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JORDAN NORMAL FORM VIA ODE'S

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Abstract: In text books on differential equations the system of ordinary differential equations with constant coefficients $X' = AX$ is often solved by reduction (by an invertible change of variables $X = PY$) to the simpler system $Y' = JY$ where J is the Jordan canonical form of A. Here we do things the other way around and deduce the existence of J and P by comparing two types of solutions of the system $X' = AX$. The proof provides a straightforward algorithm for calculating the matrices J and P above. Apart from some elementary considerations on (formal) solutions of systems of ODE's with constant coefficients, the main ingredient of the proof (and of the resulting algorithm) is one which comes up in other approaches, namely the reduction of polynomial matrices (in one variable) to diagonal form by row and column operations.

1. Introduction

A well known application of Jordan normal form is to solving the system $X' = AX$. This system of ODE's can also be solved directly ussing the classical method of reduction to diagonal form via row and column operations and the purpose of this note is to show how, conversely, this solution leads, in a natural way, to the existence of the Jordan normal form for the matrix A.

The idea is the following: the derivative DX of any solution X of a system (with constant coeffcients) is also a solution. A diagonal system has a well known basis relative to which the matrix of D is a Jordan matrix J. Reduction of $DX = AX$ to diagonal form takes this basis into a basis for

$DX = AX$ relative to which the matrix of D is also J. But $DX = AX$ has a well known basis relative to which the matrix of D is A. Thus A and J are similar.

We begin by considering a nilpotent matrix A. In this case one need only consider polynomial solutions $X(t)$. In the general (not necessarily nilpotent) case, formal power series must be used instead. But questions of convergence and differentiability of solutions never arise. Thus, despite the appearence of concepts related to differential equations, only elementary algebraic properties (namely linearity and the Leibniz rule) of formal differentiation D are used.

2. Polinomial Solutions of Systems of ODE's

The derivative $D\phi$ of the polynomial $\phi(t)$ is defined formally (without any reference to limits) by the usual formula. If $F(x)$ is a polynomial matrix then, setting x equal to D, we get a matrix operator $F(D)$ all of whose entries are polynomials $f(D)$ in D. For example, if I is the identity matrix then xI corresponds to the differentiation operator (also denoted by D) which has D's down the diagonal and zeros elsewhere. Clearly, for any $F(x)$, one has

$$F(D)D = DF(D).$$

Therefore, if $F(D)X = 0$ then $F(D)DX = 0$ as well.

A basis (alias fundamental matrix) for $F(D)$ is a polynomial matrix $\Phi(t)$ such that any polynomial solution $X(t)$ of $F(D)X = 0$ is a unique linear combination (with real coefficients) of the columns of Φ (in other words, the columns of Φ are a basis for the space of solutions X of the system $F(D)X = 0$). Since each column of $D\Phi$ is the derivative of a solution it is a linear combination of the columns of Φ. Thus there is a (unique) constant matrix M (say) such that

$$D\Phi = \Phi M.$$

In other words, M is the matrix of D relative to the basis Φ. Similarly, if Φ and Ψ are two bases for $F(D)$ then there is a constant invertible matrix P such that

$$\Psi = \Phi P.$$

Applying D, we have

$$D\Psi = D(\Phi P) = D(\Phi)P = \Phi MP = \Psi P^{-1}MP,$$

so that the matrices of D relative to any two bases for $F(D)$ are similar. There are two cases where bases are well known. First, if A is any nilpotent constant matrix then

$$\Theta = I + tA + \frac{1}{2}t^2A^2 + ...$$

is a polynomial matrix which is easily seen to be a basis for $D - A$. Clearly

$$D\Theta = \Theta A.$$

Secondly, if a diagonal matrix $\mathcal{D}(D)$ has a basis then none of its diagonal entries can be zero, for otherwise there would be infinitely many linearly independent solutions. Conversely, let $f(D)$ be a diagonal entry of $\mathcal{D}(D)$ of the form D^k+ higher powers of D where $k > 0$. Then the row vector

$$\left(1\ t\ ...\ \frac{t^{k-1}}{(k-1)!}\right)$$

is a basis for $f(D)$. Such bases may be assembled into a basis Δ for $\mathcal{D}(D)$. Clearly

$$D\Delta = \Delta J$$

where J is direct sum of Jordan blocks, each consisting of ones just above the diagonal and zeros elsewhere.

In general, a basis for $F(D)$ may be obtained by reducing F to diagonal form by row and column operations. This method, which is based on long division of polynomials, is explained in various textbooks, for example [Jordan] (Vol.3, section 141) and goes as follows.

Let g be a nonzero entry of F. By exchanging rows of F and then columns of F we may assume that g is in the upper left hand corner of F (i.e. in the first row and column). If g divides every entry in its row and column then, by row and column operations, all the entries in the first row and the first column of F except g may be reduced to zero. In other words F may be reduced to $g \oplus G$ (say). Otherwise, F is reduced to a matrix with a nonzero entry of lower degree and the procedure is then repeated. Since the degree cannot be reduced beyond zero, F will eventually be reduced to $h \oplus H$ (say). Next, H is a similarly reduced and so on. In the end, a diagonal matrix $\mathcal{D}$ is obtained. Thus there are polynomial matrices $U(x)$ and $V(x)$ (with polynomial inverses) such that

$$UFV = D$$

Since U and V are invertible, Φ is a basis for $\mathcal{D}(D)$ if and only if $V(D)\Phi$ is a basis for $F(D)$.

3. Jordan Normal Form for Nilpotent Matrices

Let A be a nilpotent matrix. By reducing $xI - A$ to diagonal form, one can calculate explicitly U, V and $\mathcal{D}$ such that

$$U(D)(D - A)V(D) = \mathcal{D}(D)$$

Now $D - A$ has a basis, namely Θ. So $\mathcal{D}(D)$ also has a basis. Hence Δ, as defined above, is a basis for $\mathcal{D}(D)$. Thus $V(D)\Delta$ is a basis for $D - A$. Therefore

$$V(D)\Delta = \Theta P$$

where P is an invertible constant matrix. Applying D, we have

$$\Theta AP = D\,(\Theta P) = DV\,(D)\,\Delta = V\,(D)\,D\Delta = (V(D)\Delta)\,J = \Theta PJ.$$

Since Θ is a basis, this implies that $AP = PJ$, as required.

If needed, the matrix P may be easily calculated, because it is the constant term of ΘP and hence also of $V(D)\Delta$.

4. The General Case

Here we switch from polynomial solutions to formal power series solutions with D defined formally by the usual formula. Many textbooks (for example Jordan's Cours d'analyse) prove the basic result (due to Euler) [Euler] that the polynomial $f(D)$ has a basis consisting of $\frac{1}{j!}t^j e^{at}$ for each factor $(x-a)^k$ of f and $\frac{1}{j!}t^j e^{at}\sin bt$ and $\frac{1}{j!}t^j e^{at}\cos bt$ for each factor $\left((x-a)^2+b^2\right)^k$ where $0 \leqslant j < k$. For diagonal $\mathcal{D}$ such bases may be assembled in the obvious way into a basis Δ for $\mathcal{D}(D)$. Again, $D\Delta = \Delta J$ where J is a Jordan matrix. The rest of the argument goes through as in the nilpotent case.

References

[1] **C. Jordan,** *Cours d'analyse*, Gauthier.Villars.

[2] **L. Euler,** *De integratione aequatiorum differentialium altiorum graduum*, Opera Omnia Vol. XXII p 108-149.