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Optimal Effort in Heterogeneous Agents Population with Global and Local Interactions

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ABSTRACT

A game where agents interact in small teams is proposed; the interaction is examined when the population consists of different types of agent and a reward mechanism devised to increase competition is introduced. We prove that such a mechanism may expand the set of Nash equilibria and, in particular, reduce the production level of some agents. Finally, we extend our results to heterogeneous populations by means of agents based modeling. This way we can study the dynamics of adjustment of agents response and extend our results when considering local interaction and a egocentric knowledge of the population composition.

RESUMEN

Se propone un juego en el cual agentes interactuan en un pequeño grupo, la interacción es examinada cuando la población contituye diversos tipos de agentes y se introduce un mecanismo de recompenza para aumentar la competición. Se demuestra que tal mecanismo puede expandir el conjunto de equilibrio de Nash y, en particular, reduce el nivel de produción de algunos agentes. Finalmente, extendemos nuestros resultados a poblaciones heteregeneas mediante un modelo de agente artificial. De esta manera es posible estudiar la dinámica de ajustamiento de respuesta de agentes y estender los resultados considerando interacción local y un conocimiento egocentrico de la composición de la población.

Key words and phrases: *Bounded rationality, Mathematical Organization Theory, Public Goods, Heterogeneous agents, Agent Based Simulation.*

AMS Classification: *91A80,91B18,91A26,91D10.*

1 Introduction

The problem of incentives and compensation is cardinal in modern economic literature; many papers address this problem considering moral hazard in agency relationships. Other papers address the problem of optimal form of hierarchy in firms and give interesting results (for a survey, see e.g. [6] and [10]).

The approach we use is different; we provide a model of firm which is very simple, and do not consider hierarchy explicitly. The model of firm we introduce may be interpreted as an Interaction Game [8] with agents playing bounded rationality strategies. This approach may fail to take into account some relevant phenomena but allows us to shed light on aspects that are usually neglected such as the interpretation of equilibria as corporate culture.

We consider a population of $2n$ agents randomly paired in teams[1]; each member supplies a non-observable individual effort in order to produce a good. Members are rewarded according to their joint production, yet each agent bears its own private cost in providing effort. In a first analysis we limit our study to symmetric agents; when considering only rational agents this game has one single Nash equilibrium. We are interested in outcomes which are less predictable than Nash equilibria and, in particular, we consider a profile of strategies which is not only ideal from the firm's prospective but also maximizes the workers' welfare. To achieve this particular profile we introduce a class of agents which is able to commit to a coordinated effort even if they have incentive to shirk; the effort provided by these agents is the maximum feasible one and any other effort cannot be greater than this one. By contrast, we assume that some of our agents may not be fully aware of the set of alternatives from which they have to choose, or may have not the skills necessary to make whatever complicated calculations are needed to discover its optimal course of action, or, finally, do not clearly perceive the action-consequence relationship especially when they face uncertainty. In other words, we assume bounded rationality of some agents and consider their effort fixed to some level determined exogenously.

[1] It must be noted that teams we consider differ from the ones considered in [7]

Obviously, a population consisting of heterogeneous agents may affect behavior of rational agents and, consequently, the equilibria of the game. The results we provide are interesting since we find, for example, that the presence of low fixed effort individuals induces rational agents to work harder. The relative composition of the population is used to study the equilibria with heterogeneous agents and these equilibria are compared to the ones in homogeneous population.

In [1] it is argued that a thorough understanding of internal incentives is critical to develop a viable theory of the firm. Furthermore, in the literature individual vs group incentives plans have been compared. Some of the drawbacks of individual incentives are well known in the literature (see for instance [12]). In this paper we do not approach these problems from the classical point of view of incentive and contracts literature (for a first introduction see [6]), rather we propose a simple ranking policy clearly showing some of these drawbacks. This way even a simple model, like the one we propose, encompasses some of the crucial points in the comparison between individual and group incentives plans. In particular, we rank agents according to their profit and, even if this should increase competition between agents, it just expands the equilibrium set of the game including only equilibria dominated by the Nash equilibrium. Furthermore, we show that this policy, in some cases, reverses the effect of fixed low effort individuals on rational agents in terms of optimal effort.

Performing simulation with artificial agents allows us to extend the theoretical results in two more directions. First, we can consider heterogeneous populations and observe the best reply dynamical adjustments when the population compositions vary. Second, we can consider the effects of local interaction when the agents have no longer the knowledge about the global composition of the population, but they can observe only the composition of their neighborhood in order to make their decisions.

The paper is structured in the following way: in Section 2 we present the model of the game and some results about equilibria in the symmetric case; in Section 3 we discuss how the proposed game may be seen as a model of a firm; in Section 4 we introduce bounded rationality agents and some results about the case with the population partitioned in different classes of agents. Section 5 defines the ranking policy and studies how this affects the equilibrium set and the performance of rational agents. Section 6 provides a discussion of the results when the theoretical model is extended by artificial simulation. Finally, in the last section we summarize our findings and provide possible interesting directions in further research.

2 The model

A population of $2n$ agents is randomly partitioned, following a uniform distribution, in n couples of two players; the partition is not revealed to the agents. As a consequence, no agent knows who his/her mate is; this avoids some of the problems related to repeated games and simplify the analysis. In this one-shot game each team is supposed to produce a good; the final production is the only verifiable variable and depends on the vector of the non-observable efforts each agent supplies: this is a *joint production model* (for a survey, see e.g. [6]). Agent $i \in I = \{1, 2, \ldots, 2n\}$ receives a monetary payoff for the final production of its team and supplies an effort e_i which implies some cost to him/her; we assume $e_i \in E \subseteq \mathbb{R}^+$, with E compact, convex and nonempty. For sake of simplicity we consider only the utility of the payoff each agent receives: if agent i is paired with agent j and the vector of their efforts is (e_i, e_j), their utility will be $f(e_i + e_j)$; each agent's disutility of effort is $c(e)$. Furthermore, we assume:

- $f : \mathbb{R}^+ \to \mathbb{R}$
- $c : \mathbb{R}^+ \to \mathbb{R}$
- $f, c \in C^2$
- $f(0) = c(0) = 0$
- $f' > 0$, $c' > 0$
- $f'(0) > c'(0)$
- $f'' < 0$, $c'' > 0$

When agents i and j are the members of the same team, their payoffs are respectively:

$$\begin{cases} \pi_i(e_i, e_j) = f(e_i + e_j) - c(e_i) \\ \pi_j(e_i, e_j) = f(e_i + e_j) - c(e_j) \end{cases} \tag{1}$$

and they are not transferable. Since the partition is random, agent i's profit is a random variable Π_i depending also on the effort its randomly paired mate exerts. The expected value of its profits is

$$\mathrm{E}[\Pi_i(\mathbf{e}, X)] = \frac{1}{2n-1} \sum_{k=1}^{n} \pi_i(e_i, e_k), \qquad k \neq i$$

where X is the random variable determining the partition and $\mathbf{e} = (e_i, e_{-i})$.

2.1 Equilibria of the game in some particular cases

Since for all players the set of strategies E is a compact, convex and nonempty subset of $\mathbb{R}^+$, and the payoff functions are continuous and concave w.r.t. the strategies, the existence of at least one equilibrium is guaranteed.

We remark that the production function is symmetrical and agents have the same cost function. In this section we limit our study to the case in which the agents choose the same effort, i.e., we consider symmetric equilibria. Therefore, the expected profit of each agent is actually (1). In addition, if we assume that players can commit to an effort, they decide the optimal effort e^C solving the following problem:

$$\max_e \quad f(2e) - c(e), \qquad e \in E \subset \mathbb{R}^+$$

where E is the set of feasible efforts. We call this profile of strategies [2] the *coordination equilibrium* $\left(e^C, e^C\right)$.

Obviously, this is not a Nash equilibrium, in fact:

Proposition 1 *The* coordination equilibrium *is Pareto optimal, but even if the two team-mates decide to coordinate to this equilibrium, there is an incentive to shirk. Furthermore, this profile of strategies maximizes the social welfare.*

Proof: The payoff function is strictly concave so FOCs are sufficient to determine the interior coordination equilibrium. The coordination equilibrium is therefore characterized by:

$$2f'\left(2e^C\right) - c'\left(e^C\right) = 0. \tag{2}$$

Let us consider the partial derivative of agent i's profit with respect to its effort when the coordination equilibrium is played:

$$\frac{\partial}{\partial e_i}\left(\pi_i\left(e_i, e_j = e^C\right)\right)_{|e_i = e^C} = f'\left(2e^C\right) - c'\left(e^C\right) < 2f'\left(2e^C\right) - c'\left(e^C\right) = 0.$$

This partial derivative is negative: π_i decreases with respect to e_i. Therefore, choosing a lower effort close enough to e^C, player i has a larger profit. The second part is obvious since this profile of strategies is also the solution of

$$\max_{e_i, e_j} \quad 2f(e_i + e_j) - c(e_i) - c(e_j), \qquad e_i, e_j \in E \subset \mathbb{R}^+.$$

□

To obtain this equilibrium we need something more, something that makes each player commit to provide effort e^C; for example, it could be achieved by signing an enforceable contract or assuming repetition of this situation (for a deep analysis refer to [5]). We assume the existence of such agents[3] and consider this profile of strategies for the following reasons. First of all, from the social point of view it would be desirable to have this situation; then, it is an important yardstick to which compare the other suboptimal equilibria; finally, in this situation the agents exert the maximal effort and this is one of the goals of the firm.

If we do not assume the existence of such ideal agents, this game has a unique Nash equilibrium $\left(e^N, e^N\right)$ and it holds:

[2]Each symmetric profile of strategies can, at least theoretically, be coordinate between players; nevertheless, we reserve this name to this particular one.

[3]With this assumption this profile of strategies constitutes an equilibrium in the sense that, these committed agents have no incentive to deviate.

Theorem 1 *The effort exerted in the Nash equilibrium is lower than the effort exerted in the coordination equilibrium, i.e.,* $e^N < e^C$*.*

Proof: The Nash equilibrium can be found solving

$$\begin{cases} \max\limits_{e_i} & f(e_i + e_j) - c(e_i), \\ & & e_i, e_j \in E \\ \max\limits_{e_j} & f(e_i + e_j) - c(e_j), \end{cases}$$

and is characterized by the following FOC:

$$f'\left(2e^N\right) = c'\left(e^N\right). \tag{3}$$

By contradiction let $e^C \leq e^N$; since f is concave and c is convex, it follows:

$$\begin{cases} c'\left(e^C\right) \leq c'\left(e^N\right), \\ f'\left(2e^C\right) \geq f'\left(2e^N\right), \end{cases}$$

by condition (2) on the coordination equilibrium and (3), this means

$$\begin{cases} c'\left(e^C\right) \leq c'\left(e^N\right) \\ c'\left(e^C\right) = 2f'\left(2e^C\right) > f'\left(2e^C\right) \geq f'\left(2e^N\right) = c'\left(e^N\right), \end{cases}$$

clearly absurd.

□

Obviously, this game is a Public Goods Game [4].

3 The game as a model of firms

We propose this model to study the possible equilibria in a complex structure such as a firm; in fact, we can think of these equilibria as *corporate culture* (see e.g. [5]). Corporate culture may be defined as the basic assumptions and beliefs that are shared by the members of a group or organization and that are used as norm. The organization's problem is to identify a rule that allows relatively efficient transactions to take place and devise some way to communicate that rule to all current and potential trading partners.

We do not expect that all the equilibria of the game we consider may represent efficient equilibria.

The organization has an interest in preserving and promoting a good reputation to allow for future beneficial transactions; nevertheless, it is quite common to observe situations where there is no interest in reputation and members' main interest is free riding. This situation is widespread in Italian public sector offices and often it is claimed that a higher level of competition could avoid such misbehavior. The ranking procedure we introduce in Section 5 may be compared to some individual merit compensation incentives used by organizations. Although individual incentive systems often lead to improved performance, these programs may, at times, lead to employees competing with one another, with undesirable results. Finally, it must be observed that the particular ranking policy we consider may be unrealistic since the principal usually cannot observe private costs of its agents; nevertheless there are always costs to take into account and our model can be easily modified to describe situations where the agents are provided with a fixed resource and have to allocate it efficiently. Furthermore, very often there is a social mechanism which tends to reward people who obtain just a higher profit than the others', without necessarily maximizing it.

4 Some results in heterogeneous populations

Even if this model is very simple we assume that not all agents may:

- be fully aware of the set of alternatives from which they have to choose
- have the skill necessary to make whatever complicated calculations are needed to discover their optimal course of action
- do clearly perceive the action-consequence relationship especially when facing uncertainty.

For an analysis of some other motives that may conflict with the rational man paradigm the reader may refer to [11].

We are interested in considering simple "bounded rationality" strategies and how they affect the equilibria of the game with heterogeneous agents. In particular, we consider agents who *stubbornly* provide a fixed effort regardless of their results. The rationales of such a behavior may be different: for example, an agent, given the difficulties to find an optimal effort, may resolve to providing a fixed effort which it consider appropriate for the situation.

Consider then three classes of agents depending on their behaviors:

- bounded rationality agents playing a fixed effort $\bar{e} \in \left[0, e^C\right]$
- committed agents who play the optimal effort knowing the fact that all the agents of this class play the same coordinated effort
- rational agents who play the optimal effort and do not commit to any coordinated effort.

In the following, we assume that the type of agent is private information while the composition of the population is common knowledge. Let $e\left(\bar{e}\right)$ be the optimal effort an agent, either committed or rational, exerts when its team-mate is a bounded rationality agent playing effort $\bar{e}$. Then $e(\bar{e})$ is the solution of the following problem:

$$\max_{e} \quad f(e+\bar{e}) - c(e).$$

Proposition 2 *Consider a team consisting of a rational (committed) agent and a fixed effort agent. The higher is the fixed effort provided, the lower is the optimal effort of the rational (committed) agent:*

$$\bar{e}_1 < \bar{e}_2 \Longrightarrow e(\bar{e}_1) > e(\bar{e}_2).$$

Furthermore, a rational (committed) agent will always provide an effort lower than e^C.

Proof: By contradiction let $e(\bar{e}_1) \leq e(\bar{e}_2)$; this means $e(\bar{e}_1) + \bar{e}_1 < e(\bar{e}_2) + \bar{e}_2$. Since f' is decreasing:

$$c'(e(\bar{e}_1)) = f'(e(\bar{e}_1) + \bar{e}_1) > f'(e(\bar{e}_2) + \bar{e}_2) = c'(e(\bar{e}_2))$$

absurd since $e(\bar{e}_1) \leq e(\bar{e}_2) \Longrightarrow c'(e(\bar{e}_1)) < c'(e(\bar{e}_2))$.

For the second part we know by the first part of this proposition that $e(\bar{e})$ will be maximum when the fixed effort is null, so it is sufficient to prove that when $\bar{e} = 0$ then $e(0) < e^C$; $e(0)$ is characterized by the FOC $f'(e(0)) = c'(e(0))$. By contradiction assume $e(0) \geq e^C$, by monotonicity of f' and c' it follows:

$$\begin{cases} 2f'\left(e^C\right) > f'\left(e^C\right) \geq f'(e(0)) \\ c'\left(e^C\right) \leq c'(e(0)), \end{cases}$$

clearly absurd.

□

This result states the fact that a rational agent will exert higher effort when paired with somebody exerting a lower effort. This is interesting in terms of free riding: it may appear that the rational agent's behavior incentives free riding, but it must be recalled that rational agents are interested only in maximizing their profit. In Section 5 we introduce an incentive to competition and the results will be different. In the following, we consider the equilibria resulting as the composition of population varies.

4.1 Fixed effort agents vs committed agents

Let us consider the profit maximizing game; the population of $2n$ agents is partitioned in two subsets:

1. m fixed effort $\bar{e}$ agents

2. $(2n-m)$ committed agents who coordinate on the best effort knowing the fact that there are m agents providing the fixed effort.

A single committed agent does not know which kind of mate it will be paired with so it has to solve the following problem:

$$\max_{e} \quad \frac{2n-m-1}{2n-1}\left(f\left(2e\right)-c\left(e\right)\right)+\frac{m}{2n-1}\left(f\left(e+\bar{e}\right)-c\left(e\right)\right). \tag{4}$$

Let e^* be the optimal solution effort to problem (4), it holds:

Theorem 2 *As the number of committed agents increases to $2n$ the optimal effort provided by the committed agents increases to the coordination equilibrium effort e^C. As the effort $\bar{e}$ provided by the fixed effort agents increases, the optimal effort provided by the committed agents decreases.*

Proof: Let us consider FOC of problem (4):

$$\begin{aligned} &F\left(e^*,m,\bar{e}\right)= \\ &=2\left(2n-m-1\right)f'\left(2e^*\right)+mf'\left(e^*+\bar{e}\right)-\left(2n-1\right)c'\left(e^*\right)=0. \end{aligned}$$

By implicit function theorem it is possible to write:

$$\begin{aligned} &\frac{\partial e^*}{\partial m}=-\frac{\partial F/\partial m}{\partial F/\partial e^*}= \\ &=-\frac{-2f'\left(2e^*\right)+f'\left(e^*+\bar{e}\right)}{4\left(2n-m-1\right)f''\left(2e^*\right)+mf''\left(e^*+\bar{e}\right)-\left(2n-1\right)c''\left(e^*\right)}. \end{aligned}$$

By assumptions on concavity/convexity of f and c the denominator is negative while, as it concerns the numerator, consider the following strictly concave functions

$$g_1\left(e\right)=f\left(e+\bar{e}\right)-c\left(e\right),$$

$$g_2\left(e\right)=f\left(2e\right)-c\left(e\right).$$

Since by Proposition 2 we have $e\left(\bar{e}\right)<e^C$, then $\forall e$ such that $e\left(\bar{e}\right)<e<e^C$ it is:

$$\begin{cases} g_1'\left(e\right)<0, \\ g_2'\left(e\right)>0, \end{cases}$$

because $e\left(\bar{e}\right)$ is the optimal point of $g_1\left(e\right)$ and e^C is the optimal point of $g_2\left(e\right)$. It follows:

$$-g_2'\left(e^*\right)+g_1'\left(e^*\right)=-2f'\left(2e^*\right)+f'\left(e^*+\bar{e}\right)<0.$$

This way if $e\left(\bar{e}\right)<e^C$ it follows that $\partial e^*/\partial m<0$ and therefore the optimal effort to problem (4) is decreasing with respect to m. In the limit case, all agents are committed, and obviously, the optimal effort will be e^C. Finally, let us consider:

$$\frac{\partial e^*}{\partial \bar{e}} = -\frac{\partial F/\partial \bar{e}}{\partial F/\partial e^*} =$$

$$= -\frac{mf''(e^* + \bar{e})}{4(2n - m - 1) f''(2e^*) + mf''(e^* + \bar{e}) - (2n - 1) c''(e^*)} < 0.$$

and the second part of the theorem follows.

□

Figure 1 summarizes the results stated in Proposition 2 and Theorem 2.

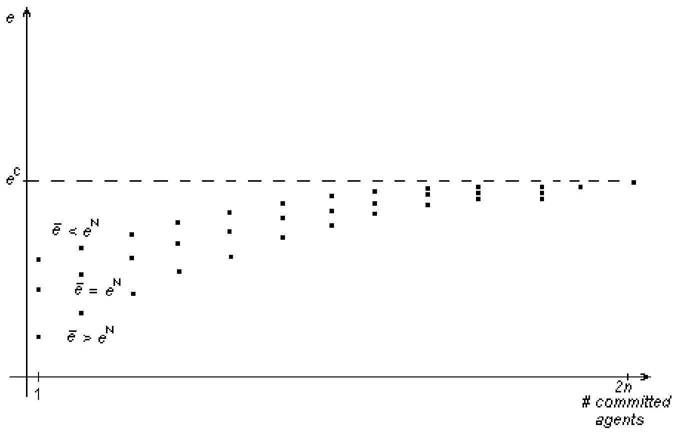

Figure 1: Fixed vs Committed

As the number of committed players approaches the whole population, their effort approximates e^C and, the higher is the effort provided by fixed effort agents, the lower is the optimal effort supplied by committed players. This result may be explained since when the population tends to consist of almost only committed agents, the probability for a single committed agent to be paired with a different kind of agent will be low. Furthermore, when few committed agents face a population of fixed effort agents, they will expect a low probability of facing another committed agent. As it concerns the second part it may be interpreted analogously to Proposition 2.

4.2 Rational agents vs committed agents

Let us consider rational agents and recall that while their goal is to maximize their profit, they are not able to coordinate as committed agents do. It is obvious that if the population consists only of *rational* agents they will play the Nash equilibrium. In this section we study the equilibria in a mixed population composed of both committed and rational agents.

Consider:

1. m rational agents playing the effort e_i , $i = 1, \dots, m$
2. $(2n - m)$ committed agents who play the best effort e_{co} knowing the fact that there are m rational agents.

A single committed agent does not know which kind of mate it will be paired with so it has to solve the following problem:

$$\max_{e_{co}} \quad \frac{2n-m-1}{2n-1}\left(f\left(2e_{co}\right)-c\left(e_{co}\right)\right)+\frac{m}{2n-1}\left(f\left(e_{co}+e_j\right)-c\left(e_{co}\right)\right) \tag{5}$$

$$j = 1, \dots, m$$

while a rational agent i will solve:

$$\max_{e_i} \quad \frac{2n-m}{2n-1}\left(f\left(e_i+e_{co}\right)-c\left(e_i\right)\right)+\frac{1}{2n-1}\sum_{j\neq i}\left(f\left(e_i+e_j\right)-c\left(e_i\right)\right). \tag{6}$$

Proposition 3 *In a mixed population composed of both committed and rational agents, the effort provided by the committed is not inferior to the effort provided by the rational agents.*

Proof: Let us consider FOC of problem (5) and (6):

$$\begin{cases} F\left(m,e_{co}\right)=\left(2n-m-1\right)\left[2f'\left(2e_{co}\right)-c'\left(e_{co}\right)\right]+m\left[f'\left(e_{co}+e_j\right)-c'\left(e_{co}\right)\right]=0, \\ G\left(m,e_i\right)=\left(2n-m\right)\left[f'\left(e_i+e_{co}\right)-c'\left(e_i\right)\right]+\sum_{j\neq i}\left[f'\left(e_i+e_j\right)-c'\left(e_i\right)\right]=0. \end{cases}$$

All rational agents solve the same problem, therefore we consider symmetrical equilibria: $e_i = e_r$, $i = 1, \dots, m$

$$\begin{cases} F\left(m,e_{co}\right)=\left(2n-m-1\right)\left[2f'\left(2e_{co}\right)-c'\left(e_{co}\right)\right]+m\left[f'\left(e_{co}+e_r\right)-c'\left(e_{co}\right)\right]=0, \\ G\left(m,e_r\right)=\left(2n-m\right)\left[f'\left(e_r+e_{co}\right)-c'\left(e_r\right)\right]+\left(m-1\right)\left[f'\left(2e_r\right)-c'\left(e_r\right)\right]=0. \end{cases} \tag{7}$$

By contradiction let $e_r > e_{co}$, it follows $2e_r > e_r + e_{co} > 2e_{co}$.

Now consider $f'\left(e_{co}+e_r\right)-c'\left(e_{co}\right)$:

- if $f'\left(e_{co}+e_r\right)-c'\left(e_{co}\right) \geq 0$, we have:

$$e_r + e_{co} > 2e_{co} \Rightarrow f'\left(2e_{co}\right) > f'\left(e_r+e_{co}\right)$$

and it follows

$$2f'\left(2e_{co}\right)-c'\left(e_{co}\right) > f'\left(2e_{co}\right)-c'\left(e_{co}\right) > 0$$

absurd since it contradicts the first FOC in (7).

- if $f'(e_{co}+e_r)-c'(e_{co})<0$, we have:

$$e_r > e_{co} \Rightarrow f'(e_r+e_{co})-c'(e_r) < f'(e_r+e_{co})-c'(e_{co}) < 0$$

and it is

$$2e_r > e_r+e_{co} \Rightarrow f'(2e_r)-c'(e_r) < f'(e_r+e_{co})-c'(e_r).$$

Putting together we obtain:

$$f'(2e_r)-c'(e_r) < f'(e_r+e_{co})-c'(e_r) < f'(e_r+e_{co})-c'(e_{co}) < 0$$

absurd since it contradicts the second FOC in (7).

It follows $e_r \le e_{co}$.

□

This may be easily interpreted since rational agents deviate and shirk in order to maximize their profit.

Theorem 3 *As the number of committed agents increases to* $2n$*, the optimal effort provided by the rational agents decreases to the best reply effort to the coordination equilibrium* $e\left(e^C\right)$ *and the optimal effort provided by the committed agents increases to the coordination equilibrium effort* e^C.

Proof: Let us consider conditions (7), by implicit function theorem it is possible to write:

$$\frac{\partial e_r}{\partial m} = -\frac{\partial G/\partial m}{\partial G/\partial e_r} = -\frac{-f'(e_r+e_{co})+f'(2e_r)}{(2n-m)\,f''(e_r+e_{co})+2\,(m-1)\,f''(2e_{co})-(2n-1)\,c''(e_r)}.$$

By assumptions on concavity/convexity of f and c the denominator is negative, and, as it concerns the numerator, since $e_r < e_{co}$ and f is a marginal decreasing function, it is positive. Therefore, $\partial e_r/\partial m > 0$. Obviously, when a single rational agent faces only committed agents it will play the best reply effort to the coordination equilibrium $e\left(e^C\right)$.

As it concerns the second part of the statement, it is obvious that when a single committed agent faces only rational agents it will play the Nash effort e^N and, vice versa, if the population consists only of committed agents, they all will play the coordinated effort e^C. Furthermore, committed agent's effort e_{co} depends on m, the number of rational agents, both directly and indirectly, via the optimal effort of rational agents:

$$e_{co} = g\,(m, e_r\,(m)).$$

We consider the infinitesimal variation in e_{co} as m increases:

$$\frac{de_{co}}{dm} = \frac{\partial g}{\partial m} + \frac{\partial g}{\partial e_r}\frac{de_r}{dm}.$$

By Theorem (2) $\partial g/\partial m$ is negative; we just proved that $de_r/dm > 0$, and by second part of Theorem (2) also $\partial g/\partial e_r < 0$. It follows the second part of the thesis.

□

Figure 2 summarizes the results stated in Proposition 3 and Theorem 3.

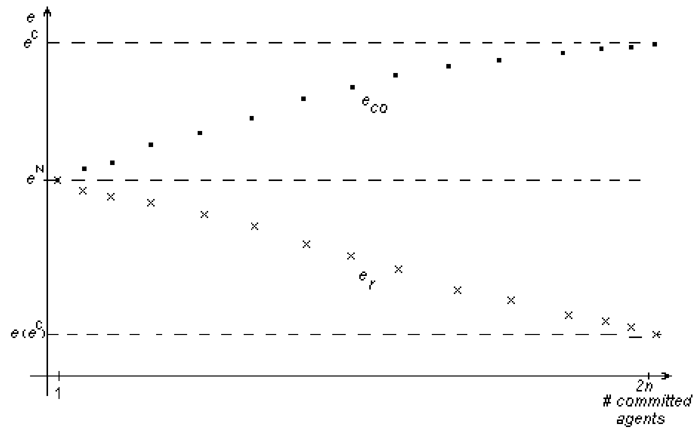

Figure 2: Rational vs Committed

When few rational agents face many committed agents, they expect that the latter will tend to exert an effort close to the committed one and, as a consequence, they will tend to play the best reply to the expected effort. Vice versa, when few committed agents face many rational agents, their effort will be lower since the probability to be paired with agents maximizing their own profit will be higher.

4.3 Fixed effort agents vs rational agents

Consider:

1. m fixed effort $\bar{e}$ agents;
2. $(2n-m)$ rational agents playing the effort e_i , $i = 1, \ldots, 2n-m$.

A single rational agent i will solve:

$$\max_{e_i} \quad \frac{m}{2n-1}\left(f\left(e_i+\bar{e}\right)-c\left(e_i\right)\right)+\frac{1}{2n-1}\sum_{j\neq i}\left(f\left(e_i+e_j\right)-c\left(e_i\right)\right). \tag{8}$$

Theorem 4 *Assume $\bar{e} > e^N$ ($\bar{e} < e^N$), as the number of fixed effort agents increases to $2n$, the optimal effort provided by the rational agents decreases (increases) to $e(\bar{e})$.*

Proof: When a single rational agent faces only fixed effort agents it will play $e(\bar{e})$. By proposition (2) it is $\bar{e} > e^N \Rightarrow e(\bar{e}) < e\left(e^N\right) = e^N$ and, obviously, $e(\bar{e}) \leq e_r \leq e^N$.

Let us consider FOC of problem (8) when we consider symmetrical equilibria: $e_i = e_r$, $i = 1, \ldots, 2n - m$:

$$H(m, e_r) = m\left[f'(e_r + \bar{e}) - c'(e_r)\right] + (2n - m - 1)\left[f'(2e_r) - c'(e_r)\right] = 0. \tag{9}$$

By implicit function theorem it is possible to write:

$$\frac{\partial e_r}{\partial m} = -\frac{\partial H/\partial m}{\partial H/\partial e_r} = -\frac{f'(e_r + \bar{e}) - f'(2e_r)}{m\left[f''(e_r + \bar{e}) - c''(e_r)\right] + (2n - m - 1)\left[2f''(2e_r) - c''(e_r)\right]}.$$

And it follows $\partial e_r/\partial m < 0$.

Similarly, the case in which $\bar{e} < e^N$ can be proved.

□

Figure 3 summarizes the results stated in Theorem 4.

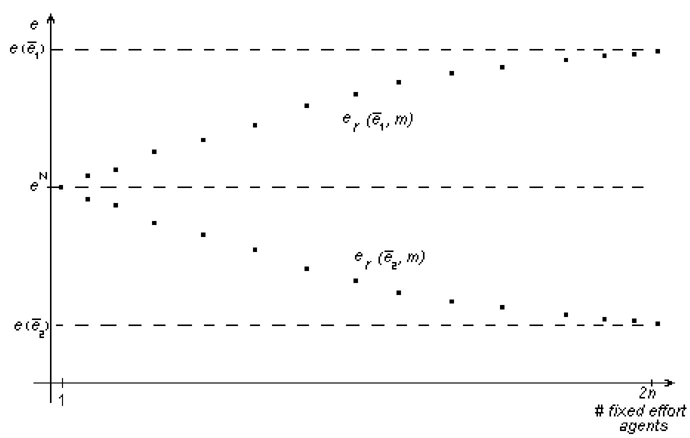

Figure 3: Rational vs Fixed

This may be explained taking into account the relative proportion of the population and the fact that the rational agents maximize their profit.

5 A ranking policy

The firm may consider to introduce some mechanism to increase productivity, for example, rewarding employees who maximize their profit. Nevertheless, such mechanisms must be devised carefully since a known problem of individual incentives is that they may lead employees competing with one another (see for instance [12]). In particular, we introduce a *ranking policy* and will show what may be some of the undesirable results for the agents. Consider the following individual incentive plan: all agents are ranked according to their payoff and normalized in the sense that the agent with the higher payoff will get 1 and the one with the lower will get 0; should all the agents have the same payoff, every agent will get 1. Formally:

$$r_i = \begin{cases} \dfrac{\pi_i - \min\limits_{j\in I}\{\pi_j\}}{\max\limits_{j\in I}\{\pi_j\} - \min\limits_{j\in I}\{\pi_j\}} & \text{if} \quad \max\limits_{j\in I}\{\pi_j\} \neq \min\limits_{j\in I}\{\pi_j\} \\ 1 & \text{if} \quad \max\limits_{j\in I}\{\pi_j\} = \min\limits_{j\in I}\{\pi_j\} \end{cases} \tag{10}$$

This policy may recall the "Employee of the Year" prize even if in a sketched way. Obviously, it must be noted that usually firms give this kind of prize observing agents' efforts and that in our model effort is not observable and costs are private. Nevertheless, this ranking policy may be viewed as a social policy where the best individual is the one with higher personal profit, and this kind of pressure may not be ignored since here we do not assume any social norm. In this section we will assume that agents want to maximize rank and examine how this ranking policy can affect the equilibrium set.

It is obvious that any profile of strategies where all players exert the same effort gives the maximum outcome. Nevertheless, some may prefer to avoid tie results and this may give incentive to deviation. This can be explained in different ways: for example, if an agent is particularly competitive it may deviate to be the only winner, while, if some others expect such a deviation, they could anticipate it and behave consequently. In particular:

Theorem 5 *Consider the team game defined in Section 2; if we introduce the ranking policy (10) the set of Nash equilibria expands to the set* $(\mathbf{e}^*)$ *where* $\mathbf{e}^* = (e^*, e^*, \ldots, e^*) \in \mathbb{R}^{2n}$, $e^* \in \left[0, e^N\right]$.

Proof: Let all the players exert the same effort $\hat{e}$, then player i will exert an effort e_i such that

$$\begin{cases} f(e_i + \hat{e}) - c(e_i) \geq f(e_i + \hat{e}) - c(\hat{e}), \\ f(e_i + \hat{e}) - c(e_i) \geq f(2\hat{e}) - c(\hat{e}). \end{cases}$$

The first condition guarantees that player i's profit will be not lower than its mate's j, while the second condition means that player i's profit is not lower than the profits of the members of the other teams. It should be noted that the first condition is equivalent to $e_i \leq \hat{e}$.

Let us consider a profile of strategies $e^* = (e^*, e^*, \ldots, e^*)$, where $e^* \in \left[0, e^N\right]$. If player i exerts effort $e_i = e^*$ his payoff is the same as the other players' and its rank will be 1. Furthermore, player i has no incentive to exert a lower effort since if we consider its profit, the partial derivative with respect to its effort is positive when $e_i = e^*$:

$$\frac{\partial}{\partial e_i}\left(\pi_i\left(e_i, e_j = e^*\right)\right)_{|e_i = e^*} = f'(2e^*) - c'(e^*) > f'\left(2e^N\right) - c'\left(e^N\right) = 0.$$

The second pure partial derivative is negative and this means that:

$$\frac{\partial}{\partial e_i}\left(\pi_i\left(e_i, e^*\right)\right) = f'\left(e_i + e^*\right) - c'\left(e_i\right) > 0 \qquad \forall e_i \in [0, e^*]$$

so there is no incentive to deviate.

Finally, let us consider a profile of strategies $\mathbf{e}^* = (e^*, e^*, \ldots, e^*)$, where $e^* > e^N$.

Player i has incentive to shirk since, considering its profit, the partial derivative with respect to its effort is positive when $e_i = e^*$:

$$\frac{\partial}{\partial e_i}\left(\pi_i\left(e_i, e_j = e^*\right)\right)_{|e_i = e^*} = f'\left(2e^*\right) - c'\left(e^*\right) < f'\left(2e^N\right) - c'\left(e^N\right) = 0$$

□

The set of Nash equilibria is expanded by the ranking policy; this adds some further problems in predicting which equilibrium will be selected. One way to overcome this could be to assume that agents have a lexicographic utility and among different equilibria would prefer the one with higher profit.

Finally, even in very simple cases we show how this policy may not increase the productivity of agents. Consider the situation in which a rational agent is employed in a firm where a corporate culture consisting in providing the same effort $\bar{e}$ is established. Without the incentive policy its optimal effort will be $e(\bar{e})$; this effort will be higher than $\bar{e}$ if and only if $\bar{e} < e^N$ and, in particular, will be the same as the other agents' if and only if $\bar{e} = e^N$.

Now, introducing the incentive policy, it holds:

Theorem 6 *Consider* $2n-1$ *fixed effort* $\bar{e}$ *agents, and a single rational agent, then the* incentive policy *never incentives the rational agent to provide an effort larger than* $\bar{e}$.

Proof: The rational agent will solve:

$$\max_e \quad f\left(e + \bar{e}\right) - c\left(e\right)$$

and take into account also its partner profit.

Let $e^* := \arg\max\left[f\left(e + \bar{e}\right) - c\left(e\right)\right]$; three cases are given:

1. $e^* < \bar{e}$: in this case the rational agent will provide this effort since $\left[f\left(e + \bar{e}\right) - c\left(e\right)\right] < \left[f\left(2\bar{e}\right) - c\left(\bar{e}\right)\right]$ and will be the only one to get payoff 1;

2. $e^* = \bar{e}$: this case is trivial since all agents will get 1;

3. $e^* > \bar{e}$: in this case should the rational agent provide this effort its partner would free ride and would be the only one to get 1.

Clearly there is no incentive to provide efforts higher than $\bar{e}$.

$\square$

This simple result is very important because it shows, in this simple model, some of the "undesirable results" [12]) of the individual incentive. In particular, if the fixed level $\bar{e}$ is low, the effort of a new rational employee is bounded by this individual incentive plan.

6 Simulation Results

In the following we propose a simulation approach in order to extend the theoretical results we derived in previous sections. The agent based simulations we performed were implemented on a customized version of the platform described in [2]. The platform has been modified simply including the behavioral classes we described in the theoretical analysis. In particular, we considered the same functional form which was used in human subject experiments in [3], that is, $f(e) = 5\sqrt{e}$ and $c(e) = e^2$; when agents i and j are in the same team, their payoff is respectively

$$\begin{cases} \pi_i(e_i, e_j) = 5\sqrt{e_i + e_j} - e_i^2, \\ \pi_j(e_i, e_j) = 5\sqrt{e_j + e_i} - e_j^2. \end{cases} \tag{11}$$

It is worth noting that, considering the one-shot game, the Nash equilibrium is $e^N = \sqrt[3]{25/32} \simeq .92100787466$ while the coordination equilibrium is $e^C = \sqrt[3]{25}/2 \simeq 1.4620088691$.

We are interested in extending the results about optimal effort when considering a mixed population of $2n$ agents which is partitioned as follows:

1. rational agents $1, 2, \dots, m_1$ who maximize their individual profit;
2. m_2 fixed effort $\bar{e}$ agents;
3. $(2n - m_1 - m_2)$ committed agents who coordinate on the best effort.

For all the agents the population composition is common knowledge. As a consequence an equilibrium configuration can be found solving

$$\begin{cases} \max\limits_{e_i} \quad \frac{2n-m_1-m_2}{2n-1} 5\sqrt{e_i + e_{co}} + \frac{1}{2n-1} \sum\limits_{\substack{j \leq m_1 \\ j \neq i}} 5\sqrt{e_i + e_j} + \frac{m_2}{2n-1} 5\sqrt{e_i + \bar{e}} - e_i^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (i = 1, 2, \dots, m_1)\ , \\ \max\limits_{e_{co}} \quad \frac{2n-m_1-m_2-1}{2n-1} 5\sqrt{2e_{co}} + \frac{m_1}{2n-1} 5\sqrt{e_{co} + e_r} + \frac{m_2}{2n-1} 5\sqrt{e_{co} + \bar{e}} - e_{co}^2. \end{cases}$$

As we assume that rational agents are symmetric an equilibrium configuration may be (e_r^*, e_{co}^*) where e_r^* indicates the optimal effort for the m_1 rational agents and e_{co}^* indicates the optimal effort for the m_2 committed agents. In the simulation, by the symmetry assumption, rational and committed agents decide their efforts dynamically as follows:

$$\begin{cases} e_r^{t+1} = \arg\max_{e_r} & \frac{2n-m_1-m_2}{2n-1}5\sqrt{e_r+e_{co}^t} + \frac{m_1-1}{2n-1}5\sqrt{e_r+e_r^t} + \frac{m_2}{2n-1}5\sqrt{e_r+\bar{e}} - e_r^2, \\ e_{co}^{t+1} = \arg\max_{e_{co}} & \frac{2n-m_1-m_2-1}{2n-1}5\sqrt{2e_{co}} + \frac{m_1}{2n-1}5\sqrt{e_{co}+e_r^t} + \frac{m_2}{2n-1}5\sqrt{e_{co}+\bar{e}} - e_{co}^2. \end{cases} \tag{12}$$

Since in our case the best reply functions are given implicitly, in the simulation we use an iterative method in order to solve the maximization problem and therefore the best reply functions for each agent are numerically evaluated. In our simulations we compute the steady state for different population compositions. Assuming $2n = 900$ agents, we represent on the $x-$axis the number of fixed agents and on the $y-$axis the number of rational agents; it follows that point (x, y), where $0 \leq x, y \leq 900$ and $x + y \leq 900$, represents a population consisting of x fixed agents, y rational agents and $900 - x - y$ committed agents. For each population composition we compute the rational agent and committed agent steady effort assuming that all agents know the population composition. The respective efforts[4] are illustrated in Figures 4 and 5.

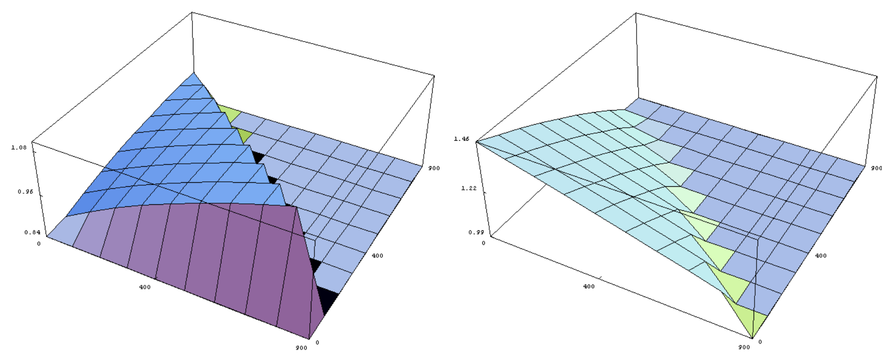

Figure 4: Rational (*left*) and Committed (*right*) agent effort with global interaction and fixed effort level $\bar{e} = 0.1$

It can be observed that the results we obtain by simulation are consistent with all the theoretical we obtained in previous sections. In fact, when considering the restrictions for zero fixed agents in Figure 4, we obtain respectively rational and committed agents' efforts; combining opportunely these restrictions we obtain the results stated in Proposition 3 and Theorem 3, as depicted in Figure 2. Observe that the same results hold for Figure 5 as we do not consider fixed effort agents.

Then, when considering the restrictions for zero rational agents in Figures 4 (*right*) and 5 (*right*), we obtain respectively the committed agents' efforts in a 0.1 and 1.2 fixed effort agents population; an opportune combination of these restrictions allows us to obtain the results stated in Proposition 2 and Theorem 2, as summarized in Figure 1.

[4]Recall that, given the population composition, efforts are defined only for $x + y \leq 900$.

Finally, when considering the restrictions for zero committed agents in Figures 4 (*left*) and 5 (*left*), i.e, $y = 900 - x, \quad x = 0, 100, 200, \ldots, 900$, we obtain respectively the rational agents' efforts in a 0.1 and 1.2 fixed effort agents population. Once again, with an appropriate combination of these restrictions we obtain the results stated in Theorem 4 and summarized in Figure 3.

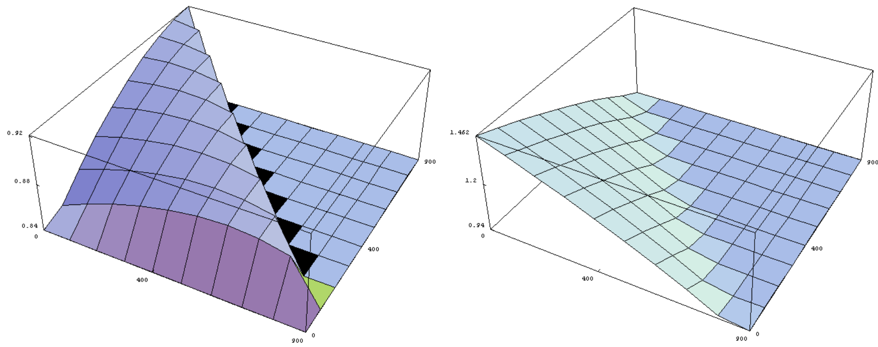

Figure 5: Rational (*left*) and Committed (*right*) agent effort with global interaction and fixed effort level $\bar{e} = 1.2$

So far, in the simulations, we have assumed that the population composition was common knowledge and that agents could interact globally, as in a complete graph (see [13]). In the following, we analyze the consequence of a *local* interaction. We assume that agents can interact in a Von Neumann neighborhood, and do not know the population composition. This has important consequences in the dynamic process described in (12) as agents must have knowledge of their opponents' best reply in order to make a decision. In particular, each agent assumes that all the agents in its neighborhood share its own information about neighborhood composition. We can say that the agents have egocentric thought about neighborhood in the sense of Piaget [9].

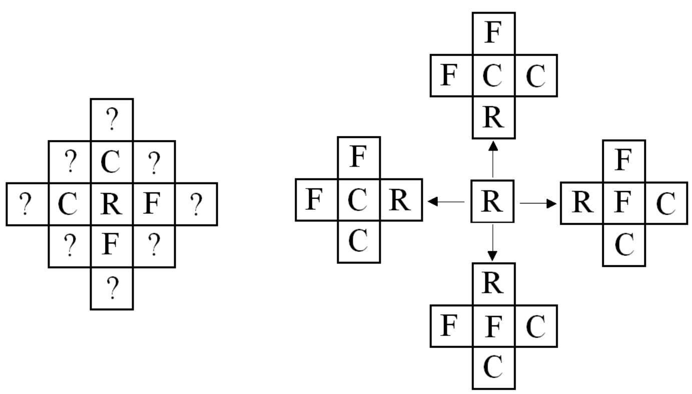

Figure 6: An example of egocentric neighborhood

For example, in Figure 6 assume that a rational agent interacts with two fixed effort agents and two committed agents. That is, its neighborhood consists of one rational, two committed and two fixed effort agents. This agent does not have the complete composition of its neighbors' neighborhood; in our model we assume that the agents consider all their neighbors having their same neighborhood composition. That is, the local composition is assumed to be constant.

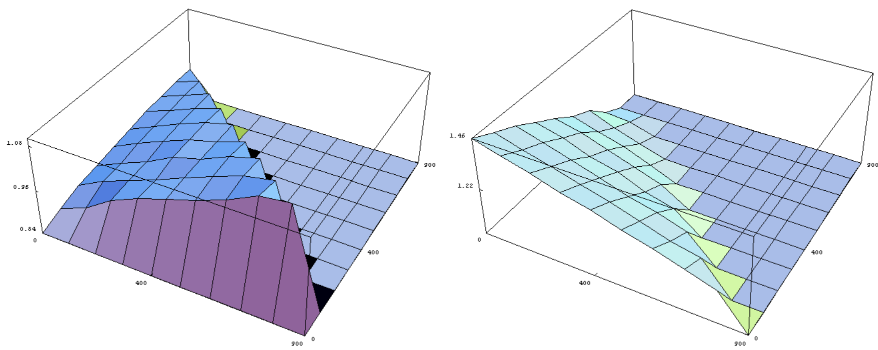

Figure 7: Rational (*left*) and Committed (*right*) agent effort with local interaction and fixed effort level $\bar{e} = 0.1$

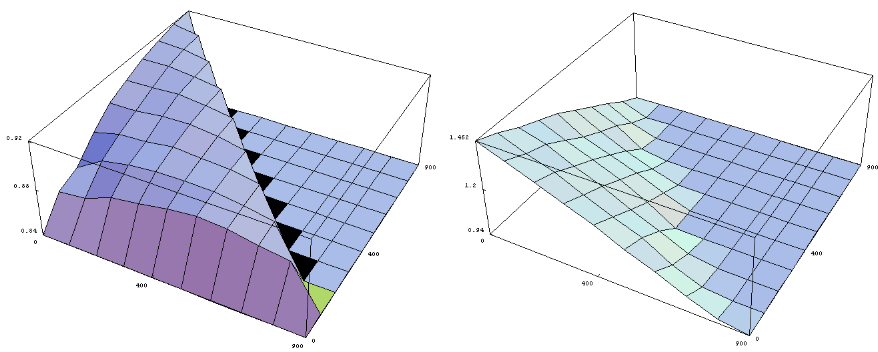

Figure 8: Rational (*left*) and Committed (*right*) agent effort with local interaction and fixed effort level $\bar{e} = 1.2$.

In our simulations the agents are randomly located on a toroidal lattice; the results we obtain with this assumption and local interaction are presented in Figures 7 and 8. Comparing these results to those presented in figures 4 and 5, we can observe that the average efforts are quite close to those in global interaction. So, global interaction behavior is a good approximation of the average behavior we obtain when considering local interaction in Von Neumann neighborhood with egocentric agents. Finally, it is interesting to observe that when agents are located according to their behavioral class instead of being randomly located, the results are quite similar to those obtained when considering homogeneous populations .

7 Conclusions and further research

In this paper we described a game which tries to model some of the transactions taking place in a firm. We discussed some possible equilibria of the game in the simpler cases and observed how the game may be compared to a version of the Prisoner's Dilemma. The results we found and the stability of equilibria are interpreted in terms of corporate culture.

We considered also some "bounded rationality" agents and the impact their presence has on the *ideal cases* equilibria.

Then, we discussed a ranking incentive devised to increase competition between agents; we showed how this ranking policy affects the performance of the subjects and found the set of equilibria of the modified game. Since the modified game has a continuum of equilibria we suggested one possible way to refine them. The results we found are interesting in terms of free riding and shirking and may help to shed light on how the different composition of employees may result in different equilibria.

Furthermore, we showed in the simple model of firm how an individual incentive plan may result in limiting the optimal effort of some agents.

Finally, by means of agent based simulation two more directions were investigated. First, we could consider heterogeneous populations with coexistence of all the kinds of behavior we studied in the theoretical analysis. This way, we could overcome the problem of agents having the best reply functions in implicit form by a numerical iterative method. The comparison of the results, when assuming that population composition was common knowledge across the agents, exhibited consistency with the theoretical results. Second, we assumed that the agents could interact only in their Von Neumann neighborhood, without knowing the global population composition. Since for practical reasons each agent must know the best reply of its opponents, we modeled a sort of egocentric thought about the neighbors' neighborhood composition. That is, we assumed that each agent had an egocentric perspective in a sense close to Piaget about the composition of its neighbors' neighborhood. In this case the local interaction results were quite similar to the global ones, and therefore coherent with the theoretical analysis.

In further research it would be interesting to analyze which equilibria may be selected in a dynamic setting of the game. It would be also interesting to consider also some more sophisticated models of bounded rationality.

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