\wedge$. This property is crucial for the characterization of domains and in the geometric study of resolvents of elliptic cone operators. For this reason, it has been incorporated in our systematic approach and is worth reviewing: Let

$$\mathbb{R}_+ \ni \varrho \mapsto \kappa_\varrho : x^{-m/2}L_b^2(Y^\wedge;E) \to x^{-m/2}L_b^2(Y^\wedge;E)$$

be the one-parameter group of isometries which on functions is defined by

$$(\kappa_\varrho f)(x,y) = \varrho^{m/2} f(\varrho x, y).$$

It is easily verified that $A_\wedge$ satisfies

$$\kappa_\varrho A_\wedge = \varrho^{-m} A_\wedge \kappa_\varrho, \tag{3.2}$$

thus the domains $\mathcal{D}_{\wedge,\min}$ and $\mathcal{D}_{\wedge,\max}$ are both κ-invariant. In particular, κ induces an action on $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$. A domain $\mathcal{D}$ for a cone operator A is said to be *stationary* if its associated domain $\mathcal{D}_\wedge$, see (3.1), is κ-invariant.

The relation (3.2) implies

$$A_\wedge - \varrho^m \lambda = \varrho^m \kappa_\varrho (A_\wedge - \lambda)\kappa_\varrho^{-1} \tag{3.3}$$

for every $\varrho > 0$ and $\lambda \in \mathbb{C}$. This property is called κ-homogeneity, see e.g. [14].

Any intermediate space $\mathcal{D}_\wedge$ with $\mathcal{D}_{\wedge,\min} \subset \mathcal{D}_\wedge \subset \mathcal{D}_{\wedge,\max}$ gives rise to a closed extension

$$A_{\wedge,\mathcal{D}_\wedge} : \mathcal{D}_\wedge \subset x^{-m/2} L_b^2(Y^\wedge; E) \to x^{-m/2} L_b^2(Y^\wedge; E).$$

As opposed to A, even if the c-symbol of $A_\wedge$ is invertible, not every such extension is Fredholm. However, for certain values of $\lambda \in \mathbb{C}$, $A_\wedge - \lambda$ is better behaved: We define the *background resolvent set* of $A_\wedge$ as

$$\text{bg-res}(A_\wedge) = \{\lambda \in \mathbb{C} : A_{\wedge,\min} - \lambda \text{ injective and } A_{\wedge,\max} - \lambda \text{ surjective}\}.$$

Using the κ-homogeneity (3.3) one can prove that this set is a union of open sectors. Moreover, if $\lambda \in \text{bg-res}(A_\wedge)$, then

$$A_{\wedge,\mathcal{D}_\wedge} - \lambda : \mathcal{D}_\wedge \subset x^{-m/2} L_b^2(Y^\wedge; E) \to x^{-m/2} L_b^2(Y^\wedge; E)$$

is Fredholm with $\operatorname{ind}(A_{\wedge,\mathcal{D}_\wedge} - \lambda) = \operatorname{ind}(A_{\wedge,\min} - \lambda) + \dim \mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min}$. The index is constant on connected components of bg-res$(A_\wedge)$.

Let Λ be a sector in bg-res$(A_\wedge)$ and consider the Grassmannian

$$\mathfrak{G} = \{\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min} : \operatorname{ind}(A_{\wedge,\mathcal{D}_\wedge} - \lambda) = 0 \text{ for } \lambda \in \Lambda\} \tag{3.4}$$

of d-dimensional subspaces of $\mathcal{D}_{\wedge,\max}/\mathcal{D}_{\wedge,\min}$, where $d = -\operatorname{ind}(A_{\wedge,\min} - \lambda)$.

One of the main reasons for considering the model operator in the context of spectral theory for cone operators is the following result:

Theorem 3.5. *Let $A \in x^{-m} \operatorname{Diff}_b^m(M; E)$ be c-elliptic with parameter in Λ. Let $\mathcal{D}$ be a domain for A and let $\mathcal{D}_\wedge$ be its associated domain. If Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$, then it is a sector of minimal growth for $A_\mathcal{D}$.*

So, the question on the existence of rays of minimal growth for a cone operator $A_\mathcal{D}$ is reduced to studying rays of minimal growth for the corresponding $A_{\wedge,\mathcal{D}_\wedge}$. The simplest case to study is when the domain $\mathcal{D}_\wedge$ is κ-invariant.

Proposition 3.6. *Suppose $\mathcal{D}_\wedge$ is κ-invariant. A sector Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$ if and only if*

$$\Lambda\backslash\{0\} \subset \text{bg-res}(A_\wedge) \text{ and } A_{\wedge,\mathcal{D}_\wedge} - \lambda_0 \text{ is invertible for some } \lambda_0 \in \Lambda\backslash\{0\}.$$

If $\mathcal{D}_\wedge$ is not κ-invariant, it generates an orbit on the Grassmannian $\mathfrak{G}$, see (3.4). In this case, we consider the attracting set of its κ-orbit as $\varrho \to 0$:

$$\Omega^-(\mathcal{D}_\wedge) = \{D \in \mathfrak{G} : \exists\, \varrho_k \to 0 \text{ such that } D = \lim_{k\to\infty} \kappa_{\varrho_k}(\mathcal{D}_\wedge/\mathcal{D}_{\wedge,\min})\}.$$

Theorem 3.7. *Let $\lambda_0 \in$ bg-res$(A_\wedge)$. The ray Γ through λ_0 is a ray of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$ iff $A_{\wedge,\mathcal{D}} - \lambda_0$ is invertible for all $\mathcal{D}$ such that $\mathcal{D}/\mathcal{D}_{\wedge,\min} \in \Omega^-(\mathcal{D}_\wedge)$.*

Remark 3.8. The above invertibility condition can be expressed in terms of the nonvanishing of a suitable finite determinant. The limiting set $\Omega^-(\mathcal{D}_\wedge)$ can be interpreted as the "principal object" associated with the domain of A.

A nice and explicit application of the previous theorem to second order regular singular operators on a metric graph can be found in [6].

4 Structure of Resolvents

Let Λ be a closed sector in $\mathbb{C}$ and assume that $A \in x^{-m}\,\text{Diff}_b^m(M;E)$ is c-elliptic with parameter in Λ. Let $A_\mathcal{D}$ be a closed extension of A in $x^{-m/2}L_b^2(M;E)$ and let $\mathcal{D}_\wedge$ be the associated domain of $\mathcal{D}$. By Theorem 3.5 we know that if Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$, then it is a sector of minimal growth for $A_\mathcal{D}$. In particular, in such a sector the resolvent $(A_\mathcal{D} - \lambda)^{-1}$ exists, thus

$$A_{\mathcal{D}_{\min}} - \lambda : \mathcal{D}_{\min} \to x^{-m/2}L_b^2 \text{ is injective and}$$
$$A_{\mathcal{D}_{\max}} - \lambda : \mathcal{D}_{\max} \to x^{-m/2}L_b^2 \text{ is surjective.}$$

Let

$$\mathcal{K}_\lambda = \ker(A_{\mathcal{D}_{\max}} - \lambda) \;\text{ and }\; \mathcal{R}_\lambda = \text{rg}(A_{\mathcal{D}_{\min}} - \lambda).$$

If $\lambda \in \text{res}(A_\mathcal{D})$, then

$$\mathcal{D}_{\max} = \mathcal{K}_\lambda \oplus \mathcal{D}. \tag{4.1}$$

Let $B_{\min}(\lambda)$ be the left-inverse of $A_{\mathcal{D}_{\min}} - \lambda$ with kernel $\mathcal{R}_\lambda^\perp$ and let $B_{\max}(\lambda)$ be the right-inverse of $A_{\mathcal{D}_{\max}} - \lambda$ with range $\mathcal{K}_\lambda^\perp$. We then have (see [3, Section 5])

$$(A_\mathcal{D} - \lambda)^{-1} = B_{\max}(\lambda) + \big[1 - B_{\min}(\lambda)(A-\lambda)\big]\pi_{\mathcal{K}_\lambda,\mathcal{D}}\, B_{\max}(\lambda), \tag{4.2}$$

where $\pi_{\mathcal{K}_\lambda,\mathcal{D}}$ is the projection on $\mathcal{K}_\lambda$ with kernel $\mathcal{D}$ according to (4.1). In fact, the projection can be replaced by $\pi_{\max}\pi_{\mathcal{K}_\lambda,\mathcal{D}}\pi_{\max}$, where $\pi_{\max}$ is the projection onto the orthogonal complement of

$\mathcal{D}_{\min}$ in $\mathcal{D}_{\max}$. With similar computations one can also analyze the resolvent of the model operator $A_\wedge$ on $\mathcal{D}_\wedge$, see [3, Section 8].

If we are interested in the asymptotic properties of the resolvent, it is accustomed to use a suitable parameter-dependent pseudodifferential calculus to approximate the resolvent by means of a "good" parametrix. In [4] we showed:

Theorem 4.3. *If Λ is a sector of minimal growth for $A_{\wedge,\mathcal{D}_\wedge}$, then*

$$(A_{\mathcal{D}} - \lambda)^{-1} = B(\lambda) + G_{\mathcal{D}}(\lambda) \text{ for } \lambda \in \Lambda,$$

where $B(\lambda)$ is a parametrix of $A_{\mathcal{D}_{\min}} - \lambda$ with $B(\lambda)(A_{\mathcal{D}_{\min}} - \lambda) = 1$ for λ sufficiently large, and $G_{\mathcal{D}}(\lambda)$ is a smoothing operator of finite rank.

In the proof of this theorem the first major step is the construction of the parametrix $B(\lambda)$. An important aspect of our parametrix is that it is an actual left-inverse for λ sufficiently large. The family $G_{\mathcal{D}}(\lambda)$ is then constructed as follows. Under the given assumptions, there is an operator family $K(\lambda) : \mathbb{C}^d \to x^{-m/2}L_b^2$, with $d = -\operatorname{ind} A_{\mathcal{D}_{\min}}$, such that

$$\begin{pmatrix}(A-\lambda) & K(\lambda)\end{pmatrix} : \begin{matrix}\mathcal{D}_{\min}\\ \oplus \\ \mathbb{C}^d\end{matrix} \to x^{-m/2}L_b^2$$

is invertible for $\lambda \in \Lambda_R$ for some $R > 0$. Its inverse can be written as

$$\begin{pmatrix}(A_{\mathcal{D}_{\min}}-\lambda) & K(\lambda)\end{pmatrix}^{-1} = \begin{pmatrix}B(\lambda)\\ T(\lambda)\end{pmatrix},$$

where $B(\lambda)$ is the parametrix of $A_{\mathcal{D}_{\min}} - \lambda$, and $T(\lambda) : x^{-m/2}L_b^2 \to \mathbb{C}^d$ is a smooth family of operators with "nice" asymptotic properties.

Let $\mathcal{E}$ be any d-dimensional complement of $\mathcal{D}_{\min}$ in $\mathcal{D}$. If we split $\mathcal{D} = \mathcal{D}_{\min} \oplus \mathcal{E}$ and write

$$A_{\mathcal{D}} - \lambda = \begin{pmatrix}(A_{\mathcal{D}_{\min}}-\lambda) & (A-\lambda)|_{\mathcal{E}}\end{pmatrix},$$

then

$$\begin{pmatrix}B(\lambda)\\ T(\lambda)\end{pmatrix}\begin{pmatrix}(A_{\mathcal{D}_{\min}}-\lambda) & (A-\lambda)|_{\mathcal{E}}\end{pmatrix} = \begin{pmatrix}1 & B(\lambda)(A-\lambda)|_{\mathcal{E}}\\ 0 & T(\lambda)(A-\lambda)|_{\mathcal{E}}\end{pmatrix},$$

so $A_{\mathcal{D}} - \lambda$ is invertible if and only if $T(\lambda)(A-\lambda) : \mathcal{E} \to \mathbb{C}^d$ is invertible. Now, since $T(\lambda)(A-\lambda)$ vanishes on $\mathcal{D}_{\min}$, it induces an operator on the quotient:

$$F(\lambda) = [T(\lambda)(A-\lambda)] : \mathcal{D}_{\max}/\mathcal{D}_{\min} \to \mathbb{C}^d,$$

and $A_{\mathcal{D}} - \lambda$ is invertible if and only if $F_{\mathcal{D}}(\lambda) = F(\lambda)|_{\mathcal{D}/\mathcal{D}_{\min}}$ is invertible.

On the other hand, $1 - B(\lambda)(A-\lambda)$ also vanishes on $\mathcal{D}_{\min}$, so it induces a map

$$\big[1 - B(\lambda)(A-\lambda)\big] : \mathcal{D}_{\max}/\mathcal{D}_{\min} \to x^{-m/2}L_b^2,$$

and we end up with the decomposition

$$(A_{\mathcal{D}} - \lambda)^{-1} = B(\lambda) + \big[1 - B(\lambda)(A-\lambda)\big] F_{\mathcal{D}}(\lambda)^{-1} T(\lambda). \tag{4.4}$$

This decomposition and the asymptotic properties of its components are crucial for the results presented in the next section.

Observe that both representations of the resolvent, (4.2) and (4.4), give a more refine picture of how the domain $\mathcal{D}$ affects it. In each case, the domain-dependent contribution is reduced to a family of linear operators acting on finite dimensional spaces. From these representations one can derive explicit Krein-like formulas.

5 Trace Expansions

Under the assumptions of the previous section, if Λ is a sector of minimal growth for $A_{\mathcal{D}}$, then for $\ell \in \mathbb{N}$ sufficiently large, $(A_{\mathcal{D}} - \lambda)^{-\ell}$ is an analytic family of trace class operators in $x^{-m/2} L^2_b(M;E)$. In this section we give a complete asymptotic expansion of $\operatorname{Tr}(A_{\mathcal{D}} - \lambda)^{-\ell}$, as $|\lambda| \to \infty$, in the case when the domain is stationary.

Theorem 5.1. *Suppose $\mathcal{D}$ is stationary. Then, for any $\varphi \in C^\infty(M; \operatorname{End}(E))$ and $\ell \in \mathbb{N}$ with $m\ell > n$,*

$$\operatorname{Tr}\big(\varphi (A_{\mathcal{D}} - \lambda)^{-\ell}\big) \sim \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} \alpha_{jk} \lambda^{\frac{n-j}{m} - \ell} \log^k \lambda \quad \textit{as } |\lambda| \to \infty,$$

with a suitable branch of the logarithm, with constants $\alpha_{jk} \in \mathbb{C}$. The numbers m_j vanish for $j < n$, and $m_n \le 1$. In general, the α_{jk} depend on φ, A, $\mathcal{D}$, and ℓ, but the coefficients α_{jk} for $j < n$ and $\alpha_{n,1}$ do not depend on $\mathcal{D}$. If both A and φ have coefficients independent of x near ∂M, then $m_j = 0$ for all $j > n$.

As mentioned in the introduction, this result has direct consequences in the asymptotic analysis of spectral functions defined by means of the resolvent.

For some $0 < \varepsilon_0 < \pi/2$, let

$$\Lambda = \{\lambda \in \mathbb{C} : |\arg \lambda| \ge \tfrac{\pi}{2} - \varepsilon_0\}$$

be a sector of minimal growth for $A_{\mathcal{D}}$. Then it is known that $-A_{\mathcal{D}}$ generates an analytic semigroup in H given by

$$e^{-tA_{\mathcal{D}}} = \frac{i}{2\pi} \int_\Gamma e^{-t\lambda} (A_{\mathcal{D}} - \lambda)^{-1}\, d\lambda \quad \text{for } t > 0, \tag{5.2}$$

where Γ is a contour in Λ such that for λ large, $|\arg \lambda| = \frac{\pi}{2} - \delta$ for some $0 < \delta < \varepsilon_0$.

If, in addition, the resolvent set of $A_{\mathcal{D}}$ contains an open neighborhood V of the origin, then for $z \in \mathbb{C}$ with $\Re z < 0$, we define

$$A^z_{\mathcal{D}} = \frac{i}{2\pi} \int_\Gamma \lambda^z (A_{\mathcal{D}} - \lambda)^{-1}\, d\lambda, \tag{5.3}$$

where Γ is an infinite path in $\Lambda \cup V$ that runs along a ray of minimal growth to a small circle centered at the origin and contained in V, then clockwise about the origin avoiding the negative real axis, and out of V along a ray of minimal growth.

In both equations (5.2) and (5.3), the path Γ is chosen to be positively oriented with respect to the spectrum of $A_{\mathcal{D}}$.

Now, Theorem 5.1 together with (5.2) give the asymptotic expansion

$$\operatorname{Tr}(\varphi e^{-tA_{\mathcal{D}}}) \sim \sum_{j=0}^{\infty} a_j t^{\frac{j-n}{m}} + \sum_{j=0}^{\infty} \sum_{k=0}^{m_j} a_{jk} t^{\frac{j}{m}} \log^k t \quad \text{as } t \to 0^+.$$

Moreover, if A is bounded from below on the minimal domain, then the ζ-function

$$\zeta_{A_{\mathcal{D}}}(s) = \operatorname{Tr}(A_{\mathcal{D}}^{-s})$$

of any selfadjoint extension with stationary domain (e.g. the Friedrichs extension) is holomorphic for $\Re s > n/m$ and has a meromorphic extension to all of $\mathbb{C}$ with poles contained in the set

$$\{\tfrac{n-j}{m} : j \in \mathbb{N}_0\}. \tag{5.4}$$

This follows from Theorem 5.1 together with (5.3), or via the formula

$$A_{\mathcal{D}}^{-s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{-tA_{\mathcal{D}}} dt, \quad \Re s > 0,$$

which implies

$$\zeta_{A_{\mathcal{D}}}(s) = \frac{1}{\Gamma(s)} \mathcal{M}(h_{\mathcal{D}})(s),$$

where $\mathcal{M}(h_{\mathcal{D}})$ denotes the Mellin transform of the function $h_{\mathcal{D}}(t) = \operatorname{Tr}(e^{-tA_{\mathcal{D}}})$.

If $\mathcal{D}$ is nonstationary, the analysis for the asymptotics passed the n-th term is considerably more involved. For instance, at the level of resolvents, these asymptotics may include rational functions in $\log\lambda$ and complex powers of λ. This case is discussed in [8]. With the results from [7], one gets the partial expansion

$$\operatorname{Tr}\big(\varphi(A_{\mathcal{D}} - \lambda)^{-\ell}\big) \sim \sum_{j=0}^{n-1} \alpha_{j,0} \lambda^{\frac{n-j}{m} - \ell} + \alpha_{n,1} \lambda^{-\ell} \log\lambda + O(|\lambda|^{-\ell}) \quad \text{as } |\lambda| \to \infty,$$

which implies

$$\operatorname{Tr}(\varphi e^{-tA_{\mathcal{D}}}) \sim \sum_{j=0}^{n-1} a_j t^{\frac{j-n}{m}} + a_{n,1} \log t + O(1) \quad \text{as } t \to 0^+.$$

Consequently, we get that $\zeta_{A_{\mathcal{D}}}(s)$ extends meromorphically to $\Re s > 0$, but we do not know in general how this function behaves in all of $\mathbb{C}$.

The complexity of the nonstationary case has already been observed in simple situations. There are examples on the half-line (see [2]) where the ζ-function extends meromorphically with additional poles not contained in the set (5.4). Moreover, for partial differential operators of Laplace type (with coefficients independent of the radial variable x), the ζ-function may not admit a meromorphic extension to all of $\mathbb{C}$ due to the presence of logarithmic singularities, see e.g. [11].

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