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A Generalization of Wiman and Valiron's theory to the Clifford analysis setting

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ABSTRACT

The classical notions of growth orders, maximum term and the central index provide powerful tools to study the asymptotic growth behavior of complex-analytic functions.

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This leads to much insight into the structure of the solutions to many two dimensional partial differential equations that are related to boundary value problems from harmonic analysis in the plane. In this overview paper we show how the classical techniques and results from Wiman and Valiron can be extended to the Clifford analysis setting in order to treat successfully analogous higher dimensional problems.

RESUMEN

Las nociones clásicas de orden de crecimiento, término máximo y de índice central proporcionan herramientas poderosas para estudiar el comportamiento de crecimiento asintótico de funciones complejas analíticas. Esto nos revela la estructura de las soluciones de varias ecuaciones diferenciales parciales de dimensión dos que son relacionadas con problemas de valores en la frontera venidos de análisis armónico en el plano. Mostramos como las técnicas clásicas y resultados de Wiman y Valiron pueden ser extendidas al contexto de análisis de Clifford para tratar con éxito problemas análogos de dimensión grande.

Key words and phrases: *monogenic functions, growth orders, growth type, maximum term, central index, Valiron's inequalities, asymptotic growth, partial differential equations.*

Math. Subj. Class.: *30G35, 30D15.*

1 Introduction

The study of the asymptotic growth behavior of holomorphic and meromorphic functions in one and several complex variables is one of the central topics in complex analysis. This line of investigation started with early works of E. Lindelöf [22], A. Pringsheim [24], A. Wiman [26] and G. Valiron [25] and had its major breakthrough in the 1920s by works of R. Nevanlinna [23] and his school. Their results turned out to be very useful in the study of complex partial differential equations, see e.g. [20, 21] and elsewhere.

This provides a strong motivation to also develop analogous methods for other function classes and in higher dimensions. One natural higher dimensional generalization of complex analysis is Clifford analysis. In this context one considers Clifford algebra valued solutions of the generalized Cauchy-Riemann system

$$Df := \frac{\partial f}{\partial x_0} + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} e_i = 0. \tag{1}$$

Solutions to this system are often called monogenic or Clifford holomorphic. Many classical theorems from complex analysis, such as for instance the Cauchy integral formula, the residue theorem, Laurent expansion theorems, etc. carry over to the higher dimensional context using this operator, see for instance [8, 5, 7]. Nevertheless, as far as we know, questions concerning possible generalizations of Wiman-Valiron theory remained untouched for a long time.

In [1] M.A. Abul-Ez and the first author introduced the notion of the growth order and the type for a particular subclass of entire Clifford holomorphic functions. See also the follow-up papers [2, 3]. In our recent papers [8, 9, 11, 13] we developed the basics for a generalized Wiman-Valiron theory for general entire monogenic functions and for monogenic Taylor series of finite convergence radius. We also managed to extend these techniques to the context of more general systems of partial differential equations, such as higher dimensional iterated Cauchy-Riemann systems [6, 7] and to polynomial Cauchy-Riemann systems equations with complex coefficients [10]. In this paper we give a concise overview over our results concerning the entire monogenic case. We show how the notions of growth orders, growth type, maximum term and the central index can be reasonably generalized to the Clifford analysis context. We exhibit how these tools can be applied to get insight in the asymptotics of related function classes and in the structure of solutions to related higher dimensional partial differential equations. This line of investigation should be regarded as a starting point to develop analogous methods for larger classes of functions that are in kernels of elliptic differential operators. We hope to get more insight in the structure of the solutions to larger classes of higher dimensional partial differential equations.

2 Preliminaries

We begin by introducing the basic notions and concepts. For detailed information about Clifford algebras and their function theory we refer for example to [8, 1] and [7].

2.1 Clifford algebras

By $\{e_1, e_2, \ldots, e_n\}$ we denote the canonical basis of the Euclidean vector space $\mathbb{R}^n$. The attached real Clifford algebra Cl_{0n} is the free algebra generated by $\mathbb{R}^n$ modulo the relation

$$\mathbf{x}^2 = -\|\mathbf{x}\|^2 e_0,$$

where $\mathbf{x} \in \mathbb{R}^n$ and e_0 is the neutral element with respect to multiplication of the Clifford algebra Cl_{0n}. In the Clifford algebra Cl_{0n} the following multiplication rules hold

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0, \quad i, j = 1, \cdots, n,$$

where δ_{ij} is the Kronecker symbol. A basis for the Clifford algebra Cl_{0n} is given by the set $\{e_A : A \subseteq \{1, \cdots, n\}\}$ with $e_A = e_{l_1} e_{l_2} \cdots e_{l_r}$, where $1 \leq l_1 < \cdots < l_r \leq n$, $e_\emptyset = e_0 = 1$. Each $a \in Cl_{0n}$ can be written in the form $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$. Two examples of real Clifford algebras are the complex number field $\mathbb{C}$ and the Hamiltonian skew field $\mathbb{H}$.

The conjugation anti-automorphism in the Clifford algebra Cl_{0n} is defined by $\overline{a} = \sum_A a_A \overline{e}_A$, where $\overline{e}_A = \overline{e}_{l_r}\overline{e}_{l_{r-1}} \cdots \overline{e}_{l_1}$ and $\overline{e}_j = -e_j$ for $j = 1, \cdots, n$, $\overline{e}_0 = e_0 = 1$. The linear subspace $span_{\mathbb{R}}\{1, e_1, \cdots, e_n\} = \mathbb{R} \oplus \mathbb{R}^n \subset Cl_{0n}$ is the so-called space of paravectors $z = x_0 + x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$ which we simply identify with $\mathbb{R}^{n+1}$. The term $x_0 =: Sc(z)$ is called the scalar part of the paravector z and $\mathbf{x} := x_1 e_1 + \cdots + x_n e_n =: Vec(z)$ its vector part.

A scalar product between two Clifford numbers $a, b \in Cl_{0n}$ is defined by $\langle a, b \rangle := Sc(a\overline{b})$ and the Clifford norm of an arbitrary $a = \sum\limits_A a_A e_A$ is $\|a\| = (\sum\limits_A |a_A|^2)^{1/2}$.

Any paravector $z \in \mathbb{R}^{n+1} \backslash \{0\}$ has an inverse element in $\mathbb{R}^{n+1}$ given by $z^{-1} = \overline{z}/\|z\|^2$.

In order to present the calculations in a more compact form, the following notations will be used, where $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ is an n-dimensional multi-index:

$$\mathbf{x}^{\mathbf{m}} := x_1^{m_1} \cdots x_n^{m_n}, \quad \mathbf{m}! := m_1! \cdots m_n!, \quad |\mathbf{m}| := m_1 + \cdots + m_n.$$

By $\tau(i)$ we denote the multi-index $(m_1, \ldots, m_n)$ with $m_j = \delta_{ij}$ for $1 \leq j \leq n$.

2.2 Clifford analysis

One way to generalize complex function theory to higher dimensional hypercomplex spaces is offered by the Riemann approach which considers Clifford algebra valued functions defined in $\mathbb{R}^{n+1}$ that are annihilated by the generalized Cauchy-Riemann operator

$$D := \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i}. \tag{2}$$

If $U \subset \mathbb{R}^{n+1}$ is an open set, then a real differentiable function $f : U \to Cl_{0n}$ is called left (right) monogenic or Clifford holomorphic at a point $z \in U$ if $Df(z) = 0$ (or $fD(z) = 0$). Functions that are left monogenic in the whole space are also called left entire.

The notion of left (right) monogenicity in $\mathbb{R}^{n+1}$ provides indeed a powerful generalization of the concept of complex analyticity to Clifford analysis. Many classical theorems from complex analysis could be generalized to higher dimensions by this approach, we refer e.g. to [8]. One important tool is the generalized Cauchy integral formula.

Let us denote by A_{n+1} the n-dimensional surface "area" of the $(n+1)$-dimensional unit ball, and by $q_{\mathbf{0}}(z) = \frac{\overline{z}}{\|z\|^{n+1}}$ the Cauchy kernel function.

Then every function f that is left monogenic in a neighbourhood of the closure $\overline{\mathcal{D}}$ of a domain $\mathcal{D}$ satisfies

$$f(z) = \frac{1}{A_{n+1}} \int_{\partial \mathcal{D}} q_{\mathbf{0}}(z - w)\, d\sigma(w)\, f(w), \tag{3}$$

where $d\sigma(w)$ is the paravector-valued outer normal surface measure, i.e.,

$$d\sigma(w) = \sum_{j=0}^{n} (-1)^j e_j dw_0 \wedge \cdots \wedge \widehat{dw_j} \wedge \cdots \wedge dw_n$$

with $\widehat{dw_j} = dw_0 \wedge \cdots \wedge dw_{i-1} \wedge dw_{i+1} \wedge \cdots \wedge dw_n$. It is important to mention that the set of left (right) monogenic functions forms only a Clifford right (left) module for $n > 1$.

In contrast to complex analysis, the ordinary powers of the hypercomplex variables are not null-solutions to the generalized Cauchy-Riemann system. In Clifford analysis these are substituted by the Fueter polynomials. These are defined by

$$\mathcal{P}_{\mathbf{m}}(z) = \frac{1}{|\mathbf{m}|!} \sum_{\pi \in perm(\mathbf{m})} z_{\pi(m_1)} \cdots z_{\pi(m_n)}$$

where $perm(\mathbf{m})$ is the set of all permutations of the sequence $(m_1, \ldots, m_n)$ and $z_i := x_i - x_0 e_i$ for $i = 1, \ldots, n$ and $\mathcal{P}_{\mathbf{0}}(z) := 1$.

In this paper we prefer to work with the slightly modified Fueter polynomials

$$V_{\mathbf{m}}(z) := \mathbf{m}! \mathcal{P}_{\mathbf{m}}(z) \tag{4}$$

which turns out to be more convenient in our calculations.

These polynomials play the analogous role of the complex power functions in the Taylor series representation of a monogenic function. More precisely, if f is a left monogenic function in a ball $\|z\| < R$, then for all $\|z\| \leq r$ with $0 < r < R$

$$f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} V_{\mathbf{m}}(z) a_{\mathbf{m}},$$

where the elements $a_{\mathbf{m}}$ are Clifford numbers which — as a consequence of Cauchy's integral formula (3) — are uniquely defined by

$$a_{\mathbf{m}} = \frac{1}{\mathbf{m}! A_{n+1}} \int_{\|z\|<r} q_{\mathbf{m}}(\zeta) d\sigma(\zeta) f(\zeta)$$

where

$$q_{\mathbf{m}}(z) = \frac{\partial^{m_0+m_1+\cdots+m_n}}{\partial x_0^{m_0} \partial x_1^{m_1} \cdots \partial x_n^{m_n}} q_{\mathbf{0}}(z) \tag{5}$$

are the generalized negative power functions. An optimal upper bound estimate for the general negative power functions (5) is given in (cf. [5]) by

$$\|q_{\mathbf{m}}(z)\|_{\|z\|=r} \leq \frac{n(n+1)\cdots(n+|\mathbf{m}|-1)}{r^{|\mathbf{m}|+n}}. \tag{6}$$

We thus have the following upper bound estimate on the Taylor coefficients

$$\|a_{\mathbf{m}}\| \leq M(r,f)\frac{c(n,\mathbf{m})}{r^{|\mathbf{m}|}}.$$

Here, and in all that follows,

$$M(r,f) := \max_{\|z\| r}\{\|f(z)\|\}$$

denotes the maximum modulus of the function f in the closed ball with radius r and

$$c(n,\mathbf{m}) := \frac{n(n+1)\cdots(n+|\mathbf{m}|-1)}{\mathbf{m}!}. \tag{7}$$

3 Order of growth of monogenic functions in $\mathbb{R}^{n+1}$

In this overview paper we restrict us to treat entire monogenic functions. These are represented by Taylor series with infinite convergence radius. Many of the results that we are going to present here can be adapted to the case of monogenic Taylor series of finite convergence radius. However, this requires in many circumstances a much more technical treatment. We refer the reader who is interested in this particular topic to our recent paper [11].

The starting point for the following investigation is that monogenic functions $f : \mathbb{R}^{n+1} \to Cl_{0n}$ satisfy a maximum principle (cf. e.g. [8]). As a consequence, the function

$$M(r,f) := \max_{\|z\|=r}\{\|f(z)\|\}, \qquad r \geq 0 \tag{8}$$

is a well-defined, continuous and strictly monotonic increasing function whenever f is non-constant. From Cauchy's inequality one can deduce a direct generalization of the classical Liouville theorem (cf. [14, 16]). This states that every left entire monogenic function that is bounded in $\mathbb{R}^{n+1}$ is a constant. Cauchy's inequality furthermore permits us to deduce the following more general version of Liouville's theorem (see [13])

Theorem 1. *Suppose that $f : \mathbb{R}^{n+1} \to Cl_{0n}$ is left entire. If there exists an index $\mathbf{s} \in \mathbb{N}_0^n$ with $|\mathbf{s}| > 0$ satisfying*

$$\liminf_{r\to\infty} \frac{M(r,f)}{r^{|\mathbf{s}|}} = L < \infty, \tag{9}$$

then

$$f(z) = \sum_{|\mathbf{m}|=0}^{|\mathbf{s}|} V_{\mathbf{m}}(z)a_{\mathbf{m}}.$$

In order to characterize larger classes of monogenic functions by their asymptotic growth behavior it turned out to be convenient to introduce growth orders for monogenic functions [1, 3, 13]. For convenience we recall its definition. First we need, cf. e.g. [20]:

Definition 1. *Let $\alpha \geq 0$. Then the plus logarithm is defined by*

$$\log^+(\alpha) := \max\{0, \log(\alpha)\}. \tag{10}$$

In the same way as in the planar case (see [20]) one introduces in the Clifford analysis setting (see also [2, 13]):

Definition 2. *(Order of growth)*
Let $f : \mathbb{R}^{n+1} \to Cl_{0n}$ be an entire function. Then

$$\rho(f) = \rho := \limsup_{r\to\infty} \frac{\log^+(\log^+ M(r,f))}{\log(r)}, \quad 0 \leq \rho \leq \infty \tag{11}$$

is called the order of growth of the function f. We further introduce

$$\lambda(f) = \lambda : \liminf_{r\to\infty} \frac{\log^+(\log^+ M(r,f))}{\log(r)}, \quad 0 \leq \lambda \leq \infty \tag{12}$$

as the inferior order of growth of f.
If $\rho = \lambda$, then we say that f is a function of regular growth. If $\rho > \lambda$ then f has irregular growth.

To get a finer classification of the growth behavior within the set of monogenic functions that have the same growth order, one further introduces the growth type of a monogenic function as follows, cf. [9].

Definition 3. *For an entire monogenic function $f : \mathbb{R}^{n+1} \to Cl_{0n}$ of order ρ $(0 < \rho < \infty)$ the (growth) type is defined by*

$$\tau(f) = \tau := \limsup_{r\to\infty} \frac{\log^+ M(r,f)}{r^\rho}.$$

Let us start with discussing some particular examples. In [13] we have proved that the following higher dimensional generalizations of the exponential function all have growth order equal to 1:

(i) The monogenic plane wave exponential function from [1] defined for $\mathbf{m} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ by

$$f_1(\mathbf{m}, z) := (|\mathbf{m}| + i\mathbf{m})e^{-|\mathbf{m}|x_0} e^{i<\mathbf{m},\mathbf{x}>},$$

(ii) The monogenic generalization exponential function from [8]

$$\begin{aligned} f_2(z) = \exp(x_0, x_1, \ldots, x_n) &= e^{x_1+\cdots+x_n}\cos(x_0\sqrt{n}) \\ &- e^{x_1+\cdots+x_n}\frac{1}{\sqrt{n}}(e_1+\cdots+e_n)\sin(x_0\sqrt{n}) \end{aligned}$$

(iii) The quaternion-valued 3-fold periodic exponential function from [17] given by

$$f_3(z) := Exp_0(z) + e_1 Exp_1(z) + e_2 Exp_2(z) + e_3 Exp_3(z)$$

where

$$Exp_0(z) = e^{x_0}(\cos(\frac{x_1}{\sqrt{3}})\cos(\frac{x_2}{\sqrt{3}})\cos(\frac{x_3}{\sqrt{3}}) - \sin(\frac{x_1}{\sqrt{3}})\sin(\frac{x_2}{\sqrt{3}})\sin(\frac{x_3}{\sqrt{3}}))$$

$$Exp_1(z) = e^{x_0}\frac{\sqrt{3}}{3}(\sin(\frac{x_1}{\sqrt{3}})\cos(\frac{x_2}{\sqrt{3}})\cos(\frac{x_3}{\sqrt{3}}) + \cos(\frac{x_1}{\sqrt{3}})\sin(\frac{x_2}{\sqrt{3}})\sin(\frac{x_3}{\sqrt{3}}))$$

$$Exp_2(z) = e^{x_0}\frac{\sqrt{3}}{3}(\cos(\frac{x_1}{\sqrt{3}})\sin(\frac{x_2}{\sqrt{3}})\cos(\frac{x_3}{\sqrt{3}}) + \sin(\frac{x_1}{\sqrt{3}})\cos(\frac{x_2}{\sqrt{3}})\sin(\frac{x_3}{\sqrt{3}}))$$

$$Exp_3(z) = e^{x_0}\frac{\sqrt{3}}{3}(\sin(\frac{x_1}{\sqrt{3}})\sin(\frac{x_2}{\sqrt{3}})\cos(\frac{x_3}{\sqrt{3}}) + \cos(\frac{x_1}{\sqrt{3}})\cos(\frac{x_2}{\sqrt{3}})\sin(\frac{x_3}{\sqrt{3}})).$$

However, not all of these higher dimensional analogues of the exponential function turn out to be of the same *type*.

For the first and the second example we can determine the value of $M(r,f)$ exactly.

We obtain that $M(r,f_1) = \||\mathbf{m}| + i\mathbf{m}\|e^{|\mathbf{m}|r}$, thus $\tau(f_1) = |\mathbf{m}|$. For f_2 we obtain that $\|f_2(z)\| = e^{x_1+\cdots+x_n}$ which implies that $M(r,f_2)e^{nr}$ and therefore $\tau(f_2) = n$.

For the third example we are able to establish a useful lower and upper bound estimate for the maximum modulus. By a direct calculation we obtain that $\frac{\sqrt{3}}{3}e^r \leq M(r,f_3) \leq e^r$ so that $\tau(f_3) = 1$. When $|\mathbf{m}| = 1$, f_1 and f_3 thus share the same growth order and growth type.

After having discussed some concrete examples, let us now turn to the more general framework. As a consequence of Cauchy's integral formula we can establish, cf. [13]:

Theorem 2. *Let f be a left entire function in $\mathbb{R}^{n+1}$. By f_i we denote the function $f_i := \frac{\partial}{\partial x_i} f$ and $M_i(r) := \max_{\|z\|=r}\left\{\|f_i(z)\|\right\}$ where $r > 0$ and $i \in \{0,\dots,n\}$. Then*

$$\rho(f) = \rho'(f) \text{ and } \lambda(f) = \lambda'(f),$$

where

$$\rho'(f) := \limsup_{r\to\infty}\frac{\log^+(\log^+(M'(r)))}{\log(r)}$$

$$\lambda'(f) := \liminf_{r\to\infty}\frac{\log^+(\log^+(M'(r)))}{\log(r)},$$

for $M'(r) := \max_{i=0,1,\dots,n}\{M_i(r)\}$.

Proof. We consider an arbitrary rectifiable curve from the origin to z. Then

$$f(z) = f(0) + \int_0^1 \sum_{i=0}^{n} x_i\ f_i(tz)dt.$$

For $z \in \mathbb{R}^{n+1}$ with $\|z\| = r$ we get

$$\begin{aligned} \|f(z)\| &\leq \|f(0)\| + r\sum_{i=0}^{n} M_i(r) \\ &\leq \|f(0)\| + r(n+1)M'(r). \end{aligned}$$

Therefore

$$M(r) \leq \|f(0)\| + r(n+1)M'(r).$$

Applying some properties of $\log^+$ we obtain that

$$\log^+(M(r,f)) \leq \log^+(\|f(0)\|) + \log^+(r(n+1)) + \log^+(M'(r)) + \log(2).$$

This in turn leads to

$$\rho(f) \leq \rho'(f) \text{ and } \lambda(f) \leq \lambda'(f).$$

To show the inequality in the other direction, we apply on f_i Cauchy's integral formula:

$$f_i(z) = \frac{1}{A_{n+1}} \int\limits_{\|\zeta - z\| = R-r} q_{\tau(i)}(\zeta - z) d\sigma(\zeta) f(\zeta). \tag{13}$$

Applying the estimate (6) to (13) we hence obtain

$$\|f_i(z)\| \leq \frac{1}{A_{n+1}} \int\limits_{\|\zeta - z\| = R-r} \frac{n}{(R-r)^{n+1}} M(R) dS$$

from which we then infer that

$$M_i(r) \leq \frac{n}{(R-r)} M(R,f).$$

In particular, for $M'(r) := \max\limits_{i=0,1,\ldots,n} \{M_i(r)\}$ we have

$$M'(r) \leq \frac{n}{(R-r)} M(R,f). \tag{14}$$

Replacing $R = 2r$ into (14) yields:

$$M'(r) \leq \frac{n}{r} M(2r,f).$$

Thus,

$$\log^+ M'(r) \leq \log^+ M(2r,f) + \log^+\left(\frac{n}{r}\right).$$

For what follows we may assume without loss of generality that $r > n$. Hence,

$$\log^+ M'(r) \leq \log^+ M(2r,f).$$

Furthermore,

$$\begin{aligned} \frac{\log^+(\log^+ M'(r))}{\log(r)} &\leq \frac{\log^+(\log^+ M(2r,f))}{\log(r)} \\ &= \frac{\log^+(\log^+ M(2r,f))}{\log(2r)} \frac{\log^+(2r)}{\log(r)} \end{aligned}$$

Thus, we have

$$\frac{\log^+(\log^+ M'(r))}{\log(r)} \leq \frac{\log^+(\log^+ M(2r,f))}{\log(2r)}\left(\frac{\log 2}{\log(r)}+1\right)$$

from which we can infer directly that

$$\rho(f) \geq \rho'(f) \text{ and } \lambda(f) \geq \lambda'(f). \quad \square$$

After having computed the growth order $\rho(f)$ resp. $\lambda(f)$ of a monogenic function f, we know the maximal value of the growth order of all partial derivatives.

Notice that Cauchy's integral formula was an important ingredient in the proof of this statement. To establish these types of results in a more general framework it is thus indeed important to work in classes of functions that are in the kernel of a differential operator that satisfy a Cauchy type integral formula. See also our paper [7] where we treated more general solutions to higher dimensional iterated Cauchy-Riemann and Dirac operators. The class of monogenic functions actually provides us with the canonical and easiest example of a function class which satisfies a Cauchy type integral formula.

4 Generalizations of some theorems from Valiron to Clifford analysis

In this section we present some generalizations of some classical theorems from G. Valiron to the Clifford analysis setting.

To this end we first define the *maximum term* and *central index* which are associated to the Taylor series of a monogenic function.
Let us consider a left entire function

$$f(z) = \sum_{|\mathbf{l}|=0}^{+\infty} V_{\mathbf{l}}(z)a_{\mathbf{l}}.$$

Let $r > 0$ be a fixed real. If f is transcendental, i.e. infinitely many $a_{\mathbf{l}} \neq 0$, then

$$\lim_{|\mathbf{l}|\to\infty} \|a_{\mathbf{l}}\| r^{|\mathbf{l}|} = 0.$$

The following expression thus is well-defined:

Definition 4. *(Maximum term)*
Let $f : \mathbb{R}^{n+1} \to Cl_{0n}$ be a left entire function with the Taylor series representation $f(z) \sum_{|\mathbf{l}|=0}^{+\infty} V_{\mathbf{l}}(z)a_{\mathbf{l}}$. Furthermore, let $r > 0$ be a fixed real. Then the associated maximum term is defined by

$$\mu(r) := \mu(r,f) := \max_{|\mathbf{l}|\geq 0}\{\|a_{\mathbf{l}}\| r^{|\mathbf{l}|}\}. \tag{15}$$

We further introduce

Definition 5. *(Central indices)*
Let $f(z) = \sum_{|\mathbf{l}|=p}^{+\infty} V_{\mathbf{l}}(z)a_{\mathbf{l}}$ *be a left entire function. For* $r > 0$ *the index (or the indices)* $\mathbf{m}$ *with maximal length* $|\mathbf{m}|$ *with* $\mu(r)\|a_{\mathbf{m}}\|r^{|\mathbf{m}|}$ *is (are) called central index (indices) which shall be denoted by* $\nu(r) = \nu(r, f) = \mathbf{m}$*. By* $\nu(0)$ *we denote the indices* $\mathbf{l}$ *which satisfy* $|\mathbf{l}| = p$*.*

The following theorem proved in [13], providing us with a direct generalization of Valiron theorem, states a relation between the *maximum term*, *central index* and the *maximum modulus*.

Theorem 3. *If* $f : \mathbb{R}^{n+1} \to Cl_{0n}$ *is a left entire function, then for all* $0 < r < R$

$$M(r) \leq \mu(r)\Big[|\nu(R)|(1 + |\nu(R)|)^{n-1} + \left(\frac{R}{R-r}\right)^n\Big]. \tag{16}$$

Proof. The function f is assumed to be left entire. Thus, it can be represented by

$$f(z) = \sum_{|\mathbf{l}|=0}^{+\infty} V_{\mathbf{l}}(z)a_{\mathbf{l}}$$

where infinitely many $a_{\mathbf{l}} \neq 0$, since f is transcendental. From the maximum modulus theorem for monogenic functions we infer that for $0 < r < R$:

$$\begin{aligned} M(r) &\leq \sum_{|\mathbf{l}|=0}^{+\infty} \|a_{\mathbf{l}}\|r^{|\mathbf{l}|} = \sum_{|\mathbf{l}|=0}^{|\nu(R)|-1} \|a_{\mathbf{l}}\|r^{|\mathbf{l}|} + \sum_{|\mathbf{l}|=|\nu(R)|}^{+\infty} \|a_{\mathbf{l}}\|r^{|\mathbf{l}|} \\ &\leq \sum_{|\mathbf{l}|=0}^{|\nu(R)|-1} \mu(r) + \sum_{|\mathbf{l}|=|\nu(R)|}^{+\infty} \|a_{\mathbf{l}}\|r^{|\mathbf{l}|}. \end{aligned} \tag{17}$$

In view of

$$\begin{aligned} \sum_{|\mathbf{l}|=0}^{|\nu(R)|-1} 1 &= \sum_{|\mathbf{l}|=0} 1 + \sum_{|\mathbf{l}|=1} 1 + \cdots + \sum_{|\mathbf{l}|=|\nu(R)|-1} 1 \\ &= 1 + \frac{((n-1)+1)!}{(n-1)!1!} + \cdots + \frac{[(n-1)+(|\nu(R)-1)]!}{(n-1)!(|\nu(R)|-1)!} \\ &\leq |\nu(R)|\left[\frac{[(n-1)+|\nu(R)|-1]!}{(n-1)!(|\nu(R)|-1)!}\right] \end{aligned}$$

where we use that for all $n \geq 1$ the inequality

$$\frac{(n-1+k)!}{(n-1)!k!} \leq \frac{(n-1+(k+1))!}{(n-1)!(k+1)!}$$

holds, which itself can be verified by a straightforward induction over k. Further,

$$\begin{aligned}
& |\nu(R)|\left[\frac{[(n-1)+|\nu(R)|-1]!}{(n-1)!(|\nu(R)|-1)!}\right] \\
= & |\nu(R)|\left[\frac{(|\nu(R)|+n-2)(|\nu(R)|+n-3)\cdots(|\nu(R)|+1)|\nu(R)|}{(n-1)!}\right] \\
= & |\nu(R)|\left[\frac{|\nu(R)|+n-2}{n-1}\cdot\frac{|\nu(R)|+n-3}{n-2}\cdot\cdots\cdot\frac{|\nu(R)|+1}{2}\cdot\frac{|\nu(R)|}{1}\right] \\
\leq & |\nu(R)|\left[\underbrace{(1+\frac{|\nu(R)|}{n-1})}_{\leq 1+|\nu(R)|}\underbrace{(1+\frac{|\nu(R)|}{n-2})}_{\leq 1+|\nu(R)|}\cdot\cdots\cdot\underbrace{(1+\frac{|\nu(R)|}{1})}_{=1+|\nu(R)|}\right] \\
\leq & |\nu(R)|\left[(1+|\nu(R)|)^{n-1}\right].
\end{aligned}$$

Inserting these results into (17) leads to

$$\begin{aligned}
M(r) & \leq \mu(r)|\nu(R)|\left[(1+|\nu(R)|)^{n-1}\right]+\sum_{|\mathbf{l}|=|\nu(R)|}^{+\infty}\|a_{\mathbf{l}}\|r^{|\mathbf{l}|}\frac{\|a_{\nu(r)}\|r^{|\nu(r)|}R^{|\mathbf{l}+\nu(R)|}}{\|a_{\nu(r)}\|r^{|\nu(R)|}R^{|\mathbf{l}+\nu(R)|}} \\
& = \mu(r)|\nu(R)|\left[(1+|\nu(R)|)^{n-1}\right]+\mu(r)\sum_{|\mathbf{l}|=|\nu(R)|}^{+\infty}\frac{\|a_{\mathbf{l}}\|R^{|\mathbf{l}|}R^{|\nu(R)|}r^{|\mathbf{l}|}}{\|a_{\nu(R)}\|R^{|\nu(R)|}R^{|\mathbf{l}|}r^{|\nu(R)|}} \\
& \leq \mu(r)|\nu(R)|\left[(1+|\nu(R)|)^{n-1}\right]+\mu(r)\sum_{|\mathbf{l}|=|\nu(R)|}^{+\infty}\left(\frac{r}{R}\right)^{|\mathbf{l}|-|\nu(R)|} \\
& = \mu(r)\Big[|\nu(R)|\left[(1+|\nu(R)|)^{n-1}\right]+\left(\frac{R}{R-r}\right)^{n}\Big]. \qquad \square
\end{aligned}$$

G. Valiron has also proved that an entire complex-analytic function f of finite order shows the asymptotic behavior $\log(M(r,f)) \approx \log(M'(r))$ where M' is the maximum modulus of the derivative. The classical proof is based on the fact that one has the relation $\mu(r) \leq M(r,f)$ for a complex-analytic function in one complex variable. In the framework of working with Clifford algebra valued monogenic Taylor series which are built with the Fueter polynomials, we have a more complicated upper bound estimate of the form

$$\mu(r) \leq \frac{n(n+1)\cdots(n+|\nu(r)|-1)}{\nu(r)!}M(r,f)$$

for a central index $\nu(r)$. This is a consequence of the higher dimensional Cauchy's inequality. Notice that this is a sharp upper bound, cf. [5]. Adapting the classical methods based on Cauchy's inequality to the higher dimensional case provides us only with a weaker result in the Clifford analysis setting. In [13] we proved that

Proposition 1. *For a left entire function* $f : \mathbb{R}^{n+1} \to Cl_{0n}$ *of order* ρ *and inferior order* λ *set*

$$\rho_1 := \limsup_{r\to\infty} \frac{\log^+ \log^+ \mu(r)}{\log(r)}, \qquad \rho_2 := \limsup_{r\to\infty} \frac{\log^+ |\nu(r)|}{\log(r)}, \tag{18}$$

and

$$\lambda_1 := \liminf_{r\to\infty} \frac{\log^+ \log^+ \mu(r)}{\log(r)}, \qquad \lambda_2 : \liminf_{r\to\infty} \frac{\log^+ |\nu(r)|}{\log(r)}. \tag{19}$$

Then $\rho \leq \rho_1 = \rho_2$ *and* $\lambda \leq \lambda_1 = \lambda_2$*.*

Remark: In the two-dimensional complex case where we have $\mu(r) \leq M(r)$ these methods allow one to establish the stronger result $\rho = \rho_1 = \rho_2$ and $\lambda = \lambda_1 = \lambda_2$, as shown for instance in [20, Theorem 4.5].

With this proposition we may establish the following theorem. It provides us with a weaker analogy of Valiron's asymptotic result on the growth of the logarithm of the derivative of a given analytic function:

Theorem 4. *If* $f : \mathbb{R}^{n+1} \to Cl_{0n}$ *is left entire with* $\rho_2(f) < \infty$*, then*

$$\limsup_{r\to\infty} \frac{\log M_i(r)}{\log \mu(r)} \leq 1 \tag{20}$$

where $M_i(r) := \max_{\|z\|=r} \left\{ \left\| \frac{\partial}{\partial x_i} f(z) \right\| \right\}$ *for* $i = 1, \ldots, n$*.*

5 The growth behavior and the Taylor coefficients of a monogenic function

In general it is difficult to determine the precise value of the maximum modulus. In many cases it is even complicated to just get a useful estimate on $M(r, f)$ from below. In this section we present an explicit relation between the Taylor coefficients and the growth order and the *type* of an entire monogenic function. This allows us to compute the growth type directly on the knowledge of the Taylor coefficients without any knowledge on the maximum modulus of the function. Notice that Taylor series actually are a natural method to construct and to define entire monogenic functions. Recall, that the product of two monogenic functions is not monogenic anymore in general. Hence it is natural to construct entire monogenic functions in an additive way, for instance by its Taylor series.

The following two theorems provide us with higher dimensional generalizations in the Clifford analysis setting of two theorems proved by Lindelöf and Pringsheim for complex analytic functions. In [8] we established

Theorem 5. *For an entire monogenic function* $f : \mathbb{R}^{n+1} \to Cl_{0n}$*, with Taylor series representation* $f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} V_{\mathbf{m}}(z) a_{\mathbf{m}}$ *let*

$$\Pi = \limsup_{|\mathbf{m}| \to +\infty} \frac{|\mathbf{m}| \log |\mathbf{m}|}{-\log \| \frac{1}{c(n,\mathbf{m})} a_{\mathbf{m}} \|}. \tag{21}$$

Then we have $\rho(f) = \Pi$.

Remark: In the cases where $\|a_{\mathbf{m}}\| = 0$ one puts $\limsup_{|\mathbf{m}| \to +\infty} \frac{|\mathbf{m}| \log |\mathbf{m}|}{-\log \| \frac{1}{c(n,\mathbf{m})} a_{\mathbf{m}} \|} := 0$.

The following theorem, proved in [9], also relates the growth type with the Taylor coefficients of an entire monogenic function:

Theorem 6. *Let* $f : \mathbb{R}^{n+1} \to Cl_{0n}$ *be an entire monogenic function with Taylor series expansion* $f(z) = \sum_{|\mathbf{m}|=0}^{+\infty} V_{\mathbf{m}}(z) a_{\mathbf{m}}$ *with order* ρ *(*$0 < \rho < +\infty$*) and*

$$\Theta = \limsup_{|\mathbf{m}| \to +\infty} |\mathbf{m}| \Big(\|a_{\mathbf{m}}\| \Big)^{\frac{\rho}{|\mathbf{m}|}}. \tag{22}$$

Then $\Theta = \tau e \rho$*, where* τ *is the type of* f*.*

In turn, Theorem 6 allows us to construct immediately examples of entire monogenic Taylor series of non-zero finite growth order ρ of any arbitrary real growth type $0 \leq \tau \leq +\infty$. Recalling from [9], we start with

Proposition 2. *Suppose that* $f : \mathbb{R}^{n+1} \to Cl_{0n}$ *is an entire monogenic function. If* $\rho(f) = 0$*, then* $\tau(f) = \infty$ *or* f *is a constant.*

Proof. If $\rho(f) = 0$, then

$$\tau(f) = \limsup_{r \to +\infty} \log^{+} M(r, f).$$

If $\tau(f) \neq \infty$, then

$$\limsup_{r \to +\infty} M(r, f) = e^{\tau},$$

which implies that

$$\|f(z)\| \leq e^{\tau} \quad \text{for all } \ z \in \mathbb{R}^{n+1}.$$

As a consequence of Theorem 1, f must be a constant. □

Example: Consider $P(z)$ to be an arbitrary left monogenic polynomial, i.e. there exist Clifford numbers $a_{\mathbf{m}} \in Cl_{0n}$ and $N \in \mathbb{N}_0$ such that $P(z) \sum_{|\mathbf{m}|=0}^{N} V_{\mathbf{m}}(z) a_{\mathbf{m}}$. From [13, Theorem 3.1] we know that

$$\|P(z)\| \leq \left(\frac{(n-1+N)!}{(n-1)!N!} + \varepsilon \right) \|a_{\mathbf{N}}\| r^N, \tag{23}$$

where $\mathbf{N}$ is the index of length N for which $\|a_{\mathbf{N}}\| \geq \|a_{\mathbf{m}}\|$ for all $|\mathbf{m}| = N$, with an arbitrarily small $\varepsilon > 0$ for r sufficiently large. Thus, it follows with $C(N) : \left(\frac{(n-1+N)!}{(n-1)!N!} + \varepsilon\right)\|a_{\mathbf{N}}\|$ that

$$\lim_{r\to\infty} \frac{\log^+(\log^+(M(r,P))}{\log(r)} \leq \lim_{r\to\infty} \frac{\log^+(\log^+(C(N)r^N)}{\log(r)} 0.$$

Thus, all monogenic polynomials satisfy $\rho(P) = \lambda(P) = 0$, like in the complex case. In view of Proposition 2 the growth type τ equals $+\infty$.

More generally, we could establish, cf. [9]:

Proposition 3. *Let $0 < \delta < +\infty$ and $0 < \lambda < +\infty$ be arbitrary real numbers. Then*

$$f(z) = \sum_{|\mathbf{m}|=1}^{+\infty} c(n,\mathbf{m})|\mathbf{m}|^{-\frac{|\mathbf{m}|}{\delta}} V_{\mathbf{m}}\Big(\big(\frac{\lambda e\delta}{n^\delta}\big)^{\frac{1}{\delta}} z\Big) \qquad (24)$$

is an entire monogenic function of growth order $\rho = \delta$ and growth type $\tau = \lambda$.

Proof. By applying Hadamard's formula, one may directly conclude that the convergence radius of (24) is $+\infty$. Since the Fueter polynomials $V_{\mathbf{m}}$ are homogeneous polynomials of total degree $|\mathbf{m}|$, f can directly be rewritten in the form $f(z) = \sum_{|\mathbf{m}|=1}^{+\infty} V_{\mathbf{m}}(z) a_{\mathbf{m}}$ with $a_{\mathbf{m}} = c(n,\mathbf{m})|\mathbf{m}|^{-\frac{|\mathbf{m}|}{\delta}} \left(\frac{\lambda e\delta}{n^\delta}\right)^{\frac{|\mathbf{m}|}{\delta}}$.
According to Theorem 5, the growth order of f therefore equals

$$\begin{aligned}
\rho(f) &= \limsup_{|\mathbf{m}|\to+\infty} \frac{|\mathbf{m}|\log|\mathbf{m}|}{-\log\|\frac{1}{c(n,\mathbf{m})}a_{\mathbf{m}}\|} = \limsup_{|\mathbf{m}|\to+\infty} \frac{|\mathbf{m}|\log|\mathbf{m}|}{-\log\left||\mathbf{m}|^{-\frac{|\mathbf{m}|}{\delta}}\left(\frac{\lambda e\delta}{n^\delta}\right)^{\frac{|\mathbf{m}|}{\delta}}\right|} \\
&= \limsup_{|\mathbf{m}|\to+\infty} \delta \frac{\log|\mathbf{m}|}{\log|\mathbf{m}| - \log(\frac{\lambda e\delta}{n^\delta})} = \delta.
\end{aligned}$$

By Theorem 6 we indeed furthermore obtain that

$$\begin{aligned}
\tau(f) &= \frac{1}{e\delta}\limsup_{|\mathbf{m}|\to+\infty} c(n,\mathbf{m})^{\frac{\delta}{|\mathbf{m}|}} \frac{\lambda e\delta}{n^\delta} \\
&= \frac{\lambda}{n^\delta}\limsup_{M\to+\infty}\max_{|\mathbf{m}|=M} c(n,\mathbf{m})^{\frac{\delta}{M}} \\
&= \frac{\lambda}{n^\delta}\limsup_{M\to+\infty}\left[\frac{(n+M-1)!}{(n-1)!(\frac{M}{n})!^n}\right]^{\frac{\delta}{M}} \\
&= \frac{\lambda}{n^\delta}\limsup_{M\to+\infty}\Big[\frac{1}{\big[(n-1)!\big]^{\frac{\delta}{M}}} \frac{\left((n+M-1)^{n+M-1+\frac{1}{2}}e^{-(n+M-1)}\right)^{\frac{\delta}{M}}}{\left(\left(\frac{M}{n}\right)^{\frac{M}{n}+\frac{1}{2}} e^{-\frac{M}{n}}\right)^{\frac{n\delta}{M}}} \\
&= \frac{\lambda}{n^\delta}\limsup_{M\to+\infty}\left(\frac{n+M-1}{\frac{M}{n}}\right)^{\delta} \lambda.
\end{aligned}$$

By analogous calculations one can further show that

Proposition 4. *Let $0 < \rho < \infty$. The functions*

$$g(z) = \sum_{|\mathbf{m}|=2}^{+\infty} c(n,\mathbf{m}) \Big[\frac{\log|\mathbf{m}|}{|\mathbf{m}|}\Big]^{\frac{|\mathbf{m}|}{\rho}} V_{\mathbf{m}}(z)$$

$$h(z) = \sum_{|\mathbf{m}|=2}^{+\infty} c(n,\mathbf{m}) \Big[\frac{1}{|\mathbf{m}|\log|\mathbf{m}|}\Big]^{\frac{|\mathbf{m}|}{\rho}} V_{\mathbf{m}}(z)$$

are entire monogenic functions of growth order ρ and $\tau(g) = +\infty$ and $\tau(h) = 0$.

6 Applications to partial differential equations

In this section we show how the notions of the maximum term and the central indices can be applied to obtain some information on the structure of the solutions of certain class of higher dimensional partial differential equations.

To proceed in this direction it turns out to be useful to first establish a relation between the asymptotic behavior of the maximum term of a monogenic function and that of their iterated radial derivatives. In [13] we established:

Theorem 7. *Let $f : \mathbb{R}^{n+1} \to Cl_{0n}$ be a left entire function. Then for all $k \in \mathbb{N}$ holds asymptotically*

$$\frac{1}{|\nu(r)|^k} \Big\| [E^k]f(z) - f(z) \Big\| \leq C\mu(r)|\nu(r)|^{-\frac{1}{2}+\varepsilon}, \quad r \notin F \tag{25}$$

where $E := \sum_{i=0}^{n} x_i \frac{\partial}{\partial x_i}$ is the Euler operator on $\mathbb{R}^{n+1}$, C is a real positive constant and F is a set of finite logarithmical measure.

As a direct consequence of Theorem 7 one obtains

Proposition 5. *Let $0 < \delta < \frac{1}{2}$. We assume that $\|z\| = r$ and that r be sufficiently large. Suppose further that the relation*

$$\|f(z)\| > \mu(r)|\nu(r)|^{-\frac{1}{2}+\delta} \tag{26}$$

is satisfied for all those z that belong to a neighborhood $\mathcal{V}_{z_0}$ of a point z_0 in which we have $\|z_0\| = r$ and $\|f(z_0)\| = \max_{\|z\|=r}\{\|f(z)\|\}$. Then for all $k \in \mathbb{N}$ holds asymptotically

$$\frac{1}{|\nu(r)|^k}[E^k]f(z) - f(z) = o(1)f(z), \quad r \notin F, \tag{27}$$

where F is again a set of finite logarithmical measure.

Remark: This statement provides us with an analogy in the context of Clifford analysis of the classical result [20, Theorem 21.3] which states that entire complex-analytic functions that satisfy

$$\|f(z)\| > M(r,f)[\nu(r)]^{-\frac{1}{4}+\delta}$$

have the asymptotic behavior

$$f^{(m)}(z) = \Big(\frac{\nu(r)}{z}\Big)^m (1+o(1))f(z).$$

In the Clifford analysis setting one thus obtains a similar asymptotic result when substituting the complex operator $z\frac{d}{dz}$ by the higher dimensional Euler operator E.

With these tools in hand we can study the structure of the solutions to some classes of partial differential equations. As a concrete example we present the following special case of an unpublished result from [12]:

Theorem 8. *Let f be an entire monogenic function of finite order $\rho_2 < \infty$. Let $\|z\| = r$ and assume that r is sufficiently large. Suppose further that the relation*

$$\|f(z)\| > \mu(r)|\nu(r)|^{-\frac{1}{2}+\delta}, \quad r \notin F$$

is satisfied for all those z that belong to a neighborhood $\mathcal{V}_{z_0}$ of a point z_0 in which we have $\|z_0\| = r$ and $\|f(z_0)\| = \max_{\|z\|=r}\{\|f(z)\|\}$. Let

$$M_j[f] = a_j \prod_{i=0}^{k} (E^i(f))^{n_i},$$

where a_j are polynomials of degree j, and $M_j[f]$ has degree $\gamma_{M_j} = \sum_{i=0}^{k} n_i$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k} in_i$. Let

$$Q[f] = \sum_{j=0}^{s} M_j[f]$$

be of degree γ_Q and weight Γ_Q. If $\gamma_Q \gamma_{M_0}$ then the differential equation $Q[f] = 0$ has no transcendental entire solutions.

Proof. If $Q[f] = 0$, then $M_0[f] = -\sum_{j=1}^{s} M_j[f]$. >From the definition of M_j it follows that

$$a_0 \left[\prod_{i=0}^{k} (E^i(f))^{n_i}\right]_{M_0} = -\sum_{j=1}^{s} \left[a_j \prod_{i=0}^{k} (E^i(f))^{n_i}\right]_{M_j}.$$

Applying Proposition 5, we obtain that

$$\|a_0\| |\nu(r)|^{\Gamma_{M_0}} \|f(z)\|^{\gamma_{M_0}} \leq \sum_{j=1}^{s} \left(\|a_j\| |\nu(r)|^{\Gamma_{M_j}} \|f(z)\|^{\gamma_{M_j}} \right).$$

Since a_0 is a non zero constant and a_j are polynomials of degree j, taking the maximum over the norm, and applying (23) leads to

$$\begin{aligned} |\nu(r)|^{\Gamma_{M_0}} M(r,f)^{\gamma_{M_0}} &\leq |\nu(r)|^{\Gamma_Q} M(r,f)^{\gamma_Q - 1} \sum_{j=1}^{s} \max_{\|z\| r} \frac{\|a_j\|}{\|a_0\|} \\ &\leq |\nu(r)|^{\Gamma_Q} M(r,f)^{\gamma_Q - 1} r^{\alpha}. \end{aligned} \tag{28}$$

Therefore, in view of $\gamma_Q = \gamma_{M_0}$ one has

$$M(r,f) \leq |\nu(r)|^{\Gamma_Q - \Gamma_{M_0}} r^{\alpha}. \tag{29}$$

For $\Gamma_Q - \Gamma_{M_0} < 0$ it follows

$$\liminf_{r \to \infty} \frac{M(r,f)}{r^{\alpha}} \leq \liminf_{r \to \infty} |\nu(r)|^{\Gamma_Q - \Gamma_{M_0}} = 0$$

which implies that f is a polynomial, as a consequence of Theorem 1.

Let us now consider the case where $\Gamma_Q - \Gamma_{M_0} > 0$. Since $\rho_2 < \infty$, we have that $|\nu(r)| < r^{\rho_2 + \epsilon}$ for $\epsilon > 0$. Therefore, there exists a $\beta > (\Gamma_Q - \Gamma_{M_0})(\rho_2 + \epsilon)$ such that

$$\begin{aligned} \liminf_{r \to \infty} \frac{M(r,f)}{r^{\beta+\alpha}} &\leq \liminf_{r \to \infty} \frac{|\nu(r)|^{\Gamma_Q - \Gamma_{M_0}}}{r^{\beta}} \\ &\leq \liminf_{r \to \infty} r^{(\Gamma_Q - \Gamma_{M_0})(\rho_2 + \epsilon) - \beta} = 0 \end{aligned}$$

which implies that f is a polynomial, as a consequence of Theorem 1. □

Concluding remarks: One can apply these techniques to obtain analogous statements for much more general classes of partial differential equations. In our recent paper [10] we were able to prove analogous statements for far more general systems involving polynomial expressions of the Cauchy-Riemann operator with arbitrary complex coefficients and radial differential operators. This paper gives a first impression in how one can extend the classical techniques from Wiman and Valiron from complex analysis to study much larger classes of higher dimensional elliptic operators and summarizes the first basic results.

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