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Super-Halley method under majorant conditions in Banach spaces

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ABSTRACT

In this paper, we have studied local convergence of Super-Halley method in Banach spaces under the assumption of second order majorant conditions. This approach allows us to obtain generalization of earlier convergence analysis under majorizing sequences. Two important special cases of the convergence analysis based on the premises of Kantorovich and Smale type conditions have also been concluded. To show efficacy of our approach we have given three numerical examples.

RESUMEN

En este artículo, hemos estudiado la convergencia local del método Super-Halley en espacios de Banach, asumiendo condiciones mayorantes de segundo orden. Este punto de vista nos permite obtener generalizaciones de análisis de convergencia bajo sucesiones mayorantes obtenidos anteriormente. También se han concluido dos casos especiales del análisis de convergencia basados en las premisas de condiciones tipo Kantorovich y Smale. Para mostrar la eficacia de nuestro enfoque, damos tres ejemplos numéricos.

Keywords and Phrases: Nonlinear equations; Super-Halley method; Majorant conditions; Local Convergence; Semilocal Convergnce; Smale-type conditions; Kantorovich-type conditions.

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1 Introduction

Let f be a given operator that maps from some nonempty open convex subset Ω of a Banach space $\mathbb{X}$ to another Banach space $\mathbb{Y}$. Approximating a locally unique solution $\bar{x}$ of a nonlinear equation

$$f(x) = 0 \tag{1.1}$$

is widely studied in both theoretical and applied areas of mathematics. Generally, this is done by using some iterative processes. An iterative process is a mathematical procedure that, from one or several initial approximations of a solution of (1.1), a sequence of iterates $\{x_n\}_{n\in\mathbb{N}}$ is constructed so that each subsequent iterate of the sequence is a better approximation to the previous approximation to the solution of (1.1); that is, the sequence $\{\|x_n - \bar{x}\|\}_{n\in\mathbb{N}}$ is convergent to zero. Usually, in order to study convergence analysis of the method, we could consider the study of local and semilocal convergence analysis. If the convergence analysis seeks assumptions around a solution $\bar{x}$, then it is called local convergence and this type of analysis estimates the radii of convergence balls, where as we could also consider assumptions around an initial point x_0 to study convergence analysis of the method. In that case the convergence analysis is called semilocal one and it gives criteria ensuring the convergence. It is also very important to give convergence ball of an iterative method, because that shows the extent to which we can choose an initial guesses for that method.

One of the most important iterative methods to solve this problem is Newton's method given by

$$x_{n+1} = x_n - f'(x_n)^{-1} f(x_n), \quad k = 0, 1, 2, \ldots \tag{1.2}$$

where $x_0 \in \Omega$ is an initial point. As anybody can recall that one of the most famous results to study convergence of Newton's method (1.2) is the well known Kantorovich method[16], which guarantees convergence of the method to a solution, using semilocal conditions. It does not require a priori existence of a solution, proving instead the existence of the solution and its uniqueness on some region. Many researches have also done works related to Kantorovich-like method (for details see [4, 8, 10, 26, 27, 29] and references there in). Also, Smale's point theory [21] assumes that the nonlinear operator is analytic at the initial point, which is an important result concerning Newton's method. Wan and Han[25, 22] has discussed the generalization and the particular cases of Smale's point estimate theory.

For a positive number α and $x \in \mathbb{X}$, we consider $B(x, \alpha)$ to stand for the open ball with radius α and center x and $\bar{B}(x, \alpha)$ is the corresponding close ball. In [7, 6], Ferreira and Svaiter had studied the local convergence of Newton's method (1.2) under the following majorant conditions:

$$\|f'(\bar{x})^{-1}[f'(y) - f'(x)]\| \le h'(\|y - x\| + \|x - \bar{x}\|) - h'(\|x - \bar{x}\|), \tag{1.3}$$

for $x, y \in B(\bar{x}, R)$, $R > 0$, where $\|y - x\| + \|x - \bar{x}\| < R$ and $h : (0, R) \to \mathbb{R}$ is a continuously differentiable, convex and strictly increasing function that satisfies $h(0) > 0$, $h'(0) = -1$ and has a zero in

$(0, R)$. Note that by this study the assumptions for guaranteeing Q-quadratic convergence of the respective iterative methods has been relaxed and a new estimate of the Q-quadratic convergence has been obtained.

Recently, inspired by these ideas, Ling and Xu[17] have presented a new convergence analysis of Halley's method which makes a relationship of the majorizing function h and the nonlinear operator f under majorant conditions similar to given in (1.3). Argyros and Ren[1] also presented a local convergence of Halley's method which gives a ball convergence of the method under assumptions similar to (1.3).

On the other hand, one of the famous third order iterative method to solve nonlinear equation (1.1) in Banach space is the Super-Halley method denoted by

$$x_{n+1} = x_n - \left[I + \frac{1}{2}L_f(x_n)[I - L_f(x_n)]^{-1}\right]f'(x_n)^{-1}f(x_n), \tag{1.4}$$

where for $x \in \mathbb{X}$, $L_f(x)$ is the linear operator defined as

$$L_f(x) = f'(x)^{-1}f''(x)f'(x)^{-1}f(x).$$

The results concerning the convergence of this method have been studied in [2, 9, 20] under different types of assumptions by using recurrence relations. On the other hand Ezquerro and Hernández[5] and Gutiérrez and Hernández[11] have studied semilocal convergence of this method by using majorizing sequences. Now, if the nonlinear operator f is analytic at the initial point then motivated by the ideas of Argyros and Ren[1] and Ling and Xu[17], we have studied local convergence of Super-Halley method (1.4) using second order majorant condition. This majorant condition generalizes the earlier results on Super-Halley method[5, 11] using majorizing sequences. Two particular cases namely results based of affine invariant Lipschitz-type condition and Smale-type condition have also been derived. Numerical efficacy of the method has also been derived by way of a number of numerical examples.

Rest of the paper is organized as follows. Some preliminaries results are contained in section 2. In section 2.1, we studied local convergence analysis of Super-Halley method. Two special cases of main result are presented in section 3. In section 4, we have shown a number of numerical examples to show efficacy of our study. Section 5 forms the conclusion part of the paper.

2 Preliminaries

In this section we provide some basic results which is required for our convergence analysis of the method.

Assume $a > 0$ and $\phi : (0, a) \to \mathbb{R}$ be a twice continuously differentiable function. Let $x, y \in B(\bar{x}, a) \subset \Omega$, with $\|y - x\| + \|x - \bar{x}\| < a$. We say that the operator f satisfy a second order majorizing function ϕ at $\bar{x}$ if the following conditions hold on f:

$$\|f'(\bar{x})^{-1}[f''(y) - f''(x)]\| \leq \phi''(\|y-x\| + \|x-\bar{x}\|) - \phi''(\|x-\bar{x}\|), \tag{2.1}$$

with the assumptions:

$$\left.\begin{array}{l} (M1)\ \phi(0) > 0, \phi''(0) > 0, \phi'(0) = -1, \\ (M2)\ \phi'' \text{ is convex and strictly increasing in } (0,a), \\ (M3)\ \phi \text{ has atleast one zero in } (0,a) \text{ with } t^* \text{ as the smallest zero and } \phi'(t^*) < 0. \end{array}\right\} \tag{2.2}$$

and

$$\|f'(\bar{x})^{-1}f(\bar{x})\| \leq \phi(0), \quad \|f'(\bar{x})^{-1}f''(\bar{x})\| \leq \phi''(0). \tag{2.3}$$

In this paper we assume that ϕ is the majorizing function of f. Note that if we define

$$\Theta_f(x) := x - \left[I + \frac{1}{2}L_f(x)[I - L_f(x)]^{-1}\right] f'(x)^{-1}f(x) \tag{2.4}$$

where $L_f(x) = f'(x)^{-1}f''(x)f'(x)^{-1}f(x)$, then $\Theta_f(x)$ can be taken as the iterative function of Super-Halley method as it can be written as $x_{n+1} = \Theta_f(x_n)$. Also the scalar sequence $\{t_n\}$ can be generated by applying the method to $\phi(t)$. In this case we can write $t_{n+1} = \Theta_\phi(t_n)$ with

$$\Theta_\phi(t) := t - \left[1 + \frac{L_\phi(t)}{2(1-L_\phi(t))}\right]\frac{\phi(t)}{\phi'(t)}, \quad L_\phi(t) = \frac{\phi(t)\phi''(t)}{\phi'(t)^2}, \quad t \in (0,a). \tag{2.5}$$

Now we can easily establish some basic properties of the majorizing function ϕ, the iterative functions $\Theta_f(x)$ and $\Theta_\phi(t)$ which are described in the following lemmas.

Lemma 2.1. *Let* ϕ *satisfies assumptions* $(M1)-(M3)$*. Then*

(i) ϕ' *is strictly convex and strictly increasing on* $(0,a)$.

(ii) ϕ *is strictly convex on* $(0,a)$, $\phi(t) > 0$ *for* $t \in (0,t^*)$ *and equation* $\phi(t) = 0$ *has at most one root in* (t^*, a).

(iii) $-1 < \phi'(t) < 0$ *for* $t \in (0,t^*)$.

(iv) $0 \leq L_\phi(t) \leq \frac{1}{2}$ *for* $t \in [0,t^*]$.

Proof. The proof is similar to one given in [17], so omitted.

Lemma 2.2. *Let* ϕ *satisfies assumptions* $(M1)-(M3)$*. Then for all* $t \in (0,t^*)$, $t < \Theta_\phi(t) < t^*$. *Moreover,* $\phi'(t^*) < 0$ *if and only if there exist* $t \in (t^*, a)$ *such that* $\phi(t) < 0$.

Proof. It is not to be mentioned that by using Lemma 2.1, one can have $\phi(t) > 0$, $-1 < \phi'(t) < 0$ and $0 \leq L_\phi(t) \leq \frac{1}{2}$ for $t \in (0, t^*)$. This gives $\left[1 + \dfrac{L_\phi(t)}{2(1 - L_\phi(t))}\right] \dfrac{\phi(t)}{\phi'(t)} < 0$ and hence $t < \Theta_\phi(t)$.

Also, for any $t \in (0, t^*)$, from the definition of directional derivative and assumption (M2) it follows that since $\phi''(t)$ is increasing in $(0, a)$ and $t < t^* < a$, we have $\phi''(t) < \phi''(t^*)$ and $\phi''(t) > 0$ which implies that left directional derivative $D^-\phi''(t) > 0$.

As $\phi''(t)\phi(t) - 2\phi'(t)^2 \leq -4\phi''(t)\phi(t)$, we obtain $D^-\Theta_\phi(t) = 1 + \dfrac{2\phi'(t)\phi(t)D^-\phi''(t)}{(\phi''(t)\phi(t) - 2\phi'(t)^2)\phi''(t)} > 0$ for $t \in (0, t^*)$.

And this implies that $\Theta_\phi(t) < \Theta_\phi(t^*) = t^*$ for any $t \in (0, t^*)$. So the first part of this lemma is complete. Now, if $\phi'(t^*) < 0$, then it is obvious that there exists $t \in (t^*, a)$ such that $\phi(t) < 0$. Conversely, noting that $\phi'(t^*) = 0$, then we have $\phi(t) > \phi(t^*) + \phi'(t)(t^* - t)$ for $t \in (t^*, a)$, which implies that $\phi'(t^*) < 0$. This completes the proof.

Remark Following properties are implied by the condition $\phi'(t^*) < 0$ in (M3):

- $\phi(t^{**}) = 0$ for some $t^{**} \in (t^*, a)$.
- $\phi(t) < 0$ for some $t \in (t^*, a)$.

Lemma 2.3. *Let* ϕ *satisfies assumptions* $(M1) - (M3)$. *Then*

$$t^* - \Theta_\phi(t) \leq \left[\frac{1}{2}\frac{\phi''(t^*)^2}{\phi'(t^*)^2} + \frac{1}{3}\frac{D^-\phi''(t^*)}{-\phi'(t^*)}\right](t^* - t)^3, t \in (0, t^*). \tag{2.6}$$

Proof. We can derive the following relation, by using the definition of Θ_ϕ in (2.5)

$$\begin{aligned} t^* - \Theta_\phi(t) &= \frac{1}{1 - L_\phi(t)}\left[(1 - L_\phi(t))(t^* - t) + \frac{\phi(t)}{2\phi'(t)}(1 - L_\phi(t)) + \frac{\phi(t)}{2\phi'(t)}\right] \\ &= -\frac{1}{\phi'(t)(1 - L_\phi(t)}\int_0^1 [\phi''(t + \sigma(t^* - t)) - \phi''(t)](t^* - t)^2(1 - \sigma)d\sigma \\ &\quad + \frac{(t^* - t)\phi''(t)}{2(1 - L_\phi(t))\phi'(t)^2}\int_0^1 \phi''(t + \sigma(t^* - t))(t^* - t)^2(1 - \sigma)d\sigma \\ &\quad - \frac{(t^* - t)\phi''(t)}{2(1 - L_\phi(t))\phi'(t)}\left(\phi(t) + \frac{\phi(t)^2}{\phi'(t)(t^* - t)}\right) \end{aligned}$$

Since ϕ'' is convex and $t < t^*$, it follows from Lemma 2.1 that

$$\phi''(t + \sigma(t^* - t)) - \phi''(t) \leq [\phi''(t^*) - \phi''(t)]\frac{\sigma(t^* - t)}{(t^* - t)}$$

So by noting that ϕ'' is strictly increasing, we have

$$\begin{aligned} t^* - \Theta_\phi(t) \quad &\leq \quad -\frac{\phi''(t^*) - \phi''(t)}{6\phi'(t)(1 - L_\phi(t))}(t^* - t)^2 + \frac{\phi''(t^*)\phi''(t)}{4\phi'(t)^2(1 - L_\phi(t))}(t^* - t)^3 \\ &\quad -\frac{\phi(t)\phi''(t)}{2(1 - L_\phi(t))\phi'(t)}(t^* - t) - \frac{\phi(t)^2\phi''(t)}{2(1 - L_\phi(t))\phi'(t)^2} \end{aligned}$$

Since $\phi'(t) < 0$, $\phi''(0) > 0$ and ϕ', ϕ'' are strictly increasing on $(0, t^*)$ and $0 \leq L_\phi(t) \leq \frac{1}{2}$ for $t \in [0, t^*]$ by Lemma 2.1, so we have

$$t^* - \Theta_\phi(t) \leq \left[\frac{1}{2}\frac{\phi''(t^*)^2}{\phi'(t^*)^2} + \frac{1}{3}\frac{D^-\phi''(t^*)}{-\phi'(t^*)}\right](t^* - t)^3. \tag{2.7}$$

As ϕ' is increasing, $\phi'(t^*) < 0$ and $\phi'(t) < 0$ t in $(0, t^*)$, we have

$$\frac{\phi''(t^*)) - \phi''(t)}{-\phi'(t)} \leq \frac{\phi''(t^*) - \phi''(t)}{-\phi'(t^*)} = \frac{1}{-\phi'(t^*)}\frac{\phi''(t^*)) - \phi''(t)}{t^* - t}(t^* - t) \leq \frac{D^-\phi''(t^*)}{-\phi'(t^*)}(t^* - t)$$

where the last inequality follows from definitions of directional derivative. Combining the above inequality with (2.7), we conclude that (2.6) holds. This completes the proof. □

Let $\{t_k\}$ denote the majorizing sequence generated by

$$t_0 = 0, t_{k+1} = \Theta_\phi(t_k) = t_k - \left[I + \frac{L_\phi(t_k)}{2(1 - L_\phi(t_k))}\right]\frac{\phi(t_k)}{\phi'(t_k)}, \quad k = 0, 1, 2, \ldots \tag{2.8}$$

We arrive at the following theorem using Lemma 2.3 that

Theorem 2.4. *Let the sequence* $\{t_k\}$ *be defined by* (2.8). *Then* $\{t_k\}$ *is well defined, strictly increasing and is contained in* $(0, t^*)$. *Moreover,* $\{t_k\}$ *satisfies (2.6) and converges to* t^* *with* $Q - \mathsf{cubic}$, *i.e.:*

$$t^* - t_{k+1} \leq \left[\frac{1}{2}\frac{\phi''(t^*)^2}{\phi'(t^*)^2} + \frac{1}{3}\frac{D^-\phi''(t^*)}{-\phi'(t^*)}\right](t^* - t_k)^3, \quad t_k \in (0, t^*).$$

2.1 Local convergence results for Super-Halley method

This section is devoted to giving the local convergence analysis of (1.4). For that the following lemmas will play important role.

Lemma 2.5. *Assume* $\|x - \bar{x}\| \leq t < t^*$. *If* $\phi : (0, t^*) \to \mathbb{R}$ *is a twice continuously differentiable function which majorizes* f *at* $\bar{x}$, *then*

(i) $f'(x)$ *is nonsingular and*

$$\|f'(x)^{-1}f'(\bar{x})\| \leq -\frac{1}{\phi'(\|x - \bar{x}\|)} \leq -\frac{1}{\phi'(t)}. \tag{2.9}$$

(ii) $\|f'(\bar{x})^{-1}f''(x)\| \leq \phi''(\|x-\bar{x}\|) \leq \phi''(t)$.

Proof. Let us take $x \in B(\bar{x}, t)$, $0 \leq t < t^*$. Since

$$f'(x) = f'(\bar{x}) + \int_0^1 [f''(\bar{x}+\sigma(x-\bar{x})) - f''(\bar{x})](x-\bar{x})d\sigma + f''(\bar{x})(x-\bar{x}),$$

we get

$$\begin{aligned}
\|I - f'(\bar{x})^{-1}f'(x)\| &\leq \int_0^1 \|f'(\bar{x})^{-1}[f''(\bar{x}+\sigma(x-\bar{x})) - f''(\bar{x})]\|\|x-\bar{x}\|d\sigma \\
&\quad + \|f'(\bar{x})^{-1}f''(\bar{x})\|\|x-\bar{x}\| \\
&\leq \int_0^1 [\phi''(\sigma(\|(x-\bar{x})\|)) - \phi''(0)]\|x-\bar{x}\|d\sigma + \phi''(0)\|x-\bar{x}\| \\
&= \phi'(\|(x-\bar{x})\|) - \phi'(0).
\end{aligned}$$

So, we conclude that

$$\|I - f'(\bar{x})^{-1}f'(x)\| \leq \phi'(t) - \phi'(0) < 1$$

as $\phi'(0) = -1$ and $-1 < \phi'(t) < 0$ for $(0, t^*)$ using Lemma 2.1. Therefore, it follows from Banach lemma that $f'(\bar{x})^{-1}f'(x)$ is nonsingular and (2.9) holds as

$$\|f'(x)^{-1}f'(\bar{x})\| \leq \frac{1}{1-\phi'(\|x-\bar{x}\|)+\phi'(0)} = -\frac{1}{\phi'(\|x-\bar{x}\|)} \leq -\frac{1}{\phi'(t)}.$$

Thus we conclude that, f' is nonsingular in $B(\bar{x}, t^*)$. By using majorant conditions, we have

$$\begin{aligned}
\|f'(\bar{x})^{-1}f''(x)\| &\leq \|f'(\bar{x})^{-1}[f''(x) - f''(\bar{x})]\| + \|f'(\bar{x})^{-1}f''(\bar{x})\| \\
&\leq \phi''(\|x-\bar{x}\|) - \phi''(0) + \phi''(0) = \phi''(\|x-\bar{x}\|) \leq \phi''(t).
\end{aligned}$$

The last inequality holds true because of ϕ'' is strictly increasing. This completes the proof. □

Now the main local convergence result for the Super-Halley method (1.4) is presented as follows.

Theorem 2.6. *Let* f *satisfies the second order majorant conditions* (2.1)-(2.3). *Then, the sequence of iterates* $\{x_n\}$ *generated by Super-Halley method (1.4) is well defined, contained in* $B(\bar{x}, t^*)$ *and converges to the unique solution* $\bar{x}$ *of (1.1). Moreover, the following error estimate hold*

$$\|\bar{x} - x_{k+1}\| \leq (t^* - t_{k+1})\left(\frac{\|\bar{x} - x_k\|}{t^* - t_k}\right)^3, \quad k = 0, 1, 2, \ldots \tag{2.10}$$

Thus the sequence $\{x_n\}$ *generated by Super-Halley method (1.4) converges* $Q-$*cubic as follows*

$$\|\bar{x} - x_{k+1}\| \leq \left[\frac{1}{2}\frac{h''(t^*)^2}{h'(t^*)^2} + \frac{1}{3}\frac{D^-h''(t^*)}{-h'(t^*)}\right]\|\bar{x} - x_k\|^3, \quad k = 0, 1, 2, \ldots \tag{2.11}$$

Proof. By using $f(\bar{x}) = 0$ and some standard analytic techniques, one can have

$$\begin{aligned}
\bar{x} - x_{k+1} &= -\Gamma_f(x_k)f'(x_k)^{-1}[-f'(x_k)(\bar{x} - x_k) - f(x_k)] - \Gamma_f(x_k)L_f(x_k)(\bar{x} - x_k) \\
&\quad + \Big[\frac{1}{2}f'(x_k)^{-1}f(x_k) - \frac{1}{2}\Gamma_f(x_k)f'(x_k)^{-1}f(x_k)\Big] \\
&= -\Gamma_f(x_k)f'(x_k)^{-1}\int_0^1 (1-\sigma)[f''(x_k + \sigma(\bar{x} - x_k)) - f''(x_k)](\bar{x} - x_k)^2 d\sigma \\
&\quad + f''(x_k)\Gamma_f(x_k)f'(x_k)^{-1}\Big[f'(x_k)^{-1}\int_0^1 (1-\sigma)f''(x_k + \sigma(\bar{x} - x_k)) \\
&\quad \times (\bar{x} - x_k)^2 d\sigma\Big](\bar{x} - x_k) \\
&\quad - \frac{1}{2}f''(x_k)\Gamma_f(x_k)f'(x_k)^{-1}\Big[f'(x_k)^{-1}\int_0^1 (1-\sigma)f''(x_k + \sigma(\bar{x} - x_k)) \\
&\quad \times (\bar{x} - x_k)^2 d\sigma\Big]^2,
\end{aligned}$$

where $\Gamma_f(x) = (I - L_f(x))^{-1}$. Using majorant condition, we can get

$$\begin{aligned}
\int_0^1 \|f'(\bar{x})^{-1}[f''(x_k + \sigma(\bar{x} - x_k)) - f''(x_k)]\|(1-\sigma)d\sigma &\leq \int_0^1 \|[\phi''(\sigma\|\bar{x} - x_k\| + \|x_k - \bar{x}\|) \\
&\quad - \phi''(\|x_k - \bar{x}\|)]\|(1-\sigma)d\sigma.
\end{aligned}$$

Also, we know that, if $u, v, w \in (0, a)$ and $u < v < w$, then because of convexity[7] of $\phi(x)$ in $(0, a)$, we have

$$\phi(v) - \phi(u) \leq [\phi(w) - \phi(u)]\frac{v - u}{w - u}.$$

Therefore,

$$\begin{aligned}
\phi''(\sigma\|\bar{x} - x_k\| + \|x_k - \bar{x}\|) - \phi''(\|x_k - \bar{x}\|) &\leq \phi''(\sigma\|\bar{x} - x_k\| + t_k) - \phi''(t_k) \\
&\leq [\phi''(\sigma(t^* - t_k) + t_k) - \phi''(t_k)]\frac{\|\bar{x} - x_k\|}{t^* - t_k}.
\end{aligned}$$

Thus Lemma 2.5 and above inequality implies

$$\begin{aligned}
\|\bar{x} - x_{k+1}\| &\leq -\frac{1}{(1 - L_\phi(t_k))\phi'(t_k)}\Big[\int_0^1 [\phi''(\sigma(t^* - t_k) + t_k) - \phi''(t_k)](1-\sigma)d\sigma\Big]\frac{\|\bar{x} - x_k\|^3}{t^* - t_k} \\
&\quad + \frac{\phi''(t_k)}{(1 - L_\phi(t_k))\phi'(t_k)^2}\Big[\int_0^1 [\phi''(\sigma(t^* - t_k) + t_k)(1-\sigma)d\sigma\Big]\|\bar{x} - x_k\|^3 \\
&\quad + \frac{1}{2}\frac{\phi''(t_k)}{(1 - L_\phi(t_k))\phi'(t_k)^2}\Big[\int_0^1 [\phi''(\sigma(t^* - t_k) + t_k)(1-\sigma)d\sigma\Big]^2\|\bar{x} - x_k\|^4 \\
&\leq \frac{\phi(t_k)}{(1 - L_\phi(t_k))\phi'(t_k)}\Big(\frac{\|\bar{x} - x_k\|}{t^* - t_k}\Big)^3 = (t^* - t_{k+1})\Big(\frac{\|\bar{x} - x_k\|}{t^* - t_k}\Big)^3.
\end{aligned}$$

Finally, we want to show that the solution $\bar{x}$ of (1.1) is unique in $\bar{B}(\bar{x}, t^*)$. For that assume $\bar{y}$ be another solution in $\bar{B}(\bar{x}, t^*)$. Then proceeding similarly as above we get

$$\|\bar{y} - x_{k+1}\| \leq (t^* - t_{k+1})\Big(\frac{\|\bar{y} - x_k\|}{t^* - t_k}\Big)^3.$$

Since the sequence $\{x_k\}$ converges to $\bar{x}$ and $\{t_k\}$ converges to t^*, we conclude that $\bar{y} = \bar{x}$. Therefore, $\bar{x}$ is the unique zero of (1.1) in $\bar{B}(\bar{x}, t^*)$. □

Remark 2.7. *It is to be noted that if we replace $\bar{x}$ with the initial approximation x_0 in (2.1), then after some manipulations we can obtain a semilocal convergence analysis of our iteration method. This analysis approach enables us to drop out the assumption of existence of a second root for the majorizing function, still guarantee $Q-$cubic convergence rate. Thus the semilocal convergence theorem for the iteration method (1.4) is as follows:*

Theorem 2.8. *Suppose $f : \Omega \subseteq \mathbb{X} \to \mathbb{Y}$ be a twice continuously differentiable nonlinear operator, and Ω is open and convex. Consider that for a given initial guess $x_0 \in \Omega$, $f'(x_0)$ is nonsingular that is $f'(x_0)^{-1}$ exists and that ϕ is the majorizing function to f at x_0 and ϕ satisfies the assumptions $(M1)--(M3)$. Then sequence $\{x_k\}$ generated by the method (1.4) for solving equation (1.1) with a starting point x_0 is well defined, contained in $B(x_0, t^*)$ and converges to a solution $\bar{x} \in \bar{B}(x_0, t^*)$ of the Eq.(1.1). The solution is unique in $B(x_0, \sigma)$, where σ is defined as $\sigma := \sup\{t \in (t^*, R) : \phi(t) \leq 0\}$. For $k = 0, 1, 2, \ldots$, a priori error estimate and a posteriori error estimate are given respectively as*

$$\|\bar{x} - x_{k+1}\| \leq (t^* - t_{k+1})\Big(\frac{\|\bar{x} - x_k\|}{t^* - t_k}\Big)^3,$$

and

$$\|\bar{x} - x_{k+1}\| \leq (t^* - t_{k+1})\Big(\frac{\|x_{k+1} - x_k\|}{t_{k+1} - t_k}\Big)^3.$$

Also the method converges $Q-$cubically as

$$\|\bar{x} - x_{k+1}\| \leq \Big[\frac{1}{2}\frac{\phi''(t^*)^2}{\phi'(t^*)^2} + \frac{1}{3}\frac{D^-\phi''(t^*)}{-\phi'(t^*)}\Big](\|\bar{x} - x_k\|)^3.$$

3 Special cases and applications

This section consists of two special cases of the local convergence results obtained in previous section. Namely, convergence results under affine covariant Kantorovich-type condition and the Smale-type γ-condition.

3.1 Kantorovich-type

Suppose that f satisfies the affine covariant Lipschitz condition (see Han and Wang[12]) as given by:

$$\|f'(\bar{x})^{-1}[f''(y) - f''(x)]\| \leq \lambda_1\|y - x\|, \quad x, y \in \Omega. \tag{3.1}$$

and the following initial conditions

$$\|f'(\bar{x})^{-1}f(\bar{x})\| \leq \beta, \tag{3.2}$$

$$\|f'(\bar{x})^{-1}f''(\bar{x})\| \leq \lambda_2. \tag{3.3}$$

Consider the scalar valued function

$$\phi(t) = \frac{\lambda_1}{6}t^3 + \frac{\lambda_2}{2}t^2 - t + \beta. \tag{3.4}$$

This function was considered as majorizing function in [28, 5, 11] for establishing convergence of super-Halley method. If we choose the above cubic polynomial as the majorizing function ϕ in (2.1), then the majorant condition (2.1) reduced to the Lipschitz condition (3.1) and in this way the results based on Lipschitz condition have been generalized by our assumptions of majorant conditions. The assumptions (M1) and (M2) are satisfied for f if the following criterion holds

$$\beta \leq \frac{2(\lambda_2 + 2(\lambda_2^2 + 2\lambda_1)^{1/2})}{3(\lambda_2 + 2(\lambda_2^2 + 2\lambda_1)^{1/2})^2}. \tag{3.5}$$

Therefore, Theorem 2.6 reduces to the following form:

Theorem 3.1. *Suppose that f satisfies the conditions (3.1)-(3.3) with the assumptions given in (3.5). Then, the sequence $\{x_k\}$ generated by Super-Halley method (1.4) for solving equation (1.1) with a starting point x_0 is well defined, contained in $B(\bar{x}, t^*)$ and converges to a solution $\bar{x} \in \bar{B}(\bar{x}, t^*)$ of the Eq.(1.1). Note that t^* is the smallest positive root of ϕ defined by (3.4) in $[0, r_1]$ where $r_1 = (-\lambda_2 + (\lambda_2^2 + 2\lambda_1)^{1/2})/\lambda_1$ is the positive root of ϕ'. The limit $\bar{x}$ of the sequence $\{x_k\}$ is the unique zero of Eq.(1.1) in $B(\bar{x}, t^{**})$, where t^{**} is the root of ϕ in the interval $(r_1, +\infty)$. Moreover, the following error estimates holds*

$$\|\bar{x} - x_{k+1}\| \leq (t^* - t_{k+1})\Big(\frac{\|\bar{x} - x_k\|}{t^* - t_k}\Big)^3, \ k = 0, 1, 2, \ldots$$

and the sequence generated by Super-Halley method (1.4) converges Q-cubic as follows

$$\|\bar{x} - x_{k+1}\| \leq \Big[\frac{3(\lambda_1 + \lambda_2 t^*)^2 + 2\lambda_2(1 - \lambda_1 t^* - \lambda_2 t^{*2}/2)}{6(1 - \lambda_1 t^* - \lambda_2 t^{*2}/2)^2}\Big](\|\bar{x} - x_k\|)^3, \ k = 0, 1, 2, \ldots$$

3.2 Smale-type

This subsection contains the local convergence results for the Super-Halley method (1.4) under the γ-Condition.

In [21], Smale has studied the convergence and error estimation of Newton's method under the hypotheses that f is analytic and satisfies

$$\|f'(\bar{x})^{-1}f^{(n)}(\bar{x})\| \leq n!\gamma^{n-1}, \qquad n > 2$$

where

$$\gamma := \sup_{k>1} \| \frac{f'(\bar{x})^{-1} f^{(n)}(\bar{x})}{k!} \|^{\frac{1}{k-1}}$$

Smale's result is completely improved by Wang and Han[24, 25] by introducing a majorizing function

$$\phi(t) = \beta - t + \frac{\gamma t^2}{1-\gamma t}, \qquad t \in \left[0, \frac{1}{\gamma}\right) \tag{3.6}$$

where

$$\|f'(\bar{x})^{-1} f(\bar{x})\| \le \beta \tag{3.7}$$

and

$$\|f'(\bar{x})^{-1} f''(\bar{x})\| \le 2\gamma \tag{3.8}$$

If we choose this function as the majorizing function ϕ in (2.1), then it reduces to the following condition:

$$\begin{aligned} \|f'(\bar{x})^{-1}[f''(y) - f''(x)]\| \quad &\le \quad \frac{2\gamma}{(1-\gamma\|y-x\| - \gamma\|x-\bar{x}\|)^3} \\ &\quad - \frac{2\gamma}{(1-\gamma\|x-\bar{x}\|)^3}, \qquad \gamma > 0 \end{aligned} \tag{3.9}$$

where $\|y-x\| + \|x-\bar{x}\| < \frac{1}{\gamma}$, and the assumptions $(M1)$ and $(M2)$ are satisfied for ϕ. Also, if $\alpha := \beta\gamma < 3 - 2\sqrt{2}$, then assumption $(M3)$ is satisfied for ϕ. Therefore, the concrete form of Theorem 2.6 is given as follows.

Theorem 3.2. *Suppose* f *satisfies* (3.7)-(3.9). *If* $\alpha := \beta\gamma < 3 - 2\sqrt{2}$, *then the sequence* $\{x_k\}$ *generated by the super-Halley method (1.4) for solving the equation (1.1) with a starting point* x_0 *is well defined, is contained in* $B(\bar{x}, t^*)$ *and converges to a solution* $\bar{x} \in \bar{B}(\bar{x}, t^*)$ *of the Eq.(1.1). The limit* $\bar{x}$ *of the sequence* $\{x_k\}$ *is unique in* $B(\bar{x}, t^{**})$, *where* t^* *and* t^{**} *are given as*

$$t^* = \frac{\alpha + 1 - \sqrt{(\alpha+1)^2 - 8\alpha}}{4\gamma} \quad \text{and} \quad t^{**} = \frac{\alpha + 1 + \sqrt{(\alpha+1)^2 - 8\alpha}}{4\gamma}$$

respectively. Moreover, the following error bound holds: for all $k \ge 0$, *we have*

$$\|\bar{x} - x_{k+1}\| \le (t^* - t_{k+1}) \Big(\frac{\|\bar{x} - x_k\|}{t^* - t_k} \Big)^3, \ k = 0, 1, 2, \dots$$

and the sequence $\{x_k\}$ *converges* Q*-cubic as follows*

$$\|\bar{x} - x_{k+1}\| \le \frac{2\gamma^2}{[2(1-\gamma t^*)^2 - 1]^2} (\|\bar{x} - x_k\|)^3, \ k = 0, 1, 2, \dots$$

4 Numerical Examples

This section is devoted to illustrate the above theoretical results by a number of numerical examples.

Example 4.1. *Let* $X = Y = \mathbb{R}$ *with* $\Omega = B(0,1)$ *and the function* f *on* Ω *is*

$$f(x) = e^x - 1 \tag{4.1}$$

and for $\bar{x} = 0$,

$$f'(\bar{x}) = 1, f''(\bar{x}) = 1.$$

Also, we obtain that

$$\lambda_1 = e, \lambda_2 = 1, \beta = 0$$

Therefore, the convergence criterion (3.5) *holds and the Theorem 3.1 is applicable to conclude that the sequence generated by super-Halley method* (1.4) *with initial point* $x_0 = 0.25$ *converges to a root of* (4.1). *In this case, we have* $t^* = 0$ *and* $t^{**} = 1.03304078$*, that is the existence and uniqueness ball are* $B(0.25, 0)$ *and* $\bar{B}(0.25, 1.03304078)$ *respectively and the error bound is* 1.525807581.

Example 4.2. *Let* $X = C[0,1]$ *the space of continuous functions defined on interval* $[0,1]$ *equipped with max norm and let* $\Omega = U[0,1]$ *and the function* f *on* Ω *is .*

$$f(x)(s) = x(s) - 2\lambda \int_0^1 G(s,t)x(t)^3 dt \tag{4.2}$$

Therefore we have

$$f'(x)u(s) = u(s) - 6\lambda \int_0^1 G(s,t)x(t)^2 u(t) dt, \qquad u \in \Omega,$$

and

$$f''(x)[uv](s) = -\lambda \int_0^1 G(s,t)x(t)(uv)(t) dt, \qquad u, v \in \Omega,$$

Now, let $M = \max_{s \in [0,1]} \int_0^1 |G(s,t)| dt$. *Then* $M = \frac{1}{8}$.. *Also, for any* $x, y \in \Omega$, *we have*

$$\|f'(\bar{x})^{-1}[f''(x) - f''(\bar{x})]\| \leq \frac{3|\lambda|}{2} \|x - \bar{x}\|.$$

So, we obtain the values of β, λ_2 *and* λ_1 *in as follows*

$$\beta = 0, \qquad \lambda_2 = 0, \qquad \lambda_1 = \frac{3|\lambda|}{2}.$$

Therefore, the convergence criterion (3.5) holds and the Theorem 3.1 is applicable to conclude that the sequence generated by Super-Halley method (1.4) with initial point x_0 *converges to a zero of* f *defined by (4.2).*

For the different values of λ *i.e. for* $\lambda = 0.0625, 0.125, 0.25, 0.5, 1$ *and the initial point* $x_0 = 0.25$ *the corresponding domain of existence and uniqueness of solution, are given in Table-4.2.*

Table-4.2: *Domains of existence and uniqueness of solution for Super-Halley's method*

λ	*convergence ball in our work*	
	existence	*uniqueness*
0.0625	$\bar{B}(0.25, 0)$	$B(0.25, 8)$
0.125	$\bar{B}(0.25, 0)$	$B(0.25, 5.656854249)$
0.25	$\bar{B}(0.25, 0)$	$B(0.25, 4)$
0.5	$\bar{B}(0.25, 0)$	$B(0.25, 2.828427125)$
1	$\bar{B}(0.25, 0)$	$B(0.25, 2)$

Now, we present a numerical example to illustrate the Smale-type conditions.

Example 4.3. *Let* $X = Y = \mathbb{R}$ *with* $\Omega = U[0, 1]$ *and the function* f *on* Ω *is*

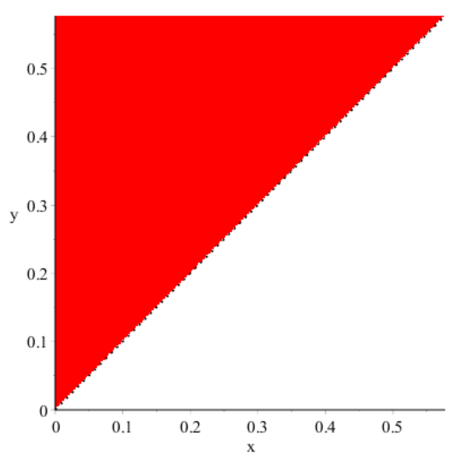
$$f(x) = e^x + 2x^2 - 1 \tag{4.3}$$

For $\bar{x} = 0$,

$$f'(\bar{x}) = 1, f''(\bar{x}) = 5$$

and we obtain that

$$\beta = 0, \gamma = 2.5$$



Therefore, the convergence criterion (3.9) holds (which can be seen from the above graph in case $y > x$*) and the Theorem 3.2 is applicable to conclude that the sequence generated by Super-Halley method (1.4) converges to a zero of* f *defined by (4.3) with* $t^* = 0$ *and* $t^{**} = 0.2$.

5 Conclusions

In this paper, the local convergence of Super-Halley method has been studied under majorant conditions on second derivative of f. Convergence ball of the method has been included. Two special cases: one Kantorovich-type conditions and another Smale-type conditions have also been studied. A number of numerical examples also given to illustrate our study.

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