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Equilibrium Cycles in a Two-Sector Economy with Sector Specific Externality

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ABSTRACT

In this paper, we study the two-sector CES economy with sector-specific externality (feedback effects). We characterize the equilibrium paths in the case that allows negative externality, and show how the degree of externality may generate equilibrium cycles around the steady state.

RESUMEN

En este artculo estudiamos economia de dos-sector CES con externalidad de sector-especifico (efecto de retroalimentacin). Nosotros caracterizamos la trajectoria de equi-

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librio en el caso que permite externalidad negativa, e demonstramos como el grado de externalidad puede generar ciclos de equilibrio alrededor del estado regular.

Key words and phrases: *Difference equations, nonlinear dynamics, bifurcation, two-periodic cycle, multiple equilibria.*

Math. Subj. Class.: *37G10, 39A11, 91B50, 91B62, 91B64, 91B66.*

1 Introduction

The aim of this paper is to show the existence of equilibrium cycles around the steady state in the two-sector model with CES production function and sector specific externality.[1] A representative agent has concrete expectations on the level of externality and make a decision assuming that the externality is not affected by his own choice of decision variables. However, externalities come from the average values of capital and labor on the market. Therefore, if a representative agent chooses values of decision variables, externalities also vary as everybody also takes the same decision.

Over the last decade, an important literature has focused on the existence of locally indeterminate equilibria in dynamic general equilibrium economies with technological external effects. Local indeterminacy means that there exists a continuum of equilibria starting from the same initial condition, all of which converging to the same steady state. It is now well-known that local indeterminacy is a sufficient condition for the existence of endogenous fluctuations generated by purely extrinsic belief shocks which do not affect the fundamentals, i.e. the preferences and technologies.[2] Indeed, in presence of local indeterminacy, by randomizing beliefs over the continuum of equilibrium paths, one may construct equilibria defined with respect to shocks on expectations, and thus provide an alternative to technology or taste shocks to get propagation mechanisms and to explain macroeconomic volatility.

Benhabib and Nishimura [3, 4] proved that indeterminacy may arise in a continuous time economy in which the production functions from the social perspective have constant return to scale. Benhabib, Nishimura and Venditti [5] studied the two-sector model with sector specific external effects in discrete time framework. They provided conditions in which indeterminacy may occur even if the production function is decreasing return to scale from the social perspective. Nishimura and Venditti [7] study the interplay between the elasticity of capital-labor substitution and the rate of depreciation of capital, and its influence on the local behavior of equilibrium paths in a neighborhood of the steady state. However, in all these contributions, the existence of local bifurcations as the degree of externalities is modified is not discussed.

In this paper, we study the model in Nishimura and Venditti [7], focusing on the external effect of capital-labor ratio in the pure capital good sector and characterize the equilibrium paths

[1] External effects are feedbacks from the other agents in the economy who also face identical maximizing problems. See Benhabib and Farmer [2] for a survey.

[2] See Cass and Shell [6].

in the case that allows negative externality, which was not discussed in their paper. We will focus on the existence of flip bifurcation, i.e. of period-two equilibrium cycles, through the existence of local indeterminacy.

In Section 2 we describe the model. We discuss the existence of a steady state and give the local characterization of the equilibrium paths around the steady state in Section 3. Section 4 contains some concluding comments.

2 The model

We consider a two-sector model with an infinitely-lived representative agent. We assume that its single period linear utility function is given by

$$u\left(c_t\right) = c_t.$$

We assume that the consumption good, c, and capital good are produced with a Constant Elasticity of Substitution (CES) production functions.

$$c_t = \left[\alpha_1 K_{ct}^{-\rho_c} + \alpha_2 L_{ct}^{-\rho_c}\right]^{-\frac{1}{\rho_c}} \tag{1}$$

$$y_t = \left[\beta_1 K_{yt}^{-\rho_2} + \beta_2 L_{yt}^{-\rho_2} + e_t\right]^{-\frac{1}{\rho_y}} \tag{2}$$

where $\rho_c,\ \rho_y > -1$ and e_t represents the time-dependent externality (feedback effects) in the capital good sector. Let the elasticity of capital-labor substitution in each sector be $\sigma_c = \frac{1}{1+\rho_c} \geq 0$ and $\sigma_y = \frac{1}{1+\rho_y} \geq 0$. We assume that the externalities are as follows:

$$e = b\bar{K}_{yt}^{-\rho_y} - b\bar{L}_{yt}^{-\rho_y}, \tag{3}$$

where $b > 0$, and $\bar{K}_y$ and $\bar{L}_y$ represents the economy-wide average values. The representative agent takes these economy-wide average values as given.

Definition 1 *We call* $y = \left[\beta_1 K_y^{-\rho_y} + \beta_2 L_y^{-\rho_y} + e\right]^{-\frac{1}{\rho_y}}$ *the production function from the private perspective, and* $y = \left[\left(\beta_1 + b\right) K_y^{-\rho_y} + \left(\beta_2 - b\right) L_y^{-\rho_y}\right]^{-\frac{1}{\rho_y}}$ *the production function from the social perspective.*

In the rest of the paper we will assume that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = 1$ so that the consumption good sector does not have externalities. Notice then that denoting $\hat{\beta}_1 = \beta_1 + b$ and $\hat{\beta}_2 = \beta_2 - b$, we get also $\hat{\beta}_1 + \hat{\beta}_2 = 1$. The investment good sector has externalities but the technology is linearly homogeneous, i.e. has constant returns, from the social perspective.

Remark 1 Notice that the externality (3) may be expressed as follows

$$e = b\bar{L}_2^{-\rho_y}\left[\left(\frac{\bar{K}_y}{\bar{L}_y}\right)^{-\rho_y} - 1\right]. \tag{4}$$

Now consider the production function from the social perspective as given in Definition 1. Dividing both sides by L_y, we get denoting $k_y = K_y/L_y$ and $\tilde{y} = y/L_y$

$$\tilde{y} = \left[(\beta_1 + b)\, k_y^{-\rho_y} + (\beta_2 - b)\right]^{-\frac{1}{\rho_y}}. \tag{5}$$

From equations (4) and (5) we derive that the externality is given in terms of the capital-labor ratio in the investment good sector.

The aggregate capital is divided between sectors,

$$k_t = K_{ct} + K_{yt},$$

and the labor endowment is normalized to one and divided between sectors,

$$L_{ct} + L_{yt} = 1.$$

The capital accumulation equation is

$$k_{t+1} = y_t,$$

as the capital depreciates completely in one period. To simplify we assume that both technologies are characterized by the same properties of substitution, i.e. $\rho_c = \rho_y = \rho$.

The consumer optimization problem will be given by

$$\begin{aligned}
\max \quad & \sum_{t=0}^{\infty} \delta^t \left[\alpha_1 K_{ct}^{-\rho} + \alpha_2 L_{ct}^{-\rho}\right]^{-\frac{1}{\rho}} \\
s.t. \quad & y_t = \left[\beta_1 K_{yt}^{-\rho} + \beta_2 L_{yt}^{-\rho} + e_t\right]^{-\frac{1}{\rho}} \\
& 1 = L_{ct} + L_{yt} \\
& k_t = K_{ct} + K_{yt} \\
& y_t = k_{t+1} \\
& k_0,\ \{e_t\}_{t=0}^{\infty} \quad given
\end{aligned} \tag{6}$$

where $\delta \in (0,1)$ is the discount factor. p_t, r_t, and w_t respectively denote the price of capital goods, the rental rate of the capital goods and the wage rate of labor at time $t \geq 0$[3]. For any sequences $\{e_t\}_{t=0}^{\infty}$ of external effects that the representative agent considers given, the Lagrangian at time

[3] We normalize the price of consumption goods to one.

$t \geq 0$ is defined as follows:

$$\begin{aligned} \mathcal{L}_t &= \left[\alpha_1 K_{ct}^{-\rho} + \alpha_2 L_{ct}^{-\rho}\right]^{-\frac{1}{\rho}} + p_t \left[\left[\beta_1 K_{yt}^{-\rho} + \beta_2 L_{yt}^{-\rho} + e_t\right]^{-\frac{1}{\rho}} - k_{t+1}\right] \\ &+ r_t \left(k_t - K_{ct} - K_{yt}\right) + w_t \left(1 - L_{ct} - L_{yt}\right). \end{aligned} \tag{7}$$

Then the first order conditions derived from the Lagrangian are as follows:

$$\frac{\partial \mathcal{L}_t}{\partial K_{ct}} = \alpha_1 \left(\frac{c_t}{K_{ct}}\right) - r_t = 0, \tag{8}$$

$$\frac{\partial \mathcal{L}_t}{\partial L_{ct}} = \alpha_2 \left(\frac{c_t}{L_{ct}}\right) - w_t = 0, \tag{9}$$

$$\frac{\partial \mathcal{L}_t}{\partial K_{yt}} = p_t \beta_1 \left(\frac{y_t}{K_{yt}}\right) - r_t = 0, \tag{10}$$

$$\frac{\partial \mathcal{L}_t}{\partial L_{yt}} = p_t \beta_2 \left(\frac{y_t}{L_{yt}}\right) - w_t = 0. \tag{11}$$

From the above first order conditions, we derive the following equation,

$$\left(\frac{\alpha_1 / \alpha_2}{\beta_1 / \beta_2}\right) = \left(\frac{K_{ct} / L_{ct}}{K_{yt} / L_{yt}}\right)^{1+\rho}. \tag{12}$$

If $\alpha_1/\alpha_2 > (<)\, \beta_1/\beta_2$, the consumption (capital) good sector is capital intensive from the private perspective.

For any value of $(k_t,\ y_t)$, solving the first order conditions with respect to K_{ct}, K_{yt}, L_{ct}, L_{yt} gives these inputs as functions of capital stock at time t and $t+1$, and external effect, namely:

$$\begin{aligned} K_{ct} &= K_c\left(k_t,\ y_t, e_t\right),\quad L_{ct} = L_c\left(k_t,\ y_t, e_t\right), \\ K_{yt} &= K_y\left(k_t,\ y_t, e_t\right),\ L_{yt} = L_y\left(k_t,\ y_t, e_t\right). \end{aligned}$$

For any given sequence $\{e_t\}_{t=0}^{\infty}$, we define the efficient production frontier as follows:

$$T^*\left(k_t,\ k_{t+1}, e_t\right) = \left[\alpha_1 K_c\left(k_t,\ y_t, e_t\right)^{-\rho} + \alpha_2 L_c\left(k_t,\ y_t, e_t\right)^{-\rho}\right]^{-\frac{1}{\rho}}.$$

Using the envelope theorem we derive the equilibrium prices,[4]

$$T_2\left(k_t, k_{t+1}, e_t\right) = -p_t, \tag{13}$$

$$T_1\left(k_t, k_{t+1}, e_t\right) = r_t. \tag{14}$$

[4]See Takayama for the envelope theorem, pp160-165. Using the envelope theorem, we get $\frac{\partial \mathcal{L}_t}{\partial k_t} = \frac{\partial T}{\partial k_t}$ and $\frac{\partial \mathcal{L}_t}{\partial k_{t+1}} = \frac{\partial T}{\partial k_{t+1}}$. This is equivalent to (13) and (14).

Next we solve the intertemporal problem (6). In this model, lifetime utility function becomes

$$U = \sum_{t=0}^{\infty} \delta^t T^* \left(k_t, k_{t+1}, e_t\right).$$

From the first order conditions with respect to k_{t+1}, we obtain the Euler equation

$$T_2 \left(k_t, k_{t+1}, e_t\right) + \delta T_1 \left(k_{t+1}, k_{t+2}, e_{t+1}\right) = 0. \tag{15}$$

The solution of equation (15) also has to satisfy the following transversality condition

$$\lim_{t\to+\infty} \delta^t k_t T_1 \left(k_t, k_{t+1}, e_t\right) = 0. \tag{16}$$

We denote the solution of this problem $\{k_t\}_{t=0}^{\infty}$. This path depends on the choice of sequence $\{e_t\}_{t=0}^{\infty}$. If the sequence $\{e_t\}_{t=0}^{\infty}$ satisfies

$$e_t = bK_y \left(k_t,\ y_t, e_t\right)^{-\rho} - bL_y \left(k_t,\ y_t, e_t\right)^{-\rho}, \tag{17}$$

then $\{\hat{k}_t\}_{t=0}^{\infty}$ is called an equilibrium path. Along an equilibrium path, the expectations of the representative agent on the externalities $\{e_t\}_{t=0}^{\infty}$ are realized.

Definition 2 *$\{k_t\}_{t=0}^{\infty}$ is an equilibrium path if $\{k_t\}_{t=0}^{\infty}$ satisfies (15), (16) and (17).*

Solving the equation (17) for e_t, we derive e_t that is given as a function of $(k_t,\ k_{t+1})$, namely $e_t = \hat{e}\left(k_t, k_{t+1}\right)$. Let us substitute $\hat{e}\left(k_t, k_{t+1}\right)$ into equations (13) and (14) and define the equilibrium prices as

$$p_t = p_t \left(k_t, k_{t+1}\right),$$

$$r_t = r_t \left(k_t, k_{t+1}\right).$$

Then the Euler equation (15) evaluated at $\{k_t\}_{t=0}^{\infty}$ is

$$-p \left(k_t, k_{t+1}\right) + \delta r \left(k_{t+1}, k_{t+2}\right) = 0. \tag{18}$$

We have the following lemma.

Lemma 1 *The partial derivatives of $T\left(k_t, k_{t+1}, e_t\right)$ with respect to k_t and k_{t+1} are given by*

$$\begin{aligned} T_1 \left(k_t, k_{t+1}, \hat{e}\left(k_t, k_{t+1}\right)\right) &= \alpha_1 \left[\alpha_1 + \alpha_2 \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1+\rho}{\rho}} \left(\frac{\left(\frac{g}{k_{t+1}}\right)^{\rho}}{\beta_2 - b} - \frac{(\beta_1+b)}{\beta_2-b}\right)^{\rho}\right]^{-\frac{1+\rho}{\rho}} \\ T_2 \left(k_t, k_{t+1}, \hat{e}\left(k_t, k_{t+1}\right)\right) &= \frac{T_1(k_t,\ k_{t+1}, \hat{e}(k_t,k_{t+1}))}{\beta_1} \left(\frac{g}{k_{t+1}}\right)^{1+\rho} \end{aligned}$$

where

$$g = g\left(k_t, k_{t+1}\right) = \left\{K_{yt} \in [0, k_t] \mid \frac{\alpha_1\beta_2}{\alpha_2\beta_1} = \left(\frac{k_t - K_{yt}}{1 - L_{yt}(K_{yt}, k_{t+1})}\right)^{1+\rho} \left(\frac{L_{yt}(K_{yt}, k_{t+1})}{K_{yt}}\right)^{1+\rho}\right\}$$

and

$$L_{yt}\left(K_{yt},k_{t+1}\right)=\left(\frac{k_{t+1}^{-\rho}-(\beta_1+b)K_{yt}^{-\rho}}{\beta_2-b}\right)^{-\frac{1}{\rho}}.$$

3 Steady state

Definition 3 *A steady state is defined by* $k_t=k_{t+1}=y_t=k^*$ *and is given by the solution of* $T_2\left(k^*,k^*,e^*\right)+\delta T_1\left(k^*,k^*,e^*\right)=0$ *with* $e^*=\hat{e}\left(k^*,k^*\right).$

In the rest of the paper we assume the following restriction on parameters' values that guarantees all the steady state values are positive.

Assumption 1 *The parameters* δ, β_1, b *and* ρ *satisfy*

$$(\delta\beta_1)^{\frac{-\rho}{1+\rho}}<\beta_1+b.$$

We obtain the steady state value.

Proposition 1 *In this model, there exists a unique stationary capital stock* k^* *such that:*

$$k^*=\left\{1+\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{-1}{1+\rho}}(\delta\beta_1)^{\frac{-1}{1+\rho}}\left[1-(\delta\beta_1)^{\frac{1}{1+\rho}}\right]\right\}^{-1}\left[\frac{1-\hat{\beta}_1(\delta\beta_1)^{\frac{-\rho}{1+\rho}}}{\hat{\beta}_2}\right]^{\frac{1}{\rho}}. \tag{19}$$

To study local behavior of the equilibrium path around the steady state k^*, we linearize the Euler equation (15) at the steady state k^* and obtain the following characteristic equation

$$\delta T_{12}\lambda^2+\left[\delta T_{11}+T_{22}\right]\lambda+T_{21}=0,$$

or

$$\delta\lambda^2+\left[\delta\frac{T_{11}}{T_{12}}+\frac{T_{22}}{T_{12}}\right]\lambda+\frac{T_{21}}{T_{12}}=0. \tag{20}$$

As shown in Nishimura and Venditti [7], the expressions of the characteristic roots are as follows:

Proposition 2 *The characteristic roots of Equation (20) are*

$$\begin{aligned}\lambda_1 &= \frac{1}{(\delta\beta_2)^{\frac{1}{1+\rho}}\left[\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1+\rho}}-\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1+\rho}}\right]}, \qquad (21)\\ \lambda_2(b) &= \frac{(\delta\beta_2)^{\frac{1}{1+\rho}}\left[\frac{\beta_1+b}{\beta_1}\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1+\rho}}-\frac{\beta_2-b}{\beta_2}\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1+\rho}}\right]}{\delta}.\end{aligned}$$

The roots of the characteristic equation determine the local behavior of the equilibrium paths. The sign of λ_1 is determined by factor intensity differences from the private perspective.[5]

We now characterize the equilibrium paths in this model. In particular we can show that the local behavior of equilibrium path around the steady state changes according to the degree of external effect in the capital good sector.

Definition 4 *A steady state k^* is called locally indeterminate if there exists ε such that for any $k_0 \in (k^* - \varepsilon, k^* + \varepsilon)$, there are infinitely many equilibrium paths converging to the steady state.*

As there is one pre-determined variable, the capital stock, local indeterminacy occurs if the stable manifold has two dimension, i.e. if the two characteristic roots are within the unit circle. We will also present conditions for local determinacy (for saddle-point stability) in which there exists a unique equilibrium path. Such a configuration occurs if the stable manifold has one dimension, i.e. if one root is outside the unit circle while the other is inside.

When the investment good is capital intensive, local indeterminacy and flip bifurcation cannot occur.

Proposition 3 *Suppose that the capital good sector is capital intensive from the private perspective, i.e. $\alpha_2\beta_1 > \alpha_1\beta_2$. Then the characteristic roots λ_1 and $\lambda_2(b)$ are positive with $\lambda_1 > 1$.*

Next we present our results assuming that the capital good is labor intensive from the private perspective, i.e. $\alpha_2\beta_1 - \alpha_1\beta_2 < 0$. Equilibrium period-two cycles may occur in this case through a flip bifurcation. We will also get local indeterminacy of equilibria. By rewriting equation (21), the characteristic roots are

$$\begin{aligned} \lambda_1 &= -\frac{1}{(\delta\beta_2)^{\frac{1}{1+\rho}}\left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1+\rho}} - \left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1+\rho}}\right]}, \\ \lambda_2(b) &= -\frac{(\delta\beta_2)^{\frac{1}{1+\rho}}\left[\frac{\beta_2-b}{\beta_2}\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1+\rho}} - \frac{\beta_1+b}{\beta_1}\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1+\rho}}\right]}{\delta}. \end{aligned} \tag{22}$$

To get $\lambda_1 \in (-1, 0)$, we need however to suppose a slightly stronger condition than simply ensuring the capital good sector to be labor intensive from the private perspective. The capital intensity difference $\alpha_1\beta_2 - \alpha_2\beta_1$ needs to be large enough and the discount factor has to be close enough to 1.

Proposition 4 *Assume that $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$ and $\delta \in (\delta_3, 1)$ with*

$$\delta_3 = \beta_2^{-1}\left[(\beta_1/\beta_2)^{\frac{1}{1+\rho}} - (\alpha_1/\alpha_2)^{\frac{1}{1+\rho}}\right]^{-1-\rho} < 1.$$

[5]If $\alpha_2\beta_1 - \alpha_1\beta_2 > 0$, the capital good sector is capital intensive from the private perspective.

Then there exist $\underline{b}(\delta) > 0$ *and* $\overline{b}(\delta) > \underline{b}(\delta)$ *such that the steady state is saddle point for* $b \in (0, \underline{b}(\delta))$*, undergoes a flip bifurcation when* $b = \underline{b}(\delta)$*, becomes locally indeterminate for* $b \in \left(\underline{b}(\delta), \overline{b}(\delta)\right)$ *and is again saddle-point stable for* $\left(\overline{b}(\delta), +\infty\right)$*. Generically, there exist period-two cycles in a left (right) neighborhood of* $\underline{b}(\delta)$ *that are locally indeterminate (saddle-point stable).*

Next we still assume that the capital good is labor intensive from the private perspective with $\alpha_2\beta_1 - \alpha_1\beta_2 < 0$, but make λ_1 an unstable root, i.e. $\lambda_1 < -1$. As a result local indeterminacy cannot occur but period-two cycles may still exist through a flip bifurcation. Two cases need to be considered: $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$ and $\delta \in (0, \delta_3)$, as well as $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} < \alpha_2^{\frac{1}{1+\rho}}$. The following result is proved along the same lines as Proposition 4.

Proposition 5 *Suppose that the capital goods sector is labor intensive from the private perspective and let*

$$\delta_4 = \beta_2^{\frac{1}{\rho}} \left[(\beta_1/\beta_2)^{\frac{1}{1+\rho}} - (\alpha_1/\alpha_2)^{\frac{1}{1+\rho}} \right]^{\frac{1+\rho}{\rho}} .$$

Assume also that one of the following sets of conditions hold:

i) $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$ *and* $\delta \in (0, \delta^*)$ *with* $\delta^* = \min\{\delta_3, \delta_4\}$*,*

ii) $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} < \alpha_2^{\frac{1}{1+\rho}}$*,* $\rho > 0$ *and* $\delta \in (0, \delta_4)$*,*

Then there exist $\underline{b}(\delta) > 0$ *and* $\overline{b}(\delta) > \underline{b}(\delta)$ *such that the steady state is totally unstable for* $b \in (0, \underline{b}(\delta))$*, undergoes a flip bifurcation when* $b = \underline{b}(\delta)$*, becomes saddle-point stable for* $b \in \left(\underline{b}(\delta), \overline{b}(\delta)\right)$ *and is again totally unstable for* $\left(\overline{b}(\delta), +\infty\right)$*. Generically, there exist period-two cycles in a left (right) neighborhood of* $\underline{b}(\delta)$ *that are locally saddle-point stable (unstable).*

Remark 2 Consider the production function from the social perspective as given in Definition 1 and recall from (5) that we can write it as follows

$$\tilde{y} = \left[(\beta_1 + b)\, k_y^{-\rho_y} + (\beta_2 - b) \right]^{-\frac{1}{\rho_y}} . \tag{23}$$

According to $b \gtrless \beta_2$, the following inequality holds: for any $\eta > 1$,

$$\begin{aligned} \left[(\beta_1 + b)(\eta k_y)^{-\rho} + (\beta_2 - b) \right]^{-\frac{1}{\rho}} &\gtrless \left[(\beta_1 + b)(\eta k_y)^{-\rho} + \eta^{-\rho}(\beta_2 - b) \right]^{-\frac{1}{\rho}} \\ &= \eta \left[(\beta_1 + b)\, k_y^{-\rho} + (\beta_2 - b) \right]^{-\frac{1}{\rho}} . \end{aligned}$$

If b is larger than β_2, the function $\tilde{y}$ exhibits increasing returns while if b is smaller than β_2 the function $\tilde{y}$ exhibits decreasing returns.

As we consider in Proposition 5 values of δ close to zero, the role of b on the local stability properties of the steady state is multiple. Indeed, starting from a low amount of externalities, an increase of b contributes to saddle-point stability and the existence of cycles through a flip bifurcation. But then if b is increased too much, total instability occurs since the returns to scale becomes increasing as shown in the previous Remark.

4 Concluding remarks

In this paper we have characterized the local dynamics of equilibrium paths depending on the size of external effects b. We have shown that when the consumption good is capital intensive, the effect of b on the local dynamics of equilibrium path depends on the value of the discount factor. If the discount factor is close enough to one and the capital intensity difference is large enough, local indeterminacy occurs for intermediary values of b while saddle-point stability is obtained when b is low enough or large enough. On the contrary, if the discount factor is low enough, local indeterminacy cannot occur. But the existence of equilibrium cycles and saddle-point stability require intermediary values of b while total instability is obtained when b is low enough or large enough.

5 Appendix

5.1 Proof of Lemma 1

We shall derive the first partial derivatives of $T\left(k_t, k_{t+1}, e_t\right)$ along an equilibrium path. The first order conditions derived from the Lagrangian are as below:

$$\alpha_1\left(\frac{c_t}{K_{ct}}\right) - r_t = 0, \tag{A1.1}$$

$$\alpha_2\left(\frac{c_t}{L_{ct}}\right) - w_t = 0, \tag{A1.2}$$

$$p_t\beta_1\left(\frac{y_t}{K_{yt}}\right) - r_t = 0, \tag{A1.3}$$

$$p_t\beta_2\left(\frac{y_t}{L_{yt}}\right) - w_t = 0. \tag{A1.4}$$

In the equilibrium the equation (2) is rewritten as

$$L_{yt} = \left(\frac{y_t^{-\rho} - \left(\beta_1 + b\right) K_{yt}^{-\rho}}{\beta_2 - b}\right)^{-\frac{1}{\rho}}. \tag{A1.5}$$

From the first order conditions (A1.1)-(A1.4),

$$\frac{\alpha_1\beta_2}{\alpha_2\beta_1} = \left(\frac{K_{ct}}{L_{ct}}\right)^{1+\rho}\left(\frac{L_{yt}}{K_{yt}}\right)^{1+\rho}.$$

Substituting $K_{ct} = k_t - K_{yt},\ L_{yt} = 1 - L_{ct}$ into the equation,

$$\frac{\alpha_1\beta_2}{\alpha_2\beta_1} = \left(\frac{k_t - K_{yt}}{1 - L_{yt}}\right)^{1+\rho}\left(\frac{L_{yt}}{K_{yt}}\right)^{1+\rho}. \tag{A1.6}$$

By solving equations (A1.5) and (A1.6) with respect to K_{yt} and substituting $y_t = k_{t+1}$, we have $K_{yt} = g\left(k_t,\ k_{t+1}\right)$. From the equation (A1.1),

$$r_t = \alpha_1\left[\alpha_1 + \alpha_2\left(\frac{K_{ct}}{L_{ct}}\right)^{\rho}\right]^{-\frac{1+\rho}{\rho}}.$$

Using the equation (A1.6) we have

$$r_t = \alpha_1\left[\alpha_1 + \alpha_2\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1+\rho}{\rho}}\left(\frac{g\left(k_t, k_{t+1}\right)}{L_{ct}}\right)^{\rho}\right]^{-\frac{1+\rho}{\rho}}.$$

And then from (A1.5) r_t can be rewritten as the following equation by substituting $\left(\frac{g(k_t,k_{t+1})}{L_{yt}}\right)^{\rho} = \frac{\left(\frac{g(k_t,k_{t+1})}{y_t}\right)^{\rho}}{\beta_2 - b} - \frac{(\beta_1+b)}{\beta_2-b}$ [6],

$$r_t = \alpha_1\left[\alpha_1 + \alpha_2\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1+\rho}{\rho}}\left(\frac{\left(\frac{g(k_t,k_{t+1})}{y_t}\right)^{\rho}}{\beta_2 - b} - \frac{(\beta_1 + b)}{\beta_2 - b}\right)^{\rho}\right]^{-\frac{1+\rho}{\rho}}. \tag{A1.7}$$

Moreover from the equation (A1.3), we have

$$p_t = \frac{r_t}{\beta_1}\left(\frac{g\left(k_t, k_{t+1}\right)}{y_t}\right)^{1+\rho}. \tag{A1.8}$$

Therefore we get T_1 and T_2 from the envelope theorem which gives

$$T_1 = r_t, \quad T_2 = -p_t.$$

■

5.2 Proof of Proposition 1

By definition k^* satisfies $T_2\left(k^*, k^*, e^*\right) + \delta T_1\left(k^*, k^*, e^*\right) = 0$ with $e^* = \hat{e}\left(k^*, k^*\right)$. In the steady state, $g^* = g\left(k^*, k^*\right)$ and $y^* = k^*$. Using Lemma 1, the Euler equation is

$$-\frac{r}{\beta_1}\left(\frac{g^*}{y^*}\right)^{1+\rho} + \delta r = 0.$$

[6] Substitute the equation (A1.5) into $\left(\frac{g}{L_{yt}}\right)^{\rho}$.

Thus,

$$g^* = (\delta\beta_1)^{\frac{1}{1+\rho}} k^*. \tag{A2.1}$$

As $y^* = k^*$ at the steady state, the equation (A1.5) becomes,

$$L_y^* = k^* \left(\frac{1-(\beta_1+b)(\delta\beta_1)^{\frac{-\rho}{1+\rho}}}{\beta_2 - b} \right)^{-\frac{1}{\rho}}. \tag{A2.2}$$

Using $K_c^* = k^* - K_y^*$ and $L_c^* = 1 - L_y^*$,

$$L_c^* = 1 - k^* \left(\frac{1-(\beta_1+b)(\delta\beta_1)^{\frac{-\rho}{1+\rho}}}{\beta_2 - b} \right)^{-\frac{1}{\rho}}, \tag{A2.3}$$

$$K_c^* = k^* \left(1 - (\delta\beta_1)^{\frac{1}{1+\rho}} \right). \tag{A2.4}$$

Then equation (A1.6) can be rewritten as follows;

$$\left(\frac{K_c^*}{L_c^*} \right)^{1+\rho} \left(\frac{L_y^*}{g^*} \right)^{1+\rho} = \frac{\alpha_1\beta_2}{\alpha_2\beta_1}. \tag{A2.5}$$

Substituting these input demand functions into the above equation and solving with respect to k^*, we can get

$$k^* = \left[1 + \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1} \right)^{\frac{-1}{1+\rho}} (\delta\beta_1)^{\frac{-1}{1+\rho}} \left(1 - (\delta\beta_1)^{\frac{1}{1+\rho}} \right) \right]^{-1} \left[\frac{1-(\beta_1+b)(\delta\beta_1)^{\frac{-\rho}{1+\rho}}}{\beta_2 - b} \right]^{\frac{1}{\rho}}.$$

Then k^* is well defined if and only if

$$(\delta\beta_1)^{\frac{-\rho}{1+\rho}} < \frac{1}{\beta_1 + b}.$$

■

5.3 Proof of Proposition 2

We give some lemmas in order to derive the characteristic roots.

Lemma 2 *At the steady state the following holds*

$$g_1 = \frac{1+\rho}{\Delta K_c},$$

$$g_2 = \left[\frac{(1+\rho) L_y^{\rho} + (1+\rho) \frac{L_y^{1+\rho}}{L_c}}{\Delta} \right] \frac{y^{-1-\rho}}{\beta_2 - b},$$

where

$$\Delta = \frac{1+\rho}{g} + \frac{1+\rho}{K_c} + \left((1+\rho) L_y^{\rho} + (1+\rho) \frac{L_y^{1+\rho}}{L_c} \right) \frac{\beta_1 + b}{\beta_2 - b} g^{-1-\rho}.$$

Proof. From equation (A2.5) we get

$$\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} = g^{-1-\rho} (k-g)^{1+\rho} \left(\frac{y^{-\rho} - (\beta_1 + b) g^{-\rho}}{\beta_2 - b} \right)^{-\frac{1+\rho}{\rho}} \left\{ 1 - \left(\frac{y^{-\rho} - (\beta_1 + b) g^{-\rho}}{\beta_2 - b} \right)^{-\frac{1}{\rho}} \right\}^{-1-\rho}.$$

Totally differentiating this equation, we have the following relationship,

$$\begin{aligned}
&[(1+\rho) g^{-1} + (1+\rho)(k-g)^{-1} + (1+\rho) \left(\frac{y^{-\rho} - (\beta_1 + b) g^{-\rho}}{\beta_2 - b} \right)^{-1} \frac{\beta_1 + b}{\beta_2 - b} g^{-1-\rho} \\
&+ (1+\rho) \left\{ 1 - \left(\frac{y^{-\rho} - (\beta_1 + b) g^{-\rho}}{\beta_2 - b} \right)^{-\frac{1}{\rho}} \right\}^{-1} \left(\frac{y^{-\rho} - (\beta_1 + b) g^{-\rho}}{\beta_2 - b} \right)^{-\frac{1+\rho}{\rho}} \frac{\beta_1 + b}{\beta_2 - b} g^{-1-\rho}] dg \\
&= (1+\rho)(k-g)^{-1} dk + (1+\rho) \left(\frac{y^{-\rho} - (\beta_1 + b) g^{-\rho}}{\beta_2 - b} \right)^{-1-1} \frac{y^{-1-\rho}}{\beta_2 - b} dy \\
&+ (1+\rho) \left\{ 1 - \left(\frac{y^{-\rho} - (\beta_1 + b) g^{-\rho}}{\beta_2 - b} \right)^{-\frac{1}{\rho}} \right\}^{-1} \left(\frac{y^{-\rho} - (\beta_1 + b) g^{-\rho}}{\beta_2 - b} \right)^{-\frac{1+\rho}{\rho}} \frac{y^{-1-\rho}}{\beta_2 - b} dy.
\end{aligned} \tag{A3.1.1}$$

Notice from equation (A1.5)

$$L_y^{-\rho} = \frac{y^{-\rho} - (\beta_1 + b) g^{-\rho}}{\beta_2 - b} \tag{A3.1.2}$$

and (A2.5)

$$\left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right)^{\frac{1}{1+\rho}} = \frac{K_c^*}{L_c^*} \left(\frac{g^*}{L_y^*} \right)^{-1}. \tag{A3.1.3}$$

Then substituting these equations and $dy_t = dk_{t+1}$ into (A3.1.1) gives

$$RHS = \frac{1+\rho}{K_c} dk_t + \left((1+\rho) L_y^{\rho} + (1+\rho) \frac{L_y^{1+\rho}}{L_c} \right) \frac{y^{1-\rho}}{\beta_2 - b} dk_{t+1},$$

$$LHS = \left[\frac{1+\rho}{g} + \frac{1+\rho}{K_c} + \left((1+\rho) L_y^{\rho} + (1+\rho) \frac{L_y^{1+\rho}}{L_c} \right) \frac{\beta_1 + b}{\beta_2 - b} g^{-1-\rho} \right] dg,$$

where we denote

$$\Delta \equiv \frac{1+\rho}{g} + \frac{1+\rho}{K_c} + \left((1+\rho) L_y^{\rho} + (1+\rho) \frac{L_y^{1+\rho}}{L_c} \right) \frac{\beta_1 + b}{\beta_2 - b} g^{-1-\rho},$$

and we derive

$$\Delta dg = \frac{1+\rho}{K_c} dk_t + \left[(1+\rho) L_y^{\rho} + (1+\rho) \frac{L_y^{1+\rho}}{L_c} \right] \frac{y^{1-\rho}}{\beta_2 - b} dk_{t+1}.$$

Therefore

$$dg = \frac{1+\rho}{\Delta K_c} dk_t + \frac{\left[(1+\rho) L_y^{\rho} + (1+\rho) \frac{L_y^{1+\rho}}{L_c} \right]}{\Delta} \frac{y^{1-\rho}}{\beta_2 - b} dk_{t+1}.$$

■

Lemma 3 *At the steady state the following holds*

$$g_1 y = (g - g_2 y) \frac{y}{g} \left(1 - \frac{K_c}{L_c} \frac{L_y}{K_y} \right)^{-1}$$

with g, K_c, L_y, L_c *respectively given by equations* $(A2.1) - (A2.4)$.

Proof. From equation (A1.5) we get

$$\left(L_y^{-\rho} + \frac{\beta_1 + b}{\beta_2 - b} K_y^{-\rho} \right) y^{-1} = \frac{y^{-\rho}}{\beta_2 - b} y^{-1}.$$

Substituting this equation into g_2,

$$g_2 = \frac{\left[(1+\rho) L_y^{\rho} + (1+\rho) \frac{L_y^{1+\rho}}{L_c} \right] (L_y^{-\rho} + \frac{\beta_1 + b}{\beta_2 - b} K_y^{-\rho}) y^{-1}}{\Delta}.$$

Using the expression of Δ we derive

$$g_2 y = g + \frac{(1+\rho)}{\Delta} \frac{L_y}{L_c} - \frac{(1+\rho)}{\Delta} \frac{g}{K_c}.$$

Then,

$$g - g_2 y = \frac{(1+\rho)}{\Delta K_c} g \left(1 - \frac{L_y}{L_c} \frac{K_c}{g} \right), \tag{A3.2.1}$$

$$g_1 y = \frac{1+\rho}{\Delta K_c} y. \tag{A3.2.2}$$

From equations (A3.2.1) and (A3.2.2), we finally get

$$g_1 y = (g - g_2 y) \frac{y}{g} \left(1 - \frac{K_c}{L_c} \frac{L_y}{K_y} \right)^{-1}.$$

■

Lemma 4 *Under Assumption* 1, *at the steady state,* $k_t = k_{t+1} = y_t = k^*$ *and the following holds*

$$\frac{V_{11}(k^*, k^*)}{V_{12}(k^*, k^*)} = -\frac{y}{g} \left(1 - \frac{K_c}{L_c} \frac{L_y}{K_y} \right)^{-1},$$

$$\frac{V_{22}(k^*,\ k^*)}{V_{12}(k^*,\ k^*)} = -\frac{g}{\beta_1 y}\left[\frac{\beta_1(\beta_2-b)}{\beta_2}\frac{K_c}{L_c}\frac{L_y}{g} + (\beta_2-b)\left(\frac{g}{L_y}\right)^{\rho} - \left(\frac{g}{y}\right)^{\rho}\right],$$

$$\frac{V_{21}(k^*,\ k^*)}{V_{12}(k^*,\ k^*)} = \frac{V_{22}(k^*,\ k^*)}{V_{12}(k^*,\ k^*)}\frac{V_{11}(k^*,\ k^*)}{V_{12}(k^*,\ k^*)},$$

where $g,\ K_c,\ L_y,\ L_c$ *are given by equations* $(A2.1)-(A2.4)$, *respectively.*

Proof. Let $V(k_t,\ k_{t+1})$ denote $T_i(k_t,\ k_{t+1}, \hat{e}(k_t,\ k_{t+1}))$ for $i=1,\ 2$. By definition,

$$\begin{aligned} V_{11}^* &= \frac{\partial T_1}{\partial k_t} = \frac{\partial r}{\partial k_t}, \\ V_{12}^* &= \frac{\partial T_1}{\partial k_{t+1}} = \frac{\partial r}{\partial k_{t+1}}, \\ V_{21}^* &= \frac{\partial T_2}{\partial k_t} = -\frac{\partial p}{\partial k_t}, \\ V_{22}^* &= \frac{\partial T_2}{\partial k_{t+1}} = -\frac{\partial p}{\partial k_{t+1}}. \end{aligned}$$

Computing the these equations, we have

$$V_{11}^* = \frac{\partial r}{\partial k_t} = -(1+\rho)\,\alpha_1^{-\frac{\rho}{1+\rho}} r^{\frac{1+2\rho}{1+\rho}} \frac{\alpha_2}{\beta_2-b}\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{\rho}{1+\rho}}\left(\frac{g}{y}\right)^{\rho}\frac{g_1}{g},$$

$$V_{12}^* = \frac{\partial r}{\partial k_{t+1}} = -(1+\rho)\,\alpha_1^{-\frac{\rho}{1+\rho}} r^{\frac{1+2\rho}{1+\rho}} \frac{\alpha_2}{\beta_2-b}\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{\rho}{1+\rho}}\left(\frac{g}{y}\right)^{\rho}\left(\frac{g_2 y - g}{yg}\right),$$

$$\frac{\partial p}{\partial k_t} = \frac{1}{\beta_1}\frac{\partial r}{\partial k_t}\left(\frac{g}{y}\right)^{1+\rho} + (1+\rho)\frac{r}{\beta_1}\left(\frac{g}{y}\right)^{1+\rho}\frac{g_1}{g},$$

$$\frac{\partial p}{\partial k_{t+1}} = \frac{1}{\beta_1}\frac{\partial r}{\partial k_{t+1}}\left(\frac{g}{y}\right)^{1+\rho} + (1+\rho)\frac{r}{\beta_1}\left(\frac{g}{y}\right)^{1+\rho}\left(\frac{g_2 y - g}{yg}\right).$$

From equation (A1.7),

$$\left(\frac{r}{\alpha_1}\right)^{\frac{\rho}{1+\rho}} = \left[\alpha_1 + \alpha_2\left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{\rho}{1+\rho}}\left(\frac{g}{L_c}\right)^{\rho}\right].$$

Substituting the above equation into V_{11}^*, and using (A3.1.2) and (A3.1.3) we obtain

$$V_{11}^* = -(1+\rho)\,r\left(\frac{g}{y}\right)^{\rho}\frac{g_1}{g}\left[\frac{\alpha_1\hat{\beta}_2}{\alpha_2}\left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\right)\frac{K_c^*}{L_c^*}\frac{L_y^*}{g^*} + \hat{\beta}_2\left(\frac{y}{L_y}\right)^{\rho}\right]^{-1},$$

where

$$\mathcal{A} \equiv \frac{\alpha_1\hat{\beta}_2}{\alpha_2}\left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}\right)\frac{K_c^*}{L_c^*}\frac{L_y^*}{g^*} + \hat{\beta}_2\left(\frac{y}{L_y}\right)^{\rho}.$$

We can calculate V_{21}^*, V_{12}^*, and V_{22}^* as we did previously,

$$V_{21}^* = -(1+\rho)\frac{r}{\beta_1}\left(\frac{g}{y}\right)^{1+\rho}\frac{g_1}{g}\left[1-\left(\frac{g}{y}\right)^{\rho}\mathcal{A}^{-1}\right],$$

$$V_{12}^* = -(1+\rho)\,r\left(\frac{g}{y}\right)^{\rho}\left(\frac{g_2y-g}{yg}\right)\mathcal{A}^{-1},$$

$$V_{22}^* = -(1+\rho)\frac{r}{\beta_1}\left(\frac{g}{y}\right)^{1+\rho}\left(\frac{g_2y-g}{yg}\right)\left[1-\left(\frac{g}{y}\right)^{\rho}\mathcal{A}^{-1}\right].$$

Then we get

$$\frac{V_{11}^*}{V_{12}^*} = \frac{g_1y}{g_2y-g},$$

$$\frac{V_{22}^*}{V_{12}^*} = \frac{g}{\beta_1 y}\left[\mathcal{A}-\left(\frac{g}{y}\right)^{\rho}\right],$$

■

We shall now prove Proposition 2. From Lemma 4 the characteristic polynomial may be rewritten as

$$G(\lambda) = \left(\lambda+\frac{V_{11}^*}{V_{12}^*}\right)\left(\delta\lambda+\frac{V_{22}^*}{V_{12}^*}\right).$$

Then the characteristic roots are

$$\lambda_1 = -\frac{V_{11}^*}{V_{12}^*},\quad \lambda_2 = -\frac{V_{22}^*}{\delta V_{12}^*}. \tag{A3.3.1}$$

We can calculate $\frac{V_{11}^*}{V_{12}^*}$ and $\frac{V_{22}^*}{V_{12}^*}$ by substituting the following relationship

$$\frac{K_c}{L_c}\frac{L_y}{g} = \left(\frac{\alpha_1\beta_2}{\alpha_2\beta_1}\right)^{\frac{1}{1+\rho}},$$

$$\frac{g}{y} = (\delta\beta_1)^{\frac{1}{1+\rho}},$$

$$\left(\frac{g}{L_y}\right)^{\rho} = \frac{(\delta\beta_1)^{\frac{\rho}{1+\rho}}-\hat{\beta}_1}{\hat{\beta}_2},$$

$$g-g_2y = \frac{(1+\rho)}{\Delta K_c}g\left(1-\frac{L_y}{L_c}\frac{K_c}{g}\right),$$

and we obtain the first root by substituting all the above equations into the expressions given in Lemma 4

$$\lambda_1 = -\frac{1}{(\delta\beta_2)^{\frac{1}{1+\rho}}\left[\left(\frac{\alpha_1}{\alpha_2}\right)^{\frac{1}{1+\rho}}-\left(\frac{\beta_1}{\beta_2}\right)^{\frac{1}{1+\rho}}\right]}.$$

Moreover we can rewrite $\mathcal{A}$ by using these equations,

$$\mathcal{A} = \hat{\beta}_2 \frac{\alpha_1}{\alpha_2} \left(\frac{\alpha_2 \beta_1}{\alpha_1 \beta_2} \right)^{-\frac{\rho}{1+\rho}} + (\delta \beta_1)^{\frac{\rho}{1+\rho}} - \hat{\beta}_1.$$

From Lemma 4, we finally have the second characteristic root,

$$\lambda_2 = -\frac{(\delta\beta_2)^{\frac{1}{1+\rho}} \left[\frac{\beta_2 - b}{\beta_2} \left(\frac{\alpha_1}{\alpha_2} \right)^{\frac{1}{1+\rho}} - \frac{\beta_1 + b}{\beta_1} \left(\frac{\beta_1}{\beta_2} \right)^{\frac{1}{1+\rho}} \right]}{\delta}.$$

■

5.4 Proof of Proposition 3

Notice from (21) that $\lambda_1 > 0$. Denoting[7]

$$\delta_1 \equiv \beta_2^{-1} \left[(\beta_1/\beta_2)^{\frac{1}{1+\rho}} - (\alpha_1/\alpha_2)^{\frac{1}{1+\rho}} \right]^{-1-\rho} > 1$$

then we obtain $\lambda_1 = (\delta_1/\delta)^{\frac{1}{1+\rho}} > 1$ for $0 < \delta < 1$. Since $(\beta_1 + b)/\beta_1 > 1$ and $(\beta_2 - b)/\beta_2 < 1$, $\lambda_2(b)$ is always positive. ■

5.5 Proof of Proposition 4

If $(\alpha_1\beta_2)^{\frac{1}{1+\rho}} - (\alpha_2\beta_1)^{\frac{1}{1+\rho}} > \alpha_2^{\frac{1}{1+\rho}}$ and $\delta \in (\delta_3, 1)$, then $-1 < \lambda_1 < 0$. The size of $\lambda_2(b)$ is determined in the following way. Notice that $\lambda_2(b)$ is increasing in b. For $b = 0$, $\lambda_2(0) = 1/\delta\lambda_1 < -1$ by the above hypothesis and for $b = \beta_2$, $\lambda_2(\beta_2) = (\delta\beta_1)^{\frac{-\rho}{1+\rho}}$.

(i) If $-1 < \rho < 0$, $\lambda_2(\beta_2) < 1$. Therefore there exist $\underline{b}(\delta) \in (0, \beta_2)$ and $\overline{b}(\delta) > \beta_2$ such that $\lambda_2 < -1$ for $b \in (0, \underline{b}(\delta))$, $-1 < \lambda_2 < 1$ for any $b \in (\underline{b}(\delta), \overline{b}(\delta))$ and $\lambda_2 > 1$ for any $b > \overline{b}(\delta)$.

(ii) If $\rho = 0$, $\lambda_2(\beta_2) = 1$. Therefore $\lambda_2(b) < -1$ for $b \in (0, \beta_2 - 2\alpha_2)$, $-1 < \lambda_2(b) < 1$ for $b \in (\beta_2 - 2\alpha_2, \beta_2)$ and $\lambda_2(b) > 1$ for $b > \beta_2$.

(iii) If $\rho > 0$, $\lambda_2(\beta_2) > 1$. Therefore there exist $\underline{b}(\delta)$ and $\overline{b}(\delta)$ in $(0, \beta_2)$ such that $\lambda_2(b) < -1$ for $b \in (0, \underline{b}(\delta))$, $-1 < \lambda_2(b) < 1$ for $b \in (\underline{b}(\delta), \overline{b}(\delta))$, and $\lambda_2(b) > 1$ for $b > \overline{b}(\delta)$.

In each of these three cases, when $b = \underline{b}(\delta)$, $\lambda_2(b) = -1$ and $\lambda_2'(b)|_{b=\underline{b}(\delta)} > 0$. It follows that $b = \underline{b}(\delta)$ is a flip bifurcation value. The result follows from the flip bifurcation Theorem (see Ruelle [8]). ■

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[7] Note that $\delta_1 = \alpha_2 \left[(\alpha_2\beta_1)^{\frac{1}{1+\rho}} - (\alpha_1\beta_2)^{\frac{1}{1+\rho}} \right]^{-1-\rho} > \frac{\alpha_2}{(\alpha_2\beta_1)} = \frac{1}{\beta_1} > 1.$

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