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An Intersection Theorem and its Applications

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ABSTRACT

In this paper we obtain a very general intersection theorem for the values of a map. From this we derive existence theorems for two types of vectorial equilibrium problems, an analytic alternative and a minimax inequality involving three real functions.

RESUMEN

En este artículo obtenemos un teorema general de intersección para los valores de una aplicación. A través de este resultado deducimos teoremas de existencia para dos tipos de problemas de equilibrio vectoriales, una alternativa analítica y una desigualdad minimax envolviendo tres funciones reales.

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1. Introduction and preliminaries

A *multimap* (or simply a *map*) $T : X \multimap Y$ is a function from a set X into the power set 2^Y of Y, that is a function with the *values* $T(x) \subseteq Y$ for $x \in X$. To a map $T : X \multimap Y$ we associate two other maps $T^c : X \multimap Y$ and $T^- : Y \multimap X$ defined by $T^c(x) = Y \setminus T(x)$, and respectively $T^-(y) = \{x \in X : y \in T(x)\}$ The values of T^- are called the fibers of T.

Let $T : X \multimap Y$ be a map. As usual the set $\{(x, y) \in X \times Y : y \in T(x)\}$ is called the *graph* of T. For $A \subseteq X$, and $B \subseteq Y$ let $T(A) = \bigcup_{x \in A} T(x)$ and $T^-(B) = \{x \in X : T(x) \cap B \neq \emptyset\}$.

For topological spaces X and Y a map $T : X \multimap Y$ is said to be: *upper semicontinuous* (u.s.c.) if for any closed set $F \subseteq Y$ the set $T^-(F)$ is closed in X; *lower semicontinuous* (l.s.c.) if for any open set $U \subseteq Y$ the set $T^-(U)$ is open in X; *compact* if $T(X)$ is contained in a compact subset of Y; *closed* if its graph is closed in $X \times Y$.

The following lemma collects known facts about u.s.c. or l.s.c. maps (see for example [7] for assertion (i), [16] for assertion (ii) and [9] for assertion (iii)).

Lemma 1 Let X and Y be topological spaces and $T : X \multimap Y$ be a map.

(i) If T has compact values, then it is u.s.c. if and only if for each $x \in X$, any net $\{x_t\}$ converging to x and any net $\{y_t\}$ with $y_t \in T(x_t)$ for all index t, there exists a subnet $\{y_{t'}\}$ of $\{y_t\}$ and $y \in T(x)$ such that $\{y_{t'}\}$ converges to y.

(ii) T is l.s.c. in $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_t\}$ converging to x, there exists a net $\{y_t\}$ converging to y, with $y_t \in T(x_t)$ for each t.

(iii) If Y is compact and T is closed, then T is u.s.c..

If X is a subset of a topological vector space we denote by coX and $\overline{X}$ the convex hull and the closure of X respectively.

Let Y be a convex set in a topological vector space and X be a topological space. The *better admissible class* $\mathcal{B}$ of mappings from Y into X (see [15]) is defined as follows:

$T \in \mathcal{B}(Y, X) \Leftrightarrow T : Y \multimap X$ is a mapping such that for any nonempty finite subset A of Y and any continuous mapping $p : T(co\ A) \to co\ A$ the composition $p \circ T_{|co\ A} : co\ A \multimap co\ A$ has a fixed point.

The class $\mathcal{B}(Y, X)$ includes many important classes of mappings, such as $U_c^k(Y, X)$ in [14], $KKM(Y, X)$ in [3] and $A(Y, X)$ in [2], as proper subclasses.

Definition 1. Let X be a convex set in a vector space and Y a vector space. A mapping $T : X \multimap Y$ is called:

(i) *quasiconvex*, if for every convex subset C of Y, $T^-(C)$ is a convex set;

(ii) *convex*, if for each $x_1, x_2 \in X$ and $\lambda \in (0, 1)$, $\lambda T(x_1) + (1 - \lambda)T(x_2) \subseteq T(\lambda x_1 + (1 - \lambda)x_2)$;

(iii) *concave*, if for each $x_1, x_2 \in X$ and $\lambda \in (0,1)$, $T(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda T(x_1) + (1-\lambda)T(x_2)$.

Lemma 2 If a map $T : X \multimap Y$ is convex then it is quasiconvex.

Proof. Let C be a convex subset of Y, $x_1, x_2 \in T^-(C)$ and $\lambda \in (0,1)$. If $y_1 \in T(x_1) \cap C$, $y_2 \in T(x_2) \cap C$, then

$$\lambda y_1 + (1-\lambda)y_2 \in (\lambda T(x_1) + (1-\lambda)T(x_2)) \cap C \subseteq T(\lambda x_1 + (1-\lambda)x_2) \cap C,$$

hence $\lambda x_1 + (1-\lambda)x_2 \in T^-(C)$. □

Let us describe in short the contents on the next sections. We obtain first a very general intersection theorem involving three maps, one of them from the class $\mathcal{B}$. Two types of applications of this result will be given in the last two sections.

The first one, offers existence theorems for the following types of vectorial equilibrium problems:

Let X be a topological space, Y be a convex set in a topological vector space, Z be a topological vector space and V be nonempty set. Let $F : Y \times Z \multimap V$, $C : Z \multimap V$ and $P : X \multimap Z$.

(I) Find $x_0 \in X$ such that $F(y,z) \subseteq C(z)$ for each $y \in Y$ and $z \in P(x_0)$;

and respectively,

(II) Find $x_0 \in X$ such that $F(y,z) \cap C(z) \neq \emptyset$ for each $y \in Y$ and $z \in P(x_0)$.

Finally, we obtain an analytic alternative and a minimax inequality involving three real functions.

From now all (topological) vector spaces will be assumed real and all topological (vector) spaces will be assumed Hausdorff.

2. An intersection theorem

Theorem 1. *Let X be a topological space, Y be a convex set in a topological vector space and Z be a nonempty set. Let $P : X \multimap Z$, $Q : Y \multimap Z$ two maps satisfying the following conditions:*

(i) for each $y \in Y$, $\{x \in X : P(x) \subseteq Q(y)\}$ is closed;

(ii) P has convex values and Q^c is quasiconvex;

(iii) there exists a compact mapping $T \in \mathcal{B}(Y,X)$ such that for each $y \in Y$, $P(T(y)) \subseteq Q(y)$.

Then there exists $x_0 \in X$ such that $P(x_0) \subseteq \bigcap_{y \in Y} Q(y)$.

Proof. Let $S : Y \multimap X$ be the map defined by

$$S(y) = \{x \in X : P(x) \nsubseteq Q(y)\}.$$

Suppose that the conclusion of theorem is false. Then $X = \bigcup_{y \in Y} S(y)$. Let $X_0 = \overline{T(Y)}$. Since X_0 is compact there exists a finite set $A = \{y_1, y_2, \dots, y_n\} \subseteq Y$ such that $X_0 = \bigcup_{i=1}^n (S(y_i) \cap X_0)$. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a partition of unity on X_0 subordinated to the cover $\{S(y_i) \cap X_0 : 0 \leq i \leq n\}$. Recall that this means that

$$\begin{cases} \alpha_i : X_0 \to [0,1] \text{ is continuous, for each } i \in \{1, 2, \dots, n\}; \\ \alpha_i(x) > 0 \Rightarrow x \in S(y_i); \\ \sum_{i=1}^n \alpha_i(x) = 1 \text{ for each } x \in X_0. \end{cases}$$

Define $f : T(co\ A) \to co\ A$ by

$$f(x) = \sum_{i=1}^n \alpha_i(x) y_i \ \text{ for all } x \in T(co\ A).$$

Since f is continuous and $T \in \mathcal{B}(Y, X)$, $f \circ T_{|A} : coA \multimap coA$ has a fixed point. Hence there exists $\widetilde{y} \in coA$ such that $\widetilde{y} \in f(T(\widetilde{y}))$. Then, for some $\widetilde{x} \in T(\widetilde{y})$ we have $\widetilde{y} = f(\widetilde{x})$. Let $I = \{i \in \{1, \dots, n\} : \alpha_i(\widetilde{x}) > 0\}$. Then $\widetilde{y} = f(\widetilde{x}) \in co\{y_i : i \in I\}$. For each $i \in I$, $\widetilde{x} \in S(y_i)$, hence $P(\widetilde{x}) \cap Q^c(y_i) \neq \emptyset$. By (ii) it follows that $P(\widetilde{x}) \cap Q^c(\widetilde{y}) \neq \emptyset$, or equivalently, $P(\widetilde{x}) \nsubseteq Q(\overline{y})$. Since $\widetilde{x} \in T(\widetilde{y})$, we get $P(T(\overline{y})) \nsubseteq Q(\overline{y})$, which contradicts (iii). □

Proposition 2. *If Z is topological space, then condition (i) in Theorem 1 is fulfilled in any of the following cases:*

(i_1) P has open fibers;

(i_2) P is l.s.c. and Q has closed values;

Proof. If P has open values then for each $y \in Y$ the set $\{x \in X : P(x) \nsubseteq Q(y)\} = \bigcup_{z \in Q^c(y)} P^-(z)$ is open, hence $\{x \in X : P(x) \subseteq Q(y)\} = X \setminus \{x \in X : P(x) \nsubseteq Q(y)\}$ is closed.

By the definition of lower semicontinuity it follows that if (i_2) holds then each set $\{x \in X : P(x) \subseteq Q(y)\}$ is closed.

□

3. Equilibrium Theorems

In [5], [6], [10-13], for a suitable choice of the sets Y, Z and V and of the maps $F : Y \times Z \multimap V$ and $C : Z \multimap V$ the authors study, all or part of the following problems:

(I) Find $z_0 \in Z$ such that $F(y, z_0) \subseteq C(z_0)$ for all $y \in Y$;

(II) Find $z_0 \in Z$ such that $F(y, z_0) \cap C(z_0) \neq \emptyset$ for all $y \in Y$;

(III) Find $z_0 \in Z$ such that $F(y, z_0) \nsubseteq C(z_0)$ for all $y \in Y$;

(IV) Find $z_0 \in Z$ such that $F(y, z_0) \cap C(z_0) = \emptyset$ for all $y \in Y$.

Each existence result concerning problem (I) (respectively, (II)), yields an existence theorem for problem (IV) (respectively, (III)), if we take into account the following equivalences: $F(y,z) \subseteq C(z) \Leftrightarrow F(y,z) \cap C^c(z) = \emptyset$ and $F(y,z) \cap C(z) \neq \emptyset \Leftrightarrow F(y,z) \nsubseteq C^c(z)$. For this reason we can fix our attention on problems (I) and (II), only.

In this section we study equilibrium problems more general than (I) and (II):

Let X be a topological space, Y be a convex set in a topological vector space, Z be a topological vector space and V be a nonempty set. Let $F : Y \times Z \multimap V$, $C : Z \multimap V$ and $P : X \multimap Z$.

(V) Find $x_0 \in X$ such that $F(y, z) \subseteq C(z)$ for each $y \in Y$ and $z \in P(x_0)$;

and respectively,

(VI) Find $x_0 \in X$ such that $F(y, z) \cap C(z) \neq \emptyset$ for each $y \in Y$ and $z \in P(x_0)$.

Of course, when $X = Z$ and $P(z) = \{z\}$ for all $z \in Z$, problem (V) (respectively (VI)), reduces to problem (I) (respectively (II)).

Theorem 3. *Suppose that the maps F, C and P satisfy the following conditions:*

(i) one of the following two requirements is fulfilled:

(i_1) P has open fibers;

(i_2) P is l.s.c., C is closed map and for each $y \in Y$, $F(y, \cdot)$ is l.s.c.

(ii) F and C^c are convex maps, P has convex values;

(iii) there exists a compact mapping $T \in \mathcal{B}(Y, X)$ such that $F(y, z) \subseteq C(z)$, for each $y \in Y$ and $z \in P(T(y))$.

Then there exists $x_0 \in X$ such that $F(y, z) \subseteq C(z)$ for each $y \in Y$ and $z \in P(x_0)$.

Proof. Let $Q : Y \multimap Z$ be the map defined by

$$Q(y) = \{z \in Z : F(x, z) \subseteq C(z)\}.$$

We prove that if (i_2) holds, then Q has closed values. Let $y \in Y$ and $\{z_t\}_{t\in\Delta}$ be a net in $Q(y)$ converging to $z \in Z$. If $v \in F(y, z)$, since $F(y, \cdot)$ is l.s.c., there exists a net $\{v_t\}_{t\in\Delta}$ converging to v such that $v_t \in F(y, z_t)$, for all $t \in \Delta$. Since $z_t \in Q(y)$, $v_t \in F(y, z_t) \subseteq C(z_t)$. The map C is closed, hence $v \in C(z)$. Thus, $F(y, z) \subseteq C(z)$, hence $z \in Q(y)$. By Proposition 2, in both cases (i_1) and (i_2), condition (i) in Theorem 1 is satisfied.

We show next that the map Q^c is convex. Let $y_1, y_2 \in Y$, $\lambda \in (0.1)$ and $z \in \lambda Q^c(y_1) + (1-\lambda)Q^c(y_2)$. There exist $z_1, z_2 \in Z$ such that $z = \lambda z_1 + (1-\lambda)z_2$ and $v_1, v_2 \in V$ such that $v_i \in F(y_i, z_i) \cap C^c(z_i)$, for $i = 1, 2$. Since the maps $F$and C^c are convex,

$\lambda v_1 + (1-\lambda)v_2 \in \lambda F(y_1, z_1) + (1-\lambda)F(y_2, z_2) \subseteq F(\lambda y_1 + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2)$,

and similarly, $\lambda v_1 + (1-\lambda)v_2 \in C^c(\lambda z_1 + (1-\lambda)z_2)$. Thus, $\lambda v_1 + (1-\lambda)v_2 \in F(\lambda y_1 + (1-\lambda)y_2, z) \cap C^c(z)$, hence $z \in Q(\lambda y_1 + (1-\lambda)y_2)$.

Hence Q^c is convex and by Lemma 2, it is quasiconvex. It is clear that condition (iii) is equivalent to the requirement similarly denoted in Theorem 1, hence all requirements of this theorem are fulfilled. Consequently, there exists $x_0 \in X$ such that $P(x_0) \subseteq \bigcap_{y \in Y} Q(y)$, that is, $F(y, z) \subseteq C(z)$, for each $y \in Y$ and $z \in P(x_0)$. □

Theorem 4. *Suppose that the maps F, C and P satisfy the following conditions:*

(i) one of the following two requirements is fulfilled:

(i_1) P has open fibers;

(i_2) P is l.s.c., C is u.s.c. with compact values and for each $y \in Y$, $F(y, \cdot)$ is closed.

(ii) F is concave map, C^c is convex map and P has convex values;

(iii) there exists a compact mapping $T \in \mathcal{B}(Y, X)$ such that $F(y, z) \cap C(z) \neq \emptyset$, for each $y \in Y$ and $z \in P(T(y))$.

Then there exists $x_0 \in X$ such that $F(y, z) \cap C(z) \neq \emptyset$ for each $y \in Y$ and $z \in P(x_0)$.

Proof. The proof is similar to that of Theorem 3. Let $Q : Y \multimap Z$ be the map defined by

$$Q(y) = \{z \in Z : F(x, z) \cap C(z) \neq \emptyset\}.$$

We show first that if (i_2) holds, then Q has closed values. Let $y \in Y$ and $\{z_t\}_{t \in \Delta}$ be a net in $Q(y)$ converging to $z \in Z$. Then, for each $t \in \Delta$, there exists $v_t \in F(y_t, z_t) \cap C(z_t)$. Since C is u.s.c. with compact values, by Lemma 1 (i), there exist a subnet $\{v_{t'}\}$ of $\{v_t\}$ and $v \in C(z)$ such that $v_{t'} \to v$. Since $F(y, \cdot)$ is closed, $v \in F(y, z)$. Therefore $F(y, z) \cap C(z) \neq \emptyset$, hence $z \in Q(y)$.

Let $y_1, y_2 \in Y$, $\lambda \in (0.1)$ and $z \in \lambda Q^c(y_1) + (1-\lambda)Q^c(y_2)$. There exist $z_1, z_2 \in Z$ such that $z = \lambda z_1 + (1-\lambda)z_2$ and $F(y_1, z_1) \subseteq C^c(z_1)$, $F(y_2, z_2) \subseteq C^c(z_2)$. By (ii) we infer that

$F(\lambda y_1 + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2) \subseteq \lambda F(y_1, z_1) + (1-\lambda)F(y_2, z_2) \subseteq \lambda C^c(z_1) + (1-\lambda)C^(z_2) \subseteq C^c(\lambda z_1 + (1-\lambda)z_2)$.

It follows that $z \in Q^c(\lambda y_1 + (1-\lambda)y_2)$, hence the map Q^c is convex. The maps P and Q satisfy all the requirements of Theorem 1 and the desired conclusion follows from this theorem. □

4. Analytic alternative, minimax inequality

Definition 2. (see [1]). Let X and Y be convex sets in two vector spaces. We say that a function $q : Y \times Z \to \overline{\mathbb{R}}$ is (y,z)-*quasiconvex* if for any finite subset $\{(y_1, z_1), \ldots, (y_n, z_n)\}$ of $Y \times Z$, and each $y \in \text{co}\,\{y_1, \ldots, y_n\}$ there exists $z \in \text{co}\,\{z_1, \ldots z_n\}$ such that $q(y,z) \leq \max_{1 \leq i \leq n} q(y_i, z_i)$.

It is clear that any function $q : Y \times Z \to \overline{\mathbb{R}}$ quasiconvex on $Y \times Z$ is (y,z)-quasiconvex but Example 2 in [1] shows that the converse is not true.

Definition 3. Let X and Z be topological spaces. A function $p : X \times Z \to \mathbb{R}$ is said to be *marginally upper semicontinuous* in x (see [8]) if for every open subset U of Z the function $x \to inf_{z \in U} p(x,z)$ is upper semicontinuous on X.

Any function upper semicontinuous in x is marginally upper semicontinuous in x but the example given in [8], p.249 shows that the converse is not true.

Theorem 5. *Let X be topological space, Y and Z be convex sets in topological vector spaces, $p : X \times Z \to \mathbb{R}$, $q : Y \times Z \to \mathbb{R}$, $t : X \times Y \to \mathbb{R}$ be functions and α, β, λ be real numbers. Suppose that the following conditions are satisfied:*

(i) one of the following requirements is fulfilled:

(i_1) for each $z \in Z$ the set $\{x \in X : p(x,z) < \alpha\}$ is open;

(i_2) p is marginally upper semicontinuous in x and for each $y \in Y$ the set $\{z \in Z : q(y,z) \geq \beta\}$ is closed;

(ii) for each $x \in X$ the set $\{z \in Z : p(x,z) < \alpha\}$ is convex;

(iii) q is (y,z)-quasiconvex;

(iv) for $x \in X$, $y \in Y$ and $z \in Z$ the following implication holds: $p(x,z) < \alpha$ and $q(y,z) < \beta \Rightarrow t(x,y) < \lambda$;

(v) the map $T : Y \multimap X$ defined by $T(y) = \{x \in X : t(x,y) \geq \lambda\}$ is compact and belongs to the class $\mathcal{B}(Y,X)$.

Then at least one of the following assertions holds:

(a) There exists $x_0 \in X$ such that $p(x_0, z) \geq \alpha$, for all $z \in Z$.

(b) There exists $z_0 \in Z$ such that $q(y, z_0) \geq \beta$, for all $y \in Y$.

Proof. Define the maps $P : X \multimap Z$, $Q : Y \multimap Z$, $T : X \multimap Y$ by

$$P(x) = \{z \in Z : p(x,z) < \alpha\}, \ Q(y) = \{z \in Z : q(y,z) \geq \beta\}, and$$

$$T(y) = \{x \in X : t(x, y) \geq \lambda\}.$$

If (i_1) holds, then P has open fibers, If (i_2) holds, then Q has closed values and we claim that P is l.s.c. Indeed, since p is marginally upper semicontinuous in x, for each open $U \subseteq Z$ the set

$$\{x \in X : P(x) \cap U \neq \emptyset\} = \{x \in X : inf_{z \in U} p(x, z) < \alpha\}$$

is open. Hence, according to Proposition 2, condition (i) in Theorem 1 holds.

Let C be a convex subset of Z, $y_1, y_2 \in Q^c(C)$ and $y \in co\{y_1, y_2\}$. Then there exist $z_1, z_2 \in C$ such that $q(y_1, z_1) < \beta$, $q(y_2, z_2) < \beta$. Since q is (y, z)-quasiconvex, there exists $z \in co\{z_1, z_2\} \subseteq C$ such that

$$q(y, z) \leq max\{q(y_1, z_1), q(y_2, z_2)] \ < \beta.$$

Thus $y \in Q^c(C)$, hence Q is quasiconvex. We prove that for each $y \in Y$, $P(T(y)) \subseteq Q(y)$. Suppose that for some $y \in Y$ there exists $x \in T(y)$ and $z \in P(x) \setminus Q(y)$. By $x \in T(y)$, we get $t(x, y) \geq \lambda$. On the other hand, since $z \in P(x) \setminus Q(y)$, we have $p(x, z) < \alpha$, $q(y, z) < \beta$ and, by (iv), we get $t(x, y) < \lambda$; a contradiction. Therefore the maps P, Q, T satisfy all the requirement of Theorem 1. According to this theorem there exists $x_0 \in X$ such that $P(x_0) \subseteq \bigcap_{y \in Y} Q(y)$. Suppose that both assertions in the conclusion of theorem are false. This means that:

(a') $P(x) \neq \emptyset$, for all $x \in X$;

(b') for each $z \in Z$ there exists $y \in Y$ such that $z \notin Q(y)$.

The following contradiction completes the proof:

$\emptyset \neq P(x_0) \subseteq \bigcap_{y \in Y} Q(y) = \emptyset.$ □

Theorem 6. *Let X be a topological compact space, Y and Z be two convex sets in topological vector spaces and $p : X \times Z \to \mathbb{R}$, $q : Y \times Z \to \mathbb{R}$, $t : X \times Y \to \mathbb{R}$ functions. Suppose that the following conditions are fulfilled:*

(i) one of the following requirements is fulfilled:

(i_1) p is u.s.c. in x;

(i_2) p is marginally upper semicontinuous in x and q is u.s.c. in z;

(ii) p is quasiconvex in z;

(iii) q is (y, z)-quasiconvex;

(iv) for $x \in X$, $y \in Y$ and $z \in Z$ the following implication holds: $t(x, y) \leq p(x, z) + q(y, z)$;

(v) for each $\lambda < \inf_{y \in Y} \sup_{x \in X} \ t(x, y)$ the map $T : Y \multimap X$, defined by $T(y) = \{x \in X : t(x, y) \geq \lambda\}$ belongs to the class $\mathcal{B}(Y, X)$.

Then,

$\inf_{y\in Y}\sup_{x\in X}\ t(x,y) \leq sup_{x\in X} inf_{z\in Z}\ p(x,z) + sup_{z\in Z} inf_{y\in Y}\ q(y,z),$

with the convention $\infty + (-\infty) = \infty$.

Proof. We may suppose that

$$\inf_{y\in Y}\sup_{x\in X}\ t(x,y) > -\infty,\ sup_{x\in X} inf_{z\in Z}\ p(x,z) < \infty,$$

$$sup_{z\in Z} inf_{y\in Y}\ q(y,z) < \infty.$$

By way of contradiction suppose that

$$\inf_{y\in Y}\sup_{x\in X}\ t(x,y) > sup_{x\in X} inf_{z\in Z}\ p(x,z) + sup_{z\in Z} inf_{y\in Y}\ q(y,z)$$

and choose $\alpha, \beta, \lambda \in \mathbb{R}$ such that $sup_{x\in X} inf_{z\in Z}\ p(x,z) < \alpha$, $sup_{z\in Z} inf_{y\in Y}\ q(y,z) < \beta$, $\lambda < \inf_{y\in Y}\sup_{x\in X}\ t(x,y)$, and $\alpha + \beta < \lambda$.

We prove that condition (iv) in Theorem 5 is fulfilled. Let $x \in X$, $y \in Y$ and $z \in Z$ such that $p(x,z) < \alpha$ and $q(y,z) < \beta$. Since $\alpha + \beta < \lambda$, by condition (iv) in the theorem that must be proved, we get $t(x,y) \leq p(x,z) + q(y,z) < \alpha + \beta < \lambda$.

It is easy to see that all the requirements of Theorem 5 are fulfilled. We prove that none of assertions (a), (b) of the conclusion of Theorem 5 can take place.

If (a) happens, then

$\alpha \leq inf_{z\in Z}\ p(x_0,z) \leq sup_{x\in X} inf_{z\in Z}\ p(x,z);$ a contradiction.

If (b) happens, then

$\beta \leq inf_{y\in Y}\ q(y,z_0) \leq sup_{z\in Z}\ inf_{y\in Y}\ q(y,z);$ a contradiction. □

Corollary 7. *Let X, Y and Z be convex subsets of three topological vector spaces, X being compact and $p : X \times Z \to \mathbb{R}$, $q : Y \times Z \to \mathbb{R}$, $t : X \times Y \to \mathbb{R}$ three functions satisfying conditions (i), (ii), (iii), (iv) of Theorem 6 and*

(v') t is upper semicontinuous on $X \times Y$ and for each $y \in Y$, $t(.,y)$ is quasiconcave on X.

Then, $\inf_{y\in Y}\sup_{x\in X}\ t(x,y) \leq sup_{x\in X} inf_{z\in Z}\ p(x,z) + sup_{z\in Z} inf_{y\in Y}\ q(y,z),$

with the convention $\infty + (-\infty) = \infty$.

Proof. It suffices to prove that condition (v) in Theorem 6 is fulfilled. Obviously for each $\lambda < \inf_{y\in Y}\sup_{x\in X}\ t(x,y)$ the map T defined in condition (v) of Theorem 6 has nonempty values. Moreover, by (v') the values of T are convex. Since t is upper semicontinuous on $X \times Y$ the map T is closed. Since X is compact, by Lemma 1, T is upper semicontinuous with compact values. Consequently T is a Kakutani map. Since, $\mathbb{K}(Y,X) \subset \mathcal{B}(Y,X)$, it follows that condition (v) from Theorem 6 is satisfied. □

The results obtained in this section generalize Theorems 19, 20 and Corollary 21 in [1], where the corresponding map T, in each result, had the KKM property. Obviously, the condition $T \in \mathcal{B}(Y, X)$ is a weaker one.

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References

[1] M. Balaj, *Coincidence and maximal element theorems and their applications to generalized equilibrium problems and minimax inequalities*, Nonlinear Anal., **68** (2008), 3962–3971.

[2] H. Ben-El-Mechaiekh, S Chebbi, M. Florenzano and J.-V. Llinares, *Abstract convexity and fixed points* J. Math. Anal. Appl. 222 (1998), 138–150.

[3] T.-H Chang and C.-L. Yen, *KKM property and fixed point theorems*, J. Math. Anal. Appl., **203** (1996), 224–235.

[4] X.P. Ding, *New H-KKM theorems and equilibria of generalized games*, Indian J. Pure Appl. Math., **27** (1996), 1057–1071.

[5] X.P. Ding and Y.J. Park, *Fixed points and generalized vector equilibrium problems in generalized convex spaces* Indian J. Pure Appl. Math., **34** (2003), 973–990.

[6] X.P. Ding and J.Y. Park, *Generalized vector equilibrium problems in generalized convex spaces*, J. Optim. Theory Appl., **120** (2004), 327–353.

[7] J.Y. Fu and A.H. Wan, *Generalized vector equilibrium problems with set-valued mappings*, Math. Methods Oper. Res., **56** (2002), 259–268.

[8] G.H. Greco and M.P. Moschen, *A minimax inequality for marginally semicontinuous functions* in Minimax Theory and Applications (B. Ricceri, S. Simons eds), Kluwer Acad. Publ., Dordrecht, 1998, pp. 41–50.

[9] M. Lassonde, *Fixed points for Kakutani factorizable multifunctions*, J. Math. Anal. Appl., **152** (1990), 46–60.

[10] L.J. Lin, Q.H. Qnsari and J.Y. Wu, *Geometric properties and coincidence theorems with applications to generalized vector equilibrium problems*, J. Optim. Theory Appl., **117** (2003), 121–137.

[11] L.J. Lin and H.L. Chen, *The study of KKM theorems with applications to vector equilibrium problems and implicit vector variational inequalities problems*, J. Global Optim., **32** (2005), 135–157.

[12] L.J. Lin, Z.T. Yu and G. Kassay, *Existence of equilibria for multivalued mappings and its application to vectorial equilibria*, J. Optim. Theory Appl., **114** (2002), 189–208.

[13] L.J. Lin and W.P. Wan, *KKM type theorems and coincidence theorems with applications to the existence of equilibrium*, J. Optim. Theory Appl., **123** (2004), 105–122.

[14] S. Park, *Foundations of the KKM theory via coincidences of composites of upper semicontinuous maps*, J. Korean Math. Soc., **31** (1994), 493–519.

[15] S. Park, *Fixed points of the better admissible multimaps*, Math. Sci. Res. Hot-Line, **1(9)** (1997), 1–6.

[16] N.X. Tan and P.N. Tinh, *On the existence of equilibrium points of vector functions*, Numerical Functional Analysis Optimiz., **19** (1998), 141–156.