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Control of Dynamic Oligopsonies with Production Factors

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ABSTRACT. Dynamic oligopsony is examined with discrete time scales. The controllability of the dynamic system is investigated where the competitive price is selected as the control variable. For the single product factor case complete characterization is given. A simple example illustrates that for the more general case no general solution can be given. We also elaborated the case when the fixed price of the production factors is the control variable. We show that the system is uncontrollable in this case.

1 Introduction

Oligopoly and related models have been examined intensively during the last 4-5 decades. This research area goes back to Cournot (1838), who is considered the founder of this field. A comprehensive summary of results up to the mid seventies is presented in Okuguchi (1976). Variants of the classical Cournot model with applications to natural resources management are discussed in Okuguchi and Szidarovszky (1999). Most studies considered the competition among the firm in the product market only. However there is a competition among the firms in the factor market as well. Oligopsonies include this kind of competition in the oligopoly models, as it is shown in Szidarovszky and Okuguchi (2001). Recently a dynamic oligopsony model was introduced by Kapolyi and Szidarovszky (2001), in which the existence, uniqueness and the global asymptotical stability of the Nash-Cournot equilibrium were proved.

In this paper we will discuss the controllability of the same dynamic system, where the competitive product price or fixed production factor price is the control variable.

2 The Mathematical Model

We consider an N-firm oligopoly with single-product firms without product differentiation. Let M denote the number of production factors, vector $\underline{l}_k$ the production factor usage of firm k, f_k the production function of firm k, $\underline{L} = \sum_{k=1}^{N} \underline{l}_k$ the total production factor usage of the industry, $\underline{w}$ the M-element price vector for the production factors, and p the competitive product price. Then the profit of firm k can be given as

$$\varphi_k(\underline{l}_1, \ldots, \underline{l}_N) = p f_k(\underline{l}_k) - \underline{l}_k^T \underline{w}(\underline{L}). \tag{1}$$

For the sake of mathematical simplicity we assume that both f_k and $\underline{w}$ are linear:

(A) $f_k(\underline{l}_k) = \underline{c}_k^T \underline{l}_k + \gamma_k$

(B) $\underline{w}(\underline{L}) = \underline{A}\underline{L} + \underline{a}$,

where $\underline{c}_k$ and $\underline{a}$ are M-element vectors, $\underline{A}$ is an $M \times M$ real matrix and γ_k is a scalar. It is also assumed that the feasible set S_k for $\underline{l}_k$ is convex, bounded, and closed in R_+^M.

In Kapolyi and Szidarovszky (2001) it has been shown that if $\underline{A} + \underline{A}^T$ is positive definite, then there is a unique Nash-Cournot equilibrium of the N-person game $G = \{n, S_1, \ldots, S_N, \varphi_1, \ldots, \varphi_N\}$ which is globally asymptotically stable.

If at any time period the game is at an equilibrium, then the interest of all players is to keep this equilibrium. However, if the game is not at the equilibrium, then it is the interest of at least one player to change strategy into its profit maximizing strategy. In this way a dynamic process is evolved.

Notice that under assumption (A) and (B),

$$\varphi_k(\underline{l}_1, \ldots, \underline{l}_N) = p(\underline{c}_k \underline{l}_k + \gamma_k) - \underline{l}_k^T \left(\underline{A} \sum_{i=1}^{N} \underline{l}_i + a \right), \tag{2}$$

so, assuming interior optimum, the first order conditions imply that

$$p\underline{c}_k - (\underline{A} + A^T)\underline{l}_k - \underline{A} \sum_{i \neq k} \underline{l}_i - \underline{a} = 0$$

so the best reply of firm k is given as

$$\underline{l}_k = -(\underline{A} + \underline{A}^T)^{-1}\underline{A}\sum_{i \neq k} \underline{l}_i + (\underline{A} + \underline{A}^T)^{-1}(-\underline{a} + p\underline{c}_k).$$

Therefore the system will evolve as it is driven by the following system of difference equations

$$\underline{l}_k(t+1) = -(\underline{A} + \underline{A}^T)^{-1}\underline{A}\sum_{i \neq k} \underline{l}_i(t) + (\underline{A} + \underline{A}^T)^{-1}(-\underline{a} + p\underline{c}_k). \tag{3}$$

In the next section we will investigate the controllability of this system.

3 Control by Competitive Price

Let us define p, the competitive product price as the control variable. This price can be influenced by incentives, marketing, advertisements, etc, so its choice as the control variable is reasonable. Introduce the following notation:

$$\underline{B} = -(\underline{A} + \underline{A}^T)^{-1}\underline{A}, \quad \underline{d}_k = (\underline{A} + \underline{A}^T)^{-1}\underline{c}_k$$

$$\underline{H} = \begin{pmatrix} \underline{0} & \underline{B} & \cdots & \underline{B} \\ \underline{B} & \underline{0} & \cdots & \underline{B} \\ \vdots & \vdots & & \vdots \\ \underline{B} & \underline{B} & \cdots & \underline{0} \end{pmatrix}, \quad c = \begin{pmatrix} \underline{d}_1 \\ \underline{d}_2 \\ \vdots \\ \underline{d}_N \end{pmatrix}.$$

Then it is well known (see for example, Szidarovszky and Bahill, 1998) that system (3) is controllable with the control variable p if and only if the Kalman matrix

$$\underline{K} = (\underline{c}, \underline{H}\underline{c}, \underline{H}^2\underline{c}, \ldots, \underline{H}^{MN-1}\underline{c})$$

has full rank.

Consider for the sake of simplicity the single production factor case, $M = 1$. Then $\underline{d}_k$ is a scalar for all k, and $\underline{B}$ is a 1×1 matrix, also a scalar.

Assume first that $N = 2$. Then

$$\underline{K} = \begin{pmatrix} d_1 & Bd_2 \\ d_2 & Bd_1 \end{pmatrix}$$

which has full rank if and only if $d_1^2 \neq d_2^2$. Since by economic considerations $A > 0$ and $c_k > 0$ for all k, and d_1 and d_2 are both positive, the system is controllable if and only if $d_1 \neq d_2$, or equivalently $c_1 \neq c_2$. If $N \geq 3$, then we will prove that the system is not controllable. Notice first that

$$\underline{H} = \underline{B}(\underline{1} - \underline{I})$$

where $\underline{I}$ is the $N \times N$ identity matrix, and all elements of $\underline{1}$ are equal to 1. Then

$$\underline{H}^2 = \underline{B}^2((N-2)\underline{1} + \underline{I})$$

which is a linear combination of $\underline{I}$ and $\underline{H}$, and similarly, $\underline{H}^3, \underline{H}^4, \ldots$ are all linear combinations of $\underline{I}$ and $\underline{H}$. Therefore $\underline{H}^2\underline{c}, \underline{H}^3\underline{c}, \ldots$ are linear combinations of $\underline{c}$ and $\underline{H}\underline{c}$, so the rank of $\underline{K}$ is at most two. Hence the system is not controllable.

Next we show that for $M > 1$ we cannot give a general answer. To illustrate the problem assume that $N = 2$, and matrix $\underline{B}$ is diagonal. So let

$$\underline{B} = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \underline{d}_1 = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \quad \underline{d}_2 = \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}.$$

Then

$$\underline{B}^2 = \begin{pmatrix} \alpha^2 & 0 \\ 0 & \beta^2 \end{pmatrix}, \quad \underline{B}^3 = \begin{pmatrix} \alpha^3 & 0 \\ 0 & \beta^3 \end{pmatrix},$$

furthermore

$$\underline{H}\underline{c} = \begin{pmatrix} \underline{0} & \underline{B} \\ \underline{B} & \underline{0} \end{pmatrix} \begin{pmatrix} \underline{d}_1 \\ \underline{d}_2 \end{pmatrix} = \begin{pmatrix} \underline{B}\underline{d}_2 \\ \underline{B}\underline{d}_1 \end{pmatrix},$$

$$\underline{H}^2\underline{c} = \begin{pmatrix} \underline{0} & \underline{B} \\ \underline{B} & \underline{0} \end{pmatrix} \begin{pmatrix} \underline{B}\underline{d}_2 \\ \underline{B}\underline{d}_1 \end{pmatrix} = \begin{pmatrix} \underline{B}^2\underline{d}_1 \\ \underline{B}^2\underline{d}_2 \end{pmatrix},$$

and

$$\underline{H}^3\underline{c} = \begin{pmatrix} \underline{0} & \underline{B} \\ \underline{B} & \underline{0} \end{pmatrix} \begin{pmatrix} \underline{B}^2\underline{d}_1 \\ \underline{B}^2\underline{d}_2 \end{pmatrix} = \begin{pmatrix} \underline{B}^3\underline{d}_2 \\ \underline{B}^3\underline{d}_1 \end{pmatrix}.$$

Therefore the Kalman-matrix has the special form

$$\underline{K} = \begin{pmatrix} d_1 & \alpha d_3 & \alpha^2 d_1 & \alpha^3 d_3 \\ d_2 & \beta d_4 & \beta^2 d_2 & \beta^3 d_4 \\ d_3 & \alpha d_1 & \alpha^2 d_3 & \alpha^3 d_1 \\ d_4 & \beta d_2 & \beta^2 d_4 & \beta^3 d_2 \end{pmatrix}.$$

Subtract the α^2-multiple of the first column from the third column, and the α^2-multiple of the second column from the last column to have matrix

$$\begin{pmatrix} d_1 & \alpha d_3 & 0 & 0 \\ d_2 & \beta d_4 & (\beta^2 - \alpha^2) d_2 & \beta(\beta^2 - \alpha^2) d_4 \\ d_3 & \alpha d_1 & 0 & 0 \\ d_4 & \beta d_2 & (\beta^2 - \alpha^2) d_4 & \beta(\beta^2 - \alpha^2) d_2 \end{pmatrix}$$

with determinant (expanded with respect to its last column)

$$
\begin{aligned}
&-\beta(\beta^2-\alpha^2)d_2(\beta^2-\alpha^2)d_2(\alpha d_1^2-\alpha d_3^2)\\
&+\beta(\beta^2-\alpha^2)d_4(\beta^2-\alpha^2)d_4(\alpha d_1^2-\alpha d_3^2)\\
&=\alpha\beta(\beta^2-\alpha^2)^2(d_1^2-d_3^2)(d_4^2-d_2^2)
\end{aligned}
$$

which is zero if $\alpha=\beta$, or $d_1=d_3$, or $d_2=d_4$, otherwise nonzero. In the first case the system is not controllable, and in the second case it is.

For nondiagonal $\underline{B}$ and $N>2$ matrix $\underline{K}$ is more complicated, and therefore no general conditions can be given.

4 Control by Fixed Labor Cost

Consider again the dynamic system (3) and assume that the fixed price $\underline{a}$ of the production factors is controlled. If all factors are controlled in the same way, then $\underline{a}$ has to be replaced by $a\cdot u(t)$, where $u(t)$ shows the control. If the production factors are controlled differently, then $\underline{a}$ is replaced by diag $(a_1,\ldots,a_M)\underline{u}(t)$, where $\underline{u}(t)$ is an M-dimensional control vector. In these cases the coefficient matrix $\underline{H}$ is the same as in the previous section, however $\underline{d}_k$ $(k=1,2,\ldots,N)$ has to be replaced by either

$$-(\underline{A}+\underline{A}^T)^{-1}\underline{a}\quad\text{or}\quad-(\underline{A}+\underline{A}^T)^{-1}\,\mathrm{diag}(a_1,\ldots,a_M).$$

Notice that for $N\geq 2$ matrix $\underline{K}$ has rank at most M, since its first M rows are identical to the second M rows, which are the same as the third M rows, and so on. Therefore the system is always uncontrollable.

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