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The Fibonacci Zeta-Function is Hypertranscendental

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ABSTRACT

Applying a theorem of Reich on Dirichlet series satisfying difference-differential equations, we show that the Fibonacci zeta-function satisfies no algebraic differential equation.

RESUMEN

Aplicando el Teorema de Reich sobre series de Dirichlet satisfaziendo ecuaciones diferenciales-diferencias, nosotros mostramos que la función zeta de Fibonacci satisfaze una ecuación diferencial no algebraica.

Key words and phrases: *Hypertranscendence, Fibonacci zeta-function.*

Math. Subj. Class.: *11B39, 11M41, 34M15.*

1 Introduction

The Fibonacci numbers are recursively defined by

$$F_0 = 0,\, F_1 = 1 \qquad \text{and} \qquad F_{n+2} = F_{n+1} + F_n \quad \text{for} \quad n \in \mathbb{N}.$$

In number theory one can often obtain arithmetic information by studying a generating function of a given number theoretical object. In the case of Fibonacci numbers this is usually the corresponding Lambert series; however, in the recent past also the generating Dirichlet series was studied; this function is more interesting with respect to its analytic properties. Let s be a complex variable. For $\mathrm{Re}\, s > 0$ the Fibonacci zeta-function is defined by

$$\zeta_{\mathsf{F}}(s) = \sum_{n \in \mathbb{N}} F_n^{-s},$$

and by analytic continuation throughout the complex plane except for simple poles at $s = -2k + \pi i(2n + k)/\log \varphi$ for $n \in \mathbb{Z}, k \in \mathbb{N}_0$, where φ is the golden ratio; this was first proved by Navas [6] (and relies mainly on Binet's formula). In [1], Elsner et al. obtained several results on the algebraic independence of the values taken by ζ_{F} on the positive integers, e.g. $\zeta_{\mathsf{F}}(2), \zeta_{\mathsf{F}}(4), \zeta_{\mathsf{F}}(6)$ are algebraically independent.

In this note we show that the Fibonacci zeta-function $\zeta_{\mathsf{F}}(s)$ is hypertranscendental, i.e., it satisfies no non-trivial algebraic differential equation (that is no finite collection of derivatives of ζ_{F}) is algebraically dependent over the field of rational functions). Actually, we shall prove a slightly stronger statement by applying Reich's theorem on Dirichlet series satisfying holomorphic difference–differential equations. In order to state this result denote by $\mathcal{D}$ the set of all ordinary Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ satisfying the following two assumptions:

- the abscissa of absoulte convergence is finite: $\sigma_a(f) < \infty$,
- the set of all divisors of indices n with $a_n \neq 0$ contains infinitely many prime numbers.

Furthermore, we introduce the following abbreviation: for a non-negative integer ν we write

$$\underline{f}^{[\nu]}(s) = (f(s), f'(s), \ldots, f^{(\nu)}(s)).$$

Reich [9] proved the following theorem: *Assume that* $f \in \mathcal{D}$. *Let* $h_0 < h_1 < \ldots < h_m$ *be any real numbers,* $\nu_0, \nu_1, \ldots, \nu_m$ *be any non-negative integers, and let* $\sigma_0 > \sigma_a(f) - h_0$. *Put* $k := \sum_{j=0}^{m} (\nu_j + 1)$. *If* $\Phi : \mathbb{C}^k \to \mathbb{C}$ *is continuous and the difference-differential equation*

$$\Phi(\underline{f}^{[\nu_0]}(s + h_0), \underline{f}^{[\nu_1]}(s + h_1), \ldots, \underline{f}^{[\nu_m]}(s + h_m)) = 0$$

holds for all s *with* $\mathrm{Re}\, s > \sigma_0$*, then* Φ *vanishes identically.* To apply this result to the Fibonacci zeta-function it suffices to show that the set of all Fibonacci numbers F_n is not generated by finitely many primes. However, this follows immediately from Lucas' theorem

$$\gcd(F_m, F_n) = F_{\gcd(m,n)}$$

(see [5]), since the right-hand side is equal to $F_1 = 1$ for any pair m, n of relatively coprime integers. Thus we obtain:

Theorem 1. *Given any real numbers $h_0 < h_1 < \ldots < h_m$, any non-negative integers $\nu_0, \nu_1, \ldots, \nu_m$, and any $\sigma_0 > -h_0$, if $\Phi : \mathbb{C}^k \to \mathbb{C}$ is continuous and the difference-differential equation*

$$\Phi(\underline{\zeta_{\mathsf{F}}}^{[\nu_0]}(s+h_0), \underline{\zeta_{\mathsf{F}}}^{[\nu_1]}(s+h_1), \ldots, \underline{\zeta_{\mathsf{F}}}^{[\nu_m]}(s+h_m)) = 0$$

holds for all s with $\operatorname{Re} s > \sigma_0$, then Φ vanishes identically.

Notice that the proof does not use the meromorphic continuation of $\zeta_{\mathsf{F}}(s)$ to $\mathbb{C}$, obtained by Navas. The statement of the theorem can easily be extended to other Dirichlet series built from linear recursive sequences. Here we only need that such sequences are divisible by infinitely many prime numbers which is true except for *degenerate* cases when the characteristic polynomial has two roots whose quotient is a root of unity; since roots are counted with multiplicities, this also includes the case of repeated roots. This was first shown by Pólya [8] and has been rediscovered by several mathematicians (see [2, 10] for some history).

We conclude with a few historical remarks on hypertranscendence and an interesting question. In 1887, Hölder [4] proved that the Gamma-function is hypertranscendental. In his challenging lecture at the International Congress for Mathematicians in Paris 1900, Hilbert [3] asked in problem 18 for a description of classes of functions definable by differential equations. In this context Hilbert stated that the Riemann zeta-function $\zeta(s)$ is hypertranscendental; the first published proof was written down by Stadigh in his dissertation (cf. Ostrowski [7]). The idea is to deduce the hypertranscendence of $\zeta(s)$ from Hölder's theorem and the fact that the Gamma-function appears in the functional equation for zeta. Besides, Hilbert [3] asked for a proof of the hypertranscendence for the more general series $\sum_{n=1}^{\infty} x^n n^{-s}$. This problem was solved by Ostrowski [7] as a particular case of a more general theorem which also applies to the case when there is no functional equation at hand; his argument relies on a comparison of the differential independence with the linear independence of its frequencies. Reich's theorem [9], which we have used to prove Theorem 1, may be regarded as the most general and powerful extension of this method. A different way for proving hypertranscendence was found by Voronin. In [11], he developped a new technique to study the joint value distribution of Dirichlet L-functions to pairwise inequivalent characters and their derivatives; in [12], he extended the method in order to prove his famous universality theorem for the Riemann zeta-function: *Let $0 < r < \frac{1}{4}$ and suppose that $g(s)$ is a non-vanishing continuous function on the disk $|s| \leq r$ which is analytic in the interior. Then, for any $\epsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas} \left\{ \tau \in [0,T] \,:\, \max_{|s|\leq r} \left| \zeta\left(s + \tfrac{3}{4} + i\tau\right) - g(s) \right| < \epsilon \right\} > 0.$$

Voronin's results imply the hypertranscendence of these Dirichlet series. It is natural to ask whether the Fibonacci zeta-function shares this or some other universality property: *is it true or false that any (suitable) analytic function $g(s)$ can be uniformly approximated by certain shifts of $\zeta_{\mathsf{F}}(s)$?*

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