CUBO A Mathematical Journal
Vol.11, $N^{\underline{o}}$04, (1–13). September 2009

A New Kupka Type Continuity, λ-Compactness and Multifunctions

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ABSTRACT

In this paper, we introduce a new Kupka type function called λ-Kupka and we investigate some of its properties. Also we obtain several characterizations and some basic properties concerning upper (lower) λ-continuous multifunctions.

RESUMEN

En este artículo introducimos un nueva función del tipo Kupka llamada λ-Kupka e investigamos algunas de sus propiedades. Además, obtenemos diversas caracterizaciones y algunas propiedades básicas referentes a multifunciones continuas superiores e inferiores λ- continuas.

Key words and phrases: *Topological spaces, Λ-sets, λ-open sets, λ-closed sets, λ-Kupka continuity, multifunction.*

Math. Subj. Class.: *54B05, 54C08; Secondary: 54D05.*

1 Introduction

Maki [6] introduced the notion of Λ-sets in topological spaces. A Λ-set is a set A which is equal to its kernel(= saturated set), i.e. to the intersection of all open supersets of A. Arenas et al. [1] introduced and investigated the notion of λ-closed sets and λ-open sets by involving Λ-sets and closed sets. Kupka [5] introduced firm continuity in order to study compactness. In the same spirit, we introduce and investigate the notion of λ-Kupka continuity to study λ-compactness. Kupka inspired by a number of characterizations of UC spaces (called also Atsuji spaces) [7] to characterize compact spaces. In doing this, he asked the question that what kind of continuity should replace uniform to be sufficiently strong to characterize compact spaces. He answered to this question by introducing a new type of continuity between topological spaces called firm continuity. He obtained several characterizations of compact spaces. This enabled them to introduce a new type function called of firm contra-λ continuous and we use it to study and obtain characterizations of strong λ-closed spaces. Lastly we obtain several characterizations concerning upper (lower)λ-continuous multifuntions.

Throughout this paper we adopt the notations and terminology of [6] and [1] and the following conventions: (X,τ), (Y,σ) and (Z,ν) (or simply X, Y and Z) will always denote topological spaces on which no separation axioms are assumed unless explicitly stated. We denote the interior and the closure of a set A by $Int(A)$ and $Cl(A)$, respectively.
A subset A of a space (X,τ) is called λ-*closed* [1] if $A = L \cap D$, where L is a Λ-set and D is a closed set. The complement of a λ-closed set is called λ-*open*. We denote the collection of all λ-open sets by $\lambda O(X,\tau)$. We set $\lambda O(X,x) = \{U \mid x \in U \in \lambda O(X,\tau)\}$. A point x in a topological space (X,τ)

is called a λ-*cluster point* of A [3] if $A \cap U \neq \emptyset$ for every λ-open set U of X containing x. The set of all λ-cluster points of A is called the λ-*closure* of A and is denoted by $\lambda Cl(A)$ ([1,3]).

Definition 1. *Let B be a subset of a space (X,τ). B is a Λ-set (resp. V-set) [6] if $B = B^{\Lambda}$ (resp. $B = B^{V}$), where :*
$B^{\Lambda} = \bigcap\{U \mid U \supset B,\, U \in \tau\}$ *and* $B^{V} = \bigcup\{F \mid B \supset F,\, F^c \in \tau\}$

2 λ-compactness and λ-Kupka continuity

Definition 2. *A space (X,τ) is said to be λ-compact [2] (also called λO-compact [4]) if every cover of X by λ-open sets has a finite subcover.*

It should be noted that in this paper we use the notation λ-compact instead of λO-compact. In [4], Ganster et al. give some proper examples of λO-compact spaces and establish their relationships with some other strong compactness notions. For the convenience of the interested reader we mention an example of λO-compact spaces from [4]: Let τ_1 be the cofinite topology on X and τ_2 be the point generated topology on X with respect to a point $p \in X$. Let $\tau = \tau_1 \cap \tau_2$. Then (X,τ) is hereditarily compact and λO-compact.

Theorem 2.1. *A topological space (X,τ) is λ-compact if and only if for every family $\{A_i \mid i \in I\}$ of λ-closed sets in X satisfying $\bigcap\limits_{i\in I}\{A_i\} = \emptyset$, there is a finite subfamily $A_{i_1}, ..., A_{i_n}$ with $\bigcap\limits_{1\le k\le n}\{A_i\} = \emptyset$.*

Proof. Straightforward.

Recall that a function $f : (X,\tau) \to (Y,\sigma)$ is said to be λ-irresolute [2] if $f^{-1}(V)$ is λ-open in (X,τ) for every λ-open set V of (Y,σ).

Theorem 2.2. *If $f : (X,\tau) \to (Y,\sigma)$ is a λ-irresolute surjection and (X,τ) is a λ-compact space, then (Y,σ) is λ-compact.*

Proof. Let $\{V_i \mid i \in I\}$ be any cover of Y by λ-open sets of (Y,σ). Since f is λ-irresolute $\{f^{-1}(V_i) \mid i \in I\}$ is a cover of X by λ-open sets of (X,τ). By λ-compactness of (X,τ), there exists a finite subset I_0 of I such that $X = \bigcup\limits_{i\in I_0}\{f^{-1}(V_i)\}$. Since f is surjective, we obtain $Y = f(X) = \bigcup\limits_{i\in I_0}\{V_i\}$. This shows that (Y,σ) is λ-compact.

Definition 3. *A function $f : X \to Y$, where X and Y are topological spaces, is said to have property φ [5] if for every open cover ∇ of Y there exists a finite cover (the members of which need not be necessarily open) $\{A_1, A_2, ..., A_n\}$ of X such that for each $i \in \{1,2,...,n\}$, there exists a set $U_i \in \nabla$ such that $f(A_i) \subset U_i$.*

Definition 4. *A function $f : X \to Y$ is said to be firmly continuous [5] if for every open cover ∇ of Y there exists a finite open cover Ξ of X such that for every $U \in \Xi$ there exists a set $G \in \nabla$ such that $f(U) \subset G$.*

Definition 5. *A function $f : X \to Y$, where X and Y are topological spaces, is said to have property ψ if for every λ-open cover ∇ of Y there exists a finite cover (the members of which need not be necessarily λ-open) $\{A_1, A_2, ..., A_n\}$ of X such that for each $i \in \{1, 2, ..., n\}$, there exists a set $U_i \in \nabla$ such that $f(A_i) \subset U_i$.*

Lemma 2.3. *A topological space X is λ-compact if and only if for every topological space Y and every λ-irresolute function $f : X \to Y$, f has the property ψ.*

Proof. Suppose that the topological space X is λ-compact and the function $f : X \to Y$ is λ-irresolute. Let Ξ be a λ-open cover of Y. The set $f(X)$ is λ-compact relative to Y. This means that there exists a finite subfamily $\{U_1, U_2, ..., U_n\}$ of Ξ which cover $f(X)$. Then the sets $A_1 = f^{-1}(U_1), A_2 = f^{-1}(U_2), ..., A_n = f^{-1}(U_n)$ form a cover of X such that $f(A_i) \subset U_i$ for each $i \in \{1, 2, ..., n\}$.
Conversely, assume that X is a topological space such that for every topological space Y and every λ-irresolute function $f : X \to Y$, f has property ψ. It follows that the identity function $id_X : X \to X$ has also property ψ. Therefore for every λ-open cover Ξ of X, there exists a finite cover $A_1, A_2, ..., A_n$ of X such that for each $i \in \{1, 2, ..., n\}$ there exists a set $U_i \in \Xi$ such that $A_i = id_X(A_i) \subset U_i$. Then $\{U_1, U_2, ..., U_n\}$ is a subcover of Ξ. Since Ξ was an arbitrary λ-open cover of X, the space X is λ-compact.

Definition 6. *A function $f : X \to Y$ is said to be λ-Kupka continuous if for every λ-open cover ∇ of Y there exists a finite λ-open cover Ξ of X such that for every $U \in \Xi$, there exists a set $G \in \nabla$ such that $f(U) \subset G$.*

Remark 2.4. *It should be noted that if the topological space X is λ-compact and Y is an arbitrary topological space, then every λ-irresolute function $f : X \to Y$ is λ-Kupka continuous.*

Lemma 2.5. *Let X, Y, Z and W be topological spaces. Let $g : X \to Y$ and $h : Z \to W$ be λ-irresolute functions and let $f : Y \to Z$ be λ-Kupka continuous. Then the functions $f \circ g : X \to Z$ and $h \circ f : Y \to W$ are λ-Kupka continuous.*

Lemma 2.6. *Let X and Y be topological spaces. Suppose that $f : X \to Y$ is a λ-irresolute function which has the property ψ. Then f is λ-Kupka continuous.*

Theorem 2.7. *The following statements are equivalent for a topological space (X, τ):*
(1) X is λ-compact;
(2) The identity function $id_X : X \to X$ is λ-Kupka continuous;
(3) Every λ-irresolute function from X to X is λ-Kupka continuous;
(4) Every λ-irresolute function from X to a topological space Y is λ-Kupka continuous;
(5) Every λ-irresolute function from X to a topological space Y has the property ψ;
(6) For each topological space Y and each λ-irresolute function $f : Y \to X$, f is λ-Kupka continuous.

Proof. (1)$\Rightarrow$ (2): Suppose that X is λ-compact. The identity function $id_X : X \to X$ is λ-irresolute and by Remark 2.4 id_X is λ-Kupka continuous.

(2)$\Rightarrow$ (3): Let $f : X \to X$ be any λ-irresolute function. By (2) the identity function $id_X : X \to X$ is λ-Kupka continuous and hence by Lemma 2.5 $f = id_X \circ f : X \to X$ is λ-Kupka continuous.

(3)$\Rightarrow$ (4): Let $f : X \to Y$ be any λ-irresolute function. The identity $id_X : X \to X$ is λ-irresolute and by (3) id_X is λ-Kupka continuous. It follows from Lemma 2.5 that $f = f \circ id_X : X \to Y$ is λ-Kupka continuous.

(4) $\Rightarrow$ (5): This is obvious.

(5) $\Rightarrow$ (1): This follows immediately from Lemma 2.3.

(6) $\Rightarrow$ (2): Let $id_X : X \to X$ be the identity function. Then id_X is λ-irresolute and by (6) id_X is λ-Kupka continuous.

(1) $\Rightarrow$ (6): Let ∇ be a λ-open cover of X. By hypothesis, the space X is λ-compact. Then there is a finite subcover $\{U_1, U_2, ..., U_n\}$ of ∇. Assume that $A_i = f^{-1}(U_i)$ for $i \in I$, where $I = \{1, 2, ..., n\}$. It follows that $f(A_i) \subset U_i$ for every $i \in I$. This shows that f is λ-Kupka continuous.

Remark 2.8. *Observe that if $f : X \to Y$ is λ-irresolute, then for each point x in the space X and each λ-open set V of Y containing $f(x)$, there exists a λ-open set U containing x such that $f(U)$ is contained in V.*

3 λ-compactness and multifunctions

Let Λ be a directed set. Now we introduce the following notions which will be used in this paper. A net $\xi = \{x_\alpha \mid \alpha \in \Lambda\}$ λ-*accumulates* at a point $x \in X$ if the net is frequently in every $U \in \lambda O(X, x)$, i.e. for each $U \in \lambda O(X, x)$ and for each $\alpha_0 \in \Lambda$, there is some $\alpha \geq \alpha_0$ such that $x_\alpha \in U$. The net ξ λ-*converges* to a point x of X if it is eventually in every $U \in \lambda O(X, x)$. We say that a filterbase $\Theta = \{F_\alpha \mid \alpha \in \Gamma\}$ λ-*accumulates* at a point $x \in X$ if $x \in \bigcap_{\alpha\in\Gamma} \lambda Cl(F_\alpha)$. Given a set S with $S \subset X$, a λ-*cover* of S is a family of λ-open subsets U_α of X for each $\alpha \in I$ such that $S \subset \bigcup_{\alpha\in I} U_\alpha$. A filterbase $\Theta = \{F_\alpha \mid \alpha \in \Gamma\}$ λ-*converges* to a point x in X if for each $U \in \lambda O(X, x)$, there exists an F_α in Θ such that $F_\alpha \subset U$.

Recall that a multifunction (also called multivalued function) F on a set X into a set Y, denoted by $F : X \to Y$, is a relation on X into Y, i.e. $F \subset X \times Y$. Let $F : X \to Y$ be a multifunction. The upper and lower inverse of a set V of Y are denoted by $F^+(V)$ and $F^-(V)$:

$$F^+(V) = \{x \in X \mid F(x) \subset V\} \text{ and } F^-(V) = \{x \in X \mid F(x) \cap V \neq \emptyset\}$$

We begin with the following notions:

Definition 7. *A point x in a space X is said to be a λ-complete accumulation point of a subset*

S of X if $Card(S \cap U) = Card(S)$ for each $U \in \lambda O(X,x)$, where $Card(S)$ denotes the cardinality of S.

Definition 8. *In a topological space X, a point x is called a λ-adherent point of a filterbase Θ on X if it lies in the λ-closure of all sets of Θ.*

Theorem 3.1. *A space X is λ-compact if and only if each infinite subset of X has a λ-complete accumulation point.*

Proof. Let the space X be λ-compact and S an infinite subset of X. Let K be the set of points x in X which are not λ-complete accumulation points of S. Now it is obvious that for each point x in K, we are able to find $U(x) \in \lambda O(X,x)$ such that $Card(S \cap U(x)) \neq Card(S)$. If K is the whole space X, then $\Theta = \{U(x) \mid x \in X\}$ is a λ-cover of X. By the hypothesis X is λ-compact, so there exists a finite subcover $\Psi = \{U(x_i)\}$, where $i = 1, 2, ..., n$ such that $S \subset \bigcup\{U(x_i) \cap S \mid i = 1, 2, ..., n\}$. Then $Card(S) = max\{Card(U(x_i) \cap S) \mid i = 1, 2, ..., n\}$ which does not agree with what we assumed. This implies that S has a λ-complete accumulation point. Now assume that X is not λ-compact and that every infinite subset $S \subset X$ has a λ-complete accumulation point in X. It follows that there exists a λ-cover Ξ with no finite subcover. Set $\delta = min\{Card(\Phi) \mid \Phi \subset \Xi$, where Φ is a λ-cover of $X\}$. Fix $\Psi \subset \Xi$ for which $Card(\Psi) = \delta$ and $\bigcup\{U \mid U \in \Psi\} = X$. Let N denote the set of natural numbers. Then by hypothesis $\delta \geq Card(N)$. By well-ordering of Ψ by some minimal well-ordering " $\sim$ ", suppose that U is any member of Ψ. By minimal well-ordering " $\sim$" we have $Card(\{V \mid V \in \Psi, V \sim U\}) < Card(\{V \mid V \in \Psi\})$. Since Ψ can not have any subcover with cardinality less than δ, then for each $U \in \Psi$ we have $X \neq \bigcup\{V \mid V \in \Psi, V \sim U\}$. For each $U \in \Psi$, choose a point $x(U) \in X \setminus \bigcup\{V \cup \{x(V)\} \mid V \in \Psi, V \sim U\}$. We are always able to do this if not one can choose a cover of smaller cardinality from Ψ. If $H = \{x(U) \mid U \in \Psi\}$, then to finish the proof we will show that H has no λ-complete accumulation point in X. Suppose that z is a point of the space X. Since Ψ is a λ-cover of X then z is a point of some set W in Ψ. By the fact that $U \sim W$, we have $x(U) \in W$. It follows that $T = \{U \mid U \in \Psi$ and $x(U) \in W\} \subset \{V \mid V \in \Psi, V \sim W\}$. But $Card(T) < \delta$. Therefore $Card(H \cap W) < \delta$. But $Card(H) = \delta \geq Card(N)$ since for two distinct points U and W in Ψ, we have $x(U) \neq x(W)$. This means that H has no λ-complete accumulation point in X which contradicts our assumptions. Therefore X is λ-compact.

Theorem 3.2. *For a space X the following statements are equivalent:*
(1) X is λ-compact;
(2) Every net in X with a well-ordered directed set as its domain λ-accumulates to some point of X.

Proof. (1) $\Rightarrow$ (2): Suppose that (X,τ) is λ-compact and $\xi = \{x_\alpha \mid \alpha \in \Lambda\}$ a net with a well-ordered directed set Λ as domain. Assume that ξ has no λ-adherent point in X. Then for each point x in X, there exist $V(x) \in \lambda O(X,x)$ and an $\alpha(x) \in \Lambda$ such that $V(x) \cap \{x_\alpha \mid \alpha \geq \alpha(x)\} = \emptyset$. This implies that $\{x_\alpha \mid \alpha \geq \alpha(x)\}$ is a subset of $X \setminus V(x)$. Then the collection $C = \{V(x) \mid x \in X\}$ is a λ-cover of X. By hypothesis of the theorem, X is λ-compact and so C has a finite subfamily $\{V(x_i)\}$, where $i = 1, 2, ..., n$ such that $X = \bigcup\{V(x_i)\}$. Suppose that the corresponding elements

of Λ be $\{\alpha(x_i)\}$, where $i = 1, 2, ..., n$. Since Λ is well-ordered and $\{\alpha(x_i)\}$, where $i = 1, 2, ..., n$ is finite, the largest element of $\{\alpha(x_i)\}$ exists. Suppose it is $\{\alpha(x_l)\}$. Then for $\gamma \geq \{\alpha(x_l)\}$, we have $\{x_\delta \mid \delta \geq \gamma\} \subset \bigcap_{i=1}^{n}(X \setminus V(x_i)) = X \setminus \bigcup_{i=1}^{n} V(x_i) = \emptyset$ which is impossible. This shows that ξ has at least one λ-adherent point in X.
(2) $\Rightarrow$ (1): Now it is enough to prove that each infinite subset has a λ-complete accumulation point by utilizing Theorem 3.1. Suppose that $S \subset X$ is an infinite subset of X. According to Zorn's Lemma, the infinite set S can be well-ordered. This means that we can assume S to be a net with a domain which is a well-ordered index set. It follows that S has a λ-adherent point z. Therefore z is a λ-complete accumulation point of S. This shows that X is λ-compact.

Theorem 3.3. *A space X is λ-compact if and only if each filterbase in X has at least one λ-adherent point.*

Proof. Suppose that X is λ-compact and $\Theta = \{F_\alpha \mid \alpha \in \Gamma\}$ a filterbase in it. Since all finite intersections of F_α's are non-empty, it follows that all finite intersection of $\lambda Cl(F_\alpha)$'s are also non-empty. Now it follows from Theorem 2.1 that $\bigcap_{\alpha\in\Gamma} \lambda Cl(F_\alpha)$ is non-empty. This means that Θ has at least one λ-adherent point. Now suppose Θ is any family of λ-closed sets. Let each finite intersection be non-empty. The sets F_α with their finite intersection establish a filterbase Θ. Therefore Θ λ-accumulates to some point z in X. It follows that $z \in \bigcap_{\alpha\in\Gamma} F_\alpha$. Now we have by Theorem 2.1 that X is λ-compact.

Theorem 3.4. *A space X is λ-compact if and only if each filterbase on X with at most one λ-adherent point is λ-convergent.*

Proof. Suppose that X is λ-compact, x a point of X and Θ is a filter base on X. The λ-adherence of Θ is a subset of $\{x\}$. Then the λ-adherence of Θ is equal to $\{x\}$ by Theorem 3.3. Assume that there exists $V \in \lambda O(X, x)$ such that for all $F \in \Theta$, $F \cap (X \setminus V)$ is non-empty. Then $\Psi = \{F \setminus V \mid F \in \Theta\}$ is a filterbase on X. It follows that the λ-adherence of Ψ is non-empty. However $\bigcap_{F\in\Theta} \lambda Cl(F \setminus V) \subset (\bigcap_{F\in\Theta} \lambda Cl(F)) \cap (X \setminus V) = \{x\} \cap (X \setminus V) = \emptyset$. But this is a contradiction. Hence for each $V \in \lambda O(X, x)$, there exists an $F \in \Theta$ with $F \subset V$. This shows that Θ λ-converges to x.
To prove the converse, it suffices to show that each filterbase in X has at least one λ-accumulation point. Assume that Θ is a filterbase on X with no λ-adherent point. By hypothesis, Θ λ-converges to some point z in X. Suppose F_α is an arbitrary element of Θ. Then for each $V \in \lambda O(X, z)$, there exists $F_\beta \in \Theta$ such that $F_\beta \subset V$. Since Θ is a filterbase, there exists a γ such that $F_\gamma \subset F_\alpha \cap F_\beta \subset F_\alpha \cap V$, where F_γ non-empty. This means that $F_\alpha \cap V$ is non-empty for every $V \in \lambda O(X, z)$ and correspondingly for each α, z is a point of $\lambda Cl(F_\alpha)$. It follows that $z \in \bigcap_\alpha \lambda Cl(F_\alpha)$. Therefore z is a λ-adherent point of Θ which is a contradiction. This shows that X is λ-compact.

Now, we further investigate properties of λ-compactness by 1-lower and 1-upper λ-continuous functions. We begin with the following notions and in what follows R denotes the set of real numbers.

Definition 9. *A function $f : X \to R$ is said to be 1-lower (resp. 1-upper) λ-continuous at the point y in X if for each $\lambda > 0$, there exists a λ-open set $U(y)$ such that $f(x) > f(y) \setminus \lambda$(resp. $f(x) > f(y) + \lambda$) for every point x in $U(y)$. The function f is 1-lower (resp. 1-upper) λ-continuous in X if it has these properties for every point x of X.*

Theorem 3.5. *A function $f : X \to R$ is 1-lower λ-continuous if and only if for each $\eta \in R$, the set of all x such that $f(x) \leq \eta$ is λ-closed.*

Proof. It is obvious that the family of sets $\tau = \{(\eta, \infty) \mid \eta \in R\} \cup R$ establishes a topology on R. Then the function f is 1-lower λ-continuous if and only if $f : X \to (R, \tau)$ is λ-continuous. The interval $(-\infty, \eta]$ is closed in (R, τ). It follows that $f^{-1}((-\infty, \eta])$ is λ-closed. Therefore the set of all x such that $f(x) \leq \eta$ is equal to $f^{-1}((-\infty, \eta])$ and thus is λ-closed.

Corollary 3.6. *A subset S of X is λ-compact if and only if the characteristic function X_S is 1-lower λ-continuous.*

Theorem 3.7. *A function $f : X \to R$ is 1-upper λ-continuous if and only if for each $\eta \in R$, the set of all x such that $f(x) \geq \eta$ is λ-closed.*

Corollary 3.8. *A subset S of X is λ-compact if and only if the characteristic function X_S is 1-upper λ-continuous.*

Theorem 3.9. *If the function $F(x) = sup_{i \in I} f_i(x)$ exists, where f_i are 1-lower λ-continuous functions from X into R, then $F(x)$ is 1-lower λ-continuous.*

Proof. Suppose that $\eta \in R$. Let $F(x) < \eta$ and therefore for every $i \in I$, $f_i(x) < \eta$. It is obvious that $\{x \in X \mid F(x) \leq \eta\} = \bigcap_{i \in I} \{x \in X \mid f_i(x) \leq \eta\}$. Since each f_i is 1-lower λ-continuous, then each set of the form $\{x \in X \mid f_i(x) \leq \eta\}$ is λ-closed in X by Theorem 3.5. Since an arbitrary intersection of λ-closed sets is λ-closed, then $F(x)$ is 1-lower λ-continuous.

Theorem 3.10. *If the function $G(x) = inf_{i \in I} f_i(x)$ exists, where f_i are 1-upper λ-continuous functions from X into R, then $G(x)$ is 1-upper λ-continuous.*

Theorem 3.11. *Let $f : X \to R$ be a 1-lower λ-continuous function, where X is λ-compact. Then f assumes the value $m = inf_{x \in X} f(x)$.*

Proof. Suppose $\eta > m$. Since f is 1-lower λ-continuous, then the set $K(\eta) = \{x \in X \mid f(x) \leq \eta\}$ is a non-empty λ-closed set in X by infimum property. Hence the family $\{K(\eta) \mid \eta > m\}$ is a collection of non-empty λ-closed sets with finite intersection property in X. By Theorem 2.1 this family has non-empty intersection. Suppose $z \in \bigcap_{\eta > m} K(\eta)$. Therefore $f(z) = m$ as we wished to prove.

Theorem 3.12. *Let $f : X \to R$ be a 1-upper λ-continuous function, where X is a λ-compact space. Then f attains the value $m = sup_{x \in X} f(x)$.*

Proof. It is similar to the proof of Theorem 3.11.

It should be noted that if a function f at the same time satisfies conditions of Theorem 3.11 and Theorem 3.12, then f is bounded and attains its bound.

Here, we give some characterizations of λ-compact spaces by using lower (resp. upper) λ-continuous multifunctions.

Definition 10. *A multifunction $F : X \to Y$ is said to be lower (resp. upper) λ-continuous if $X \setminus F^-(S)$ (resp. $F^-(S)$) is λ-closed in X for each open (resp. closed) set S in Y.*

Lemma 3.13. *For a multifunction $F : X \to Y$, the following statements are equivalent:*
(1) F is lower λ-continuous;
(2) If $x \in F^-(U)$ for a point x in X and an open set $U \subset Y$, then $V \subset F^-(U)$ for some $V \in \lambda O(x)$;
(3) If $x \notin F^+(D)$ for a point x in X and a closed set $D \subset Y$, then $F^+(D) \subset K$ for some λ-closed set K with $x \notin K$;
(4) $F^-(U) \in \lambda O(X)$ for each open set $U \subset Y$.

Lemma 3.14. *For a multifunction $F : X \to Y$, the following statements are equivalent:*
(1) F is upper λ-continuous;
(2) If $x \in F^+(V)$ for a point x in X and an open set $V \subset Y$, then $F(U) \subset V$ for some $U \in \lambda O(x)$;
(3) If $x \notin F^-(D)$ for a point x in X and a closed set $D \subset Y$, then $F^-(D) \subset K$ for some λ-closed set K with $x \notin K$;
(4) $F^+(U) \in \lambda O(X)$ for each open set $U \subset Y$.

Recall that a relation, denoted by $\leq$, on a set X is said to be a partial order for X if it satisfies the following properties:
(i) $x \leq x$ holds for every $x \in X$ (reflexitivity),
(ii) If $x \leq y$ and $y \leq x$, then $x = y$ (antisymmetry),
(iii)If $x \leq y$ and $y \leq z$, then $x \leq z$ (transivity).
A set equipped with an order relation is called a partially ordered set (or poset).

Theorem 3.15. *The following two statements are equivalent for a space X:*
(1) X is λ-compact.
(2) Every lower λ-continuous multifunction from X into the closed sets of a space assumes a minimal value with respect to set inclusion relation.

Proof. (1) $\Rightarrow$ (2): Suppose that F is a lower λ-continuous multifunction from X into the closed subsets of a space Y. We denote the poset of all closed subsets of Y with the set inclusion relation "$\subseteq$" by Λ. Now we show that $F : X \to \Lambda$ is a lower λ-continuous function. We will show that $N = F^-(\{S \subset Y \mid S \in \Lambda \text{ and } S \subseteq C\})$ is λ-closed in X for each closed set C of Y. Let $z \notin N$, then $F(z) \neq S$ for every closed set S of Y. It is obvious that $z \in F^-(Y \setminus C)$, where $Y \setminus C$ is open in Y. By Lemma 3.13 (2), we have $W \subset F^-(Y \setminus C)$ for some $W \in \lambda O(z)$. Hence $F(w) \cap (Y \setminus C) \neq \emptyset$ for each w in W. So for each w in W, $F(w) \setminus C \neq \emptyset$. Consequently, $F(w) \setminus S \neq \emptyset$ for every closed subset S of Y for which $S \subseteq C$. We consider that $W \cap N = \emptyset$. This means that N is λ-closed. It is clear to observe that F assumes a minimal value.

(2) $\Rightarrow$ (1): Suppose that X is not λ-compact. It follows that we have a net $\{x_i \mid i \in \Lambda\}$, where Λ is a well-ordered set with no λ-accumulation point by [5, Theorem 3.2]. We give Λ the order topology. Let $M_j = \lambda Cl(\{x_i \mid i \geq j\})$ for every j in Λ. We establish a multifunction $F : X \to \Lambda$ where $F(x) = \{i \in \Lambda \mid i \geq j_x\}$, j_x is the first element of all those j's for which $x \notin M_j$. Since Λ has the order topology, $F(x)$ is closed. By the fact that $\{j_x \mid x \in X\}$ has no greatest element in Λ, then F does not assume any minimal value with respect to set inclusion. We now show that $F^-(U) \in \lambda O(X)$ for every open set U in Λ. If $U = \Lambda$, then $F^-(U) = X$ which is λ-open. Suppose that $U \subset \Lambda$ and $z \in F^-(U)$. It follows that $F(z) \cap U \neq \emptyset$. Suppose $j \in F(z) \cap U$. This means that $j \in U$ and $j \in F(z) = \{i \in \Lambda \mid i \geq j_x\}$. Therefore $M_j \geq M_{j_x}$. Since $z \notin M_{j_x}$, then $z \notin M_j$. There exists $W \in SO(z)$ such that $W \cap \{x_i \mid i \in \Lambda\} = \emptyset$. This means that $W \cap M_j = \emptyset$. Let $w \in W$. Since $W \cap M_j = \emptyset$, it follows that $w \notin M_j$ and since j_w is the first element for which $w \notin M_j$, then $j_w \leq j$. Therefore $j \in \{i \in \Lambda \mid i \geq j_w\} = F(w)$. By the fact that $j \in U$, then $j \in F(w) \cap U$. It follows that $F(w) \cap U \neq \emptyset$ and therefore $w \in F^-(U)$. So we have $W \subset F^-(U)$ and thus $z \in W \subset F^-(U)$. Therefore $F^-(U)$ is λ-open. This shows that F is lower λ-continuous which contradicts the hypothesis of the theorem. So the space X is λ-compact.

Theorem 3.16. *The following two statements are equivalent for a space X:*
(1) X is λ-compact.
(2) Every upper λ-continuous multifunction from X into the subsets of a T_1-space attains a maximal value with respect to set inclusion relation.

Proof. Its proof is similar to that of Theorem 3.15.

The following result concerns the existence of a fixed point for multifunctions on λ-compact spaces.

Theorem 3.17. *Suppose that $F : X \to Y$ is a multifunction from a λ-compact domain X into itself. Let $F(S)$ be λ-closed for S being a λ-closed set in X. If $F(x) \neq \emptyset$ for every point $x \in X$, then there exists a nonempty, λ-closed set C of X such that $F(C) = C$.*

Proof. Let $\Lambda = \{S \subset X \mid S \neq \emptyset, S \in \lambda C(X)$ and $F(S) \subset S\}$. It is evident that x belongs to Λ. Therefore $\Lambda \neq \emptyset$ and also it is partially ordered by set inclusion. Suppose that $\{S_\gamma\}$ is a chain in Λ. Then $F(S_\gamma) \subset S_\gamma$ for each γ. By the fact that the domain is λ-compact, $S = \bigcap_\gamma S_\gamma \neq \emptyset$ and also $S \in \lambda C(X)$. Moreover, $F(S) \subset F(S_\gamma) \subset S_\gamma$ for each γ. It follows that $F(S) \subset S_\gamma$. Hence $S \in \Lambda$ and $S = inf\{S_\gamma\}$. It follows from Zorn's lemma that Λ has a minimal element C. Therefore $C \in \lambda C(X)$ and $F(C) \subset C$. Since C is the minimal element of Λ, we have $F(C) = C$.

4 Some properties of (m,n)-λ-compact spaces

We begin with the following notions which will be used in the sequel.

Definition 11. *A space (X,τ) is said to be (m,n)-λ-compact if from every λ-open covering $\{U_i \mid i \in I\}$ of X whose cardinality I, denoted by Card I, is at most n one can select a subcovering $\{U_{i_j} \mid j \in J\}$ of X whose Card J is at most m.*

Definition 12. *A subset A of the space (X,τ) is said to be (m,n)-λ-compact relative to X if form every cover $\{U_i \mid i \in I\}$ of A by λ-open sets of X whose Card is at most n, one can select a subcover $\{U_{i_j} \mid j \in J\}$ of A whose Card J is at most m.*

Definition 13. *A space (X,τ) is said to be completely (m,n)-λ-compact if every subset of X is (m,n)-λ-compact relative to X.*

Remark 4.1. *Observe that a $(1,n)$-λ-compact space is a n-λ-compact space and $(1,\infty)$-λ-compact space is the usual λ-compact space. A $(1,\omega)$-λ-compactness is λ-compactness in the Fréchet sense and a (ω,∞)-λ-compact space is a λ-Lindelöf space.*

Definition 14. *A family $\{U_i \mid i \in I\}$ of subsets of a set X is said to have the m-intersection property if every subfamily of cardinality at most m has a non-void intersection.*

Theorem 4.2. *A space (X,τ) is (m,n)-λ-compact if and only if every family $\{P_i\}$ of λ-closed sets $P_i \subseteq X$ having the m-intersection property also has the n-intersection property.*

Proof. The proof is a consequence of the following equivalent statements:
(1) X is (m,n)-λ-compact;
(2) If $\{U_i \mid i \in I\}$ is a λ-open cover of X such that Card $I \leq n$, then there is a subcover $\{U_{i_j}\}$ of X such that Card $J \leq m$;
(3) If $\{U_i \mid i \in I\}$ is a family of λ-open sets such that $X - (\cup_i U_{i_j}) \neq \emptyset$ whenever Card $J \leq m$, then $X - (\cup_i U_{i_j}) \neq \emptyset$ whenever Card $J \leq n$;
(4) If $\{P_i \mid i \in I\}$ is a family of λ-closed sets having the m-intersection property then $\{P_i\}$ has also the n-intersection property.

Theorem 4.3. *If a space X is (m,n)-λ-compact and if Y is a λ-closed subset of X, then Y is (m,n)-λ-compact relative to X.*

Proof. Suppose that $\{U_i \mid i \in I\}$ is a cover of Y by λ-open sets of X such that Card $I \leq n$. By adding $U_j = X - Y$, we obtain a λ-open cover of X with cardinality at most n. By eliminating U_j, we have a subcover of $\{U_i\}$ whose cardinality is at most m.

Theorem 4.4. *If X is a space such that every λ-open subset of X is (m,n)-λ-compact relative to X, then X is completely (m,n)-λ-compact.*

Proof. Let $Y \subset X$ and $\{U_i \mid i \in I\}$ be a cover of Y by λ-open sets of X such that Card $I \leq n$. Then the family $\{U_i \mid i \in I\}$ is a cover of $\cup_i U_i$ by λ-open sets of X. Then, there is a subfamily $\{U_{i_j} \mid j \in J\}$ of Card $J \leq m$ which covers $\cup_i U_i$. This subfamily also covers the set Y and so Y is (m,n)-λ-compact relative to X.

Theorem 4.5. *Let X be a space and $\{Y_k \mid k \in K\}$ be a family of subsets. If every Y_k is (m,n)-λ-compact relative to X for some $m \geq$ Card K, then $\cup\{Y_k \mid k \in K\}$ is (m,n)-λ-compact relative to X.*

Proof. If $\{U_i \mid i \in I\}$ is a cover of $Y = \cup_k Y_k$ by λ-open sets of X, then it is a cover of Y_k by λ-open sets of X for every $k \in K$. If Card $I \leq n$, then $\{U_i\}$ contains a subfamily $\{U_{i_j} \mid j_k \in J_k\}$ for which Card $j_k \leq m$ and is a covering of Y_k. The union of these families is a λ-open subfamily of $\{U_i\}$ which covers Y and its cardinality is at most m.

Definition 15. *A point $x \in X$ is called an m-λ-accumulation point of a set S in X if for every λ-open set U_x containing x, we have Card $(U_x \cap S) > m$.*

Observe that if $m = 0, 1$ or ω, then the relation Card $(U_x \cap S) > m$ means that $U_x \cap S \neq \emptyset$, not finite or not countable.

Theorem 4.6. *Let X be a space and S a subset of X of cardinality greater than m (i.e. $S \subset X$ and Card $S > m$). If X is (m,n)-λ-compact for some $n > m$, then S has a λ-accumulation point in X. If X is (m,∞)-λ-compact, then S has an m-λ-accumulation point in X.*

Proof. Assume that $S \subset X$ of cardinality at most n which has no λ-accumulation points in X. Then, for each $x \in X$, there is a λ-open set U_x such that at most one point of S belongs to U_x. Suppose U is the union of all sets U_x which contain no points of S. Let U_s denote the union of all sets U_x which contain the point $s \in S$. Then U and U_s are λ-open sets. Therefore $\{U, U_s\}$ is a λ-open cover of X of cardinality at most n. If X is (m,n)-λ-compact, then this cover contains a subcover of cardinality at most n. If X is (m,n)-λ-compact, then this cover contains a subcover of cardinality at most m. But this subcover must contain every U_s since $s \in S$ is covered only by U_s. Thus Card $S \leq m$. If the cardinality of a set S is greater than m, then S has at least one λ-accumulation point in X. The two other cases can be proved by the same token with a little modification.

Received: April 2008. Revised: June 2008.

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