

Certain results on the conharmonic curvature tensor of (κ, μ) -contact metric manifolds

DIVYASHREE G.¹ AND VENKATESHA ²

¹ *Department of Mathematics,
Govt., Science College, Chitradurga-577501, Karnataka, India.*

² *Department of Mathematics, Kuvempu University,
Shankaraghatta - 577 451, Shimoga, Karnataka, India.*

gdivyashree9@gmail.com, vensmath@gmail.com

ABSTRACT

The paper presents a study of (κ, μ) -contact metric manifolds satisfying certain conditions on the conharmonic curvature tensor.

RESUMEN

El artículo presenta un estudio de variedades (κ, μ) -contacto métricas satisfaciendo ciertas condiciones sobre el tensor de curvatura conharmónico.

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1 Introduction

In 1995, Blair et al. [3] introduced the idea of a class of contact metric manifolds for which the characteristic vector field ξ belongs to the (κ, μ) -nullity distribution for some real numbers κ and μ and such type of manifolds are called (κ, μ) -contact metric manifold. The non-Sasakian (κ, μ) -contact metric manifolds have two classes, namely, the class consists of the unit tangent sphere bundles of spaces of constant curvature, equipped with the natural contact metric structure and the class contains all the three-dimensional unimodular Lie groups, except the commutative one admitting the structure of a left invariant (κ, μ) -contact metric manifold [3, 4, 9]. Boeckx [4] given a full classification of (κ, μ) -contact metric manifolds. (κ, μ) -contact metric manifolds have been studied by several authors in [5, 6, 13, 11] and others.

A rank-four tensor N that remains invariant under conharmonic transformation for a $(2n+1)$ -dimensional Riemannian manifold M is given by

$$N(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY], \quad (1.1)$$

which is also of the form

$$N(X, Y, Z, T) = R(X, Y, Z, T) - \frac{1}{2n-1}[S(Y, Z)g(X, T) - S(X, Z)g(Y, T) + g(Y, Z)g(QX, T) - g(X, Z)g(QY, T)], \quad (1.2)$$

where R, S and Q represents the Riemannian curvature tensor, Ricci tensor and Ricci operator respectively.

A manifold whose conharmonic curvature vanishes at every point of the manifold is called conharmonically flat manifold. Such a curvature tensor have been extensively studied by Siddiqui and Ahsan [12], Ozgur [8], Avijit Sarkar et al. [10], Asghari and Taleshian [7] and many others.

Our present work is organised in the following way: After introduction, section 2 includes basics related to (κ, μ) -contact metric manifold which will be used later. Section 3 deals with conharmonically flat (κ, μ) -contact metric manifolds. We proved that conharmonically locally ϕ -symmetric (κ, μ) -contact metric manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ in section 4. Section 5 and 6 are devoted to the study of h -Conharmonically semisymmetric and ϕ -Conharmonically semisymmetric non-Sasakian (κ, μ) -contact metric manifolds respectively. Finally, we have shown that if the conharmonic curvature tensor on a (κ, μ) -contact metric manifold is divergent free then the Ricci tensor S is a Codazzi tensor.

2 Preliminaries

A $(2n + 1)$ -dimensional differentiable manifold M^{2n+1} is called a contact manifold [1] if it carries a global 1-form η such that $\eta \wedge (d\eta)^{2n+1} \neq 0$ everywhere on M^{2n+1} . It is well known that a contact metric manifold admits an almost contact metric structure (ϕ, ξ, η, g) , where ϕ is a $(1, 1)$ -tensor field, ξ is the characteristic vector field, and a Riemannian metric g such that

$$\phi^2 = -I + \eta \otimes \xi, \quad g(X, \xi) = \eta(X), \tag{2.1}$$

$$\eta(\xi) = 1, \quad g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y). \tag{2.2}$$

$$d\eta(X, Y) = g(X, \phi Y), \quad g(X, \phi Y) = -g(Y, \phi X), \tag{2.3}$$

for all vector fields $X, Y \in TM^{2n+1}$ and then we call a structure as contact metric structure. A manifold M^{2n+1} with such a structure is said to be contact metric manifold and it is denoted by (ϕ, ξ, η, g) .

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad d\eta(\xi, X) = 0. \tag{2.4}$$

We define a $(1, 1)$ -tensor field h by $h = \frac{1}{2} \mathcal{L}_\xi \phi$, where \mathcal{L}_ξ is the Lie differentiation in the direction of ξ . Since the tensor field h is self-adjoint and anticommutes with ϕ , we have

$$h\xi = 0, \quad \phi h + h\phi = 0, \quad \text{tr}h = \text{tr}\phi h = 0, \tag{2.5}$$

$$\nabla_X \xi = -\phi X - \phi hX, \tag{2.6}$$

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.7}$$

where ∇ is the Levi-Civita connection and if $X \neq 0$ is an eigenvector of h corresponding to the eigenvalue λ , then ϕX is an eigenvector of h corresponding to the eigenvalue $-\lambda$. Blair et al. [3] studied the (κ, μ) -nullity condition and the (κ, μ) -nullity distribution $N(\kappa, \mu)$ of a contact metric manifold M is defined by [3]

$$N(\kappa, \mu) : p \longrightarrow N_p(\kappa, \mu) \tag{2.8}$$

$$= [Z \in T_p M : R(X, Y)Z = (\kappa I + \mu h)\{g(Y, Z)X - g(X, Z)Y\}],$$

for all $X, Y \in TM^{2n+1}$. A contact metric manifold M^{2n+1} with $\xi \in N(\kappa, \mu)$ is called a (κ, μ) -contact metric manifold. In a (κ, μ) -contact metric manifold, we have

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\}, \tag{2.9}$$

for all $X, Y \in TM^{2n+1}$.

In a (κ, μ) -contact metric manifold, the following relations hold [3, 11]:

$$h^2 = (\kappa - 1)\phi^2, \quad (2.10)$$

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \quad (2.11)$$

$$R(\xi, X)Y = \kappa[g(X, Y)\xi - \eta(Y)X] + \mu[g(hX, Y)\xi - \eta(Y)hX], \quad (2.12)$$

$$S(X, \xi) = 2n\kappa\eta(X), \quad (2.13)$$

$$S(X, Y) = [2(n-1) - n\mu]g(X, Y) + [2(n-1) + \mu]g(hX, Y) \\ + [2(1-n) + n(2\kappa + \mu)]\eta(X)\eta(Y), \quad (2.14)$$

$$QX = [2(n-1) - n\mu]X + [2(n-1) + \mu]hX \\ + [2(n-1) + n(2\kappa + \mu)], \quad (2.15)$$

$$S(\phi X, \phi Y) = S(X, Y) - 2n\kappa\eta(X)\eta(Y) - 2(2n-2 + \mu)g(hX, Y), \quad (2.16)$$

$$g(QX, Y) = S(X, Y). \quad (2.17)$$

From (2.6), we have

$$(\nabla_X \eta)Y = g(X + hX, \phi Y), \quad (2.18)$$

$$(\nabla_X h)Y = \{(1 - \kappa)g(X, \phi Y) + g(X, h\phi Y)\}\xi + \eta(Y)\{h(\phi X + \phi hX)\} \\ - \mu\eta(X)\phi hY, \quad (2.19)$$

where S is the Ricci tensor of type $(0, 2)$, Q is the Ricci operator and r is the scalar curvature of the manifold. It is well known that in a Sasakian manifold, the Ricci operator Q commutes with ϕ . But in a (κ, μ) -contact metric manifold Q does not commute with ϕ . In general, in a (κ, μ) -contact metric manifold Blair et al.[3] proved the following:

Proposition 1. *Let M^n be a (κ, μ) -contact metric manifold, then the relation*

$$Q\phi - \phi Q = 2[2(n-1) + \mu]h\phi,$$

holds.

From the definition of η -Einstein manifold, it follows easily that $Q\phi = \phi Q$. Hence from Proposition 2.1 we obtain either $\mu = -2(n-1)$, or the manifold is Sasakian. Using $\mu = -2(n-1)$, from (2.14) we obtain that the manifold is an η -Einstein manifold. Therefore Yildiz and De [13] proved the following:

Proposition 2. *In a non-Sasakian (κ, μ) -contact metric manifold, the following conditions are equivalent:*

- (i) η -Einstein manifold,
- (ii) $Q\phi = \phi Q$.

For $n = 1$, from Proposition 2.1 and Proposition 2.2, Yildiz and De [13] obtained the following:

Corolary 1. *A 3-dimensional non-Sasakian (κ, μ) -contact η -Einstein manifold is an $N(\kappa)$ -contact metric manifold.*

Lemma 2.1. [2]: *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a contact metric manifold with $R(X, Y)\xi = 0$ for all vector fields X, Y tangent to M^{2n+1} . Then M^{2n+1} is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$.*

3 Conharmonically flat (κ, μ) -contact metric manifolds

From (1.2), for a $(2n+1)$ -dimensional conharmonically flat (κ, μ) -contact metric manifold, we have

$$R(X, Y, Z, T) = \frac{1}{2n-1} [S(Y, Z)g(X, T) - S(X, Z)g(Y, T) + g(Y, Z)g(QX, T) - g(X, Z)g(QY, T)]. \quad (3.1)$$

Substituting $Z = \xi$ in (3.1) and using (2.1), (2.9) and (2.13), we obtain

$$\begin{aligned} & \kappa[\eta(Y)g(X, T) - \eta(X)g(Y, T)] + \mu[\eta(Y)g(hX, T) - \eta(X)g(hY, T)] \\ &= \frac{1}{2n-1} [2n\kappa\eta(Y)g(X, T) - 2n\kappa\eta(X)g(Y, T) + \eta(Y)g(QX, T) - \eta(X)g(QY, T)]. \end{aligned} \quad (3.2)$$

Again, by taking $Y = \xi$ and using (2.1), (2.2), (2.5) and (2.13), (3.2) becomes

$$S(X, T) = -\kappa g(X, T) + (2n+1)\kappa\eta(X)\eta(T) + (2n-1)\mu g(hX, T). \quad (3.3)$$

From the equation (3.3), it follows that if $\mu = 0$, then the manifold is an η -Einstein manifold.

Conversely, if the manifold is η -Einstein, then we can write

$$S(X, T) = a_1 g(X, T) + b_1 \eta(X)\eta(T). \quad (3.4)$$

On equating (3.3) and (3.4), we find

$$a_1 g(X, T) + b_1 \eta(X)\eta(T) = -\kappa g(X, T) + (2n+1)\kappa\eta(X)\eta(T) + (2n-1)\mu g(hX, T) \quad (3.5)$$

Now, in (3.5) replacing T by ϕX and using (2.3), we get

$$(2n-1)\mu g(hX, \phi X) = 0, \quad (3.6)$$

for all X . Consequently, $\mu = 0$.

Hence, an n -dimensional conharmonically flat (κ, μ) -contact metric manifold is an η -Einstein manifold if and only if $\mu = 0$. But from (2.14), it follows that a (κ, μ) -contact metric manifold is

η -Einstein if and only if $\{2(n-1) + \mu\} = 0$. If we consider a $(2n+1)$ -dimensional ($n > 1$) conharmonically flat η -Einstein (κ, μ) -contact metric manifold, then $n = 1$, which contradicts the fact that $n > 1$.

Hence, the theorem can be stated as follows:

Theorem 3.1. *An $(2n+1)$ -dimensional ($n > 1$) conharmonically flat (κ, μ) -contact metric manifold cannot be an η -Einstein manifold.*

4 Conharmonically locally ϕ -symmetric (κ, μ) -contact metric manifolds

Definition 4.1. *An $(2n+1)$ -dimensional ($n > 1$) (κ, μ) -contact metric manifold M^{2n+1} is said to be conharmonically locally ϕ -symmetric if it satisfies*

$$\phi^2((\nabla_W N)(X, Y)Z) = 0, \quad (4.1)$$

for all X, Y, Z, W orthogonal to ξ .

Taking covariant differentiation of (1.1), we have

$$\begin{aligned} (\nabla_W N)(X, Y)Z &= (\nabla_W R)(X, Y)Z - \frac{1}{2n-1}[(\nabla_W S)(Y, Z)X - (\nabla_W S)(X, Z)Y \\ &\quad + g(Y, Z)(\nabla_W Q)(X) - g(X, Z)(\nabla_W Q)(Y)], \end{aligned} \quad (4.2)$$

where ∇ denotes the Levi-Civita connection on the manifold.

Differentiating equations (2.8), (2.14) and (2.15) covariantly with respect to W , we obtain

$$\begin{aligned} (\nabla_W R)(X, Y)Z &= W\kappa\{g(Y, Z)X - g(X, Z)Y\} + W\mu\{g(Y, Z)hX - g(X, Z)hY\} \\ &\quad + \mu[g(Y, Z)\{(1-\kappa)g(W, \phi X) + g(W, h\phi X)\}\xi \\ &\quad + \eta(X)\{h(\phi W + \phi hW)\} - \mu\eta(W)\phi hX \\ &\quad - g(X, Z)\{(1-\kappa)g(W, \phi Y) + g(W, h\phi Y)\}\xi \\ &\quad + \eta(Y)\{h(\phi W + \phi hW)\} - \mu\eta(W)\phi hY], \end{aligned} \quad (4.3)$$

$$\begin{aligned} (\nabla_W S)(Y, Z)X &= \{2(1-n) + n(2\kappa + \mu)\}[g(W, \phi Y)\eta(Z)X \\ &\quad + g(hW, \phi Y)\eta(Z)X + g(W, \phi Z)\eta(Y)X + g(hW, \phi Z)\eta(Y)X \\ &\quad + (2(n-1) + \mu)\{(1-\kappa)g(W, \phi Y)\eta(Z)X + g(W, h\phi Y)\eta(Z)X \\ &\quad + g(h(\phi W + \phi hW), Z)\eta(Y)X\} - \mu g(\phi hY, Z)\eta(W)X \end{aligned} \quad (4.4)$$

and

$$\begin{aligned}
 (\nabla_W Q)(X) &= \{2(n-1) + \mu\} \{ (1-\kappa)g(W, \phi X) + g(W, h\phi X) \} \xi \\
 &\quad + \eta(X) \{ h(\phi W + \phi hX) \} - \mu \eta(W) \phi hX \\
 &\quad + \{2(n-1) + n(2\kappa + \mu)\} g(W, \phi X) \xi \\
 &\quad + \{2(n-1) + n(2\kappa + \mu)\} g(hW, \phi X) \xi \\
 &\quad - \{2(n-1) + n(2\kappa + \mu)\} \eta(X) \phi W \\
 &\quad - \{2(n-1) + n(2\kappa + \mu)\} \eta(X) \phi hW.
 \end{aligned} \tag{4.5}$$

Now, considering equations (4.3), (4.4) and (4.5) in (4.2) and also taking X, Y, Z, W orthogonal to ξ , we get

$$\begin{aligned}
 (\nabla_W N)(X, Y)Z &= W\kappa[g(Y, Z)X - g(X, Z)Y] + W\kappa[g(Y, Z)hX - g(X, Z)hY] \\
 &\quad + \mu \{ (1-\kappa)g(Y, Z)g(W, \phi X)\xi + (1-\kappa)g(Y, Z)g(W, h\phi X)\xi \\
 &\quad - (1-\kappa)g(X, Z)g(W, \phi Y)\xi - (1-\kappa)g(X, Z)g(W, h\phi Y)\xi \} \\
 &\quad - \frac{1}{2n-1} \{ [2(n-1) + \mu] \{ (1-\kappa)g(Y, Z)g(W, \phi X)\xi \\
 &\quad - g(X, Z)g(W, \phi Y)\xi \} + g(Y, Z)g(W, h\phi X)\xi \\
 &\quad - g(X, Z)g(W, h\phi Y)\xi \} \\
 &\quad + \{2(n-1) + n(2\kappa + \mu)\} [g(Y, Z)g(W, \phi X)\xi \\
 &\quad + g(Y, Z)g(hW, \phi X)\xi - g(X, Z)g(W, \phi Y)\xi \\
 &\quad - g(X, Z)g(hW, \phi Y)\xi].
 \end{aligned} \tag{4.6}$$

Applying ϕ^2 on both sides of (4.6), one can obtain

$$\begin{aligned}
 \phi^2((\nabla_W N)(X, Y)Z) &= \phi^2 \{ W\kappa[g(Y, Z)X - g(X, Z)Y] + W\kappa[g(Y, Z)hX \\
 &\quad - g(X, Z)hY] + \mu \{ (1-\kappa)g(Y, Z)g(W, \phi X)\xi \\
 &\quad + (1-\kappa)g(Y, Z)g(W, h\phi X)\xi - (1-\kappa)g(X, Z)g(W, \phi Y)\xi \\
 &\quad - (1-\kappa)g(X, Z)g(W, h\phi Y)\xi \} \\
 &\quad - \frac{1}{2n-1} \{ [2(n-1) + \mu] \{ (1-\kappa)g(Y, Z)g(W, \phi X)\xi \\
 &\quad - g(X, Z)g(W, \phi Y)\xi \} + g(Y, Z)g(W, h\phi X)\xi \\
 &\quad - g(X, Z)g(W, h\phi Y)\xi \} \\
 &\quad + \{2(n-1) + n(2\kappa + \mu)\} [g(Y, Z)g(W, \phi X)\xi \\
 &\quad + g(Y, Z)g(hW, \phi X)\xi - g(X, Z)g(W, \phi Y)\xi \\
 &\quad - g(X, Z)g(hW, \phi Y)\xi].
 \end{aligned} \tag{4.7}$$

From (4.1) and using (2.1), (4.7) becomes

$$\begin{aligned} & (W\kappa)[g(X, Z)Y - g(Y, Z)X] + (W\kappa)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi \\ & + (W\mu)[g(X, Z)hY - g(Y, Z)hX] = 0. \end{aligned} \quad (4.8)$$

Again, considering X, Y orthogonal to ξ , one can get

$$(W\kappa)[g(X, Z)Y - g(Y, Z)X] + (W\mu)[g(X, Z)hY - g(Y, Z)hX] = 0. \quad (4.9)$$

By taking inner product of (4.9) with V , we have

$$\begin{aligned} & (W\kappa)[g(X, Z)g(Y, V) - g(Y, Z)g(X, V)] + (W\mu)[g(X, Z)g(hY, V) \\ & - g(Y, Z)g(hX, V)] = 0. \end{aligned} \quad (4.10)$$

On contraction, the above equation yields

$$-2n(W\kappa)g(Y, Z) + (W\mu)g(Z, hY) = 0. \quad (4.11)$$

Setting $Y = \xi$ in (4.11) and using (2.5), we get

$$2n(W\kappa)\eta(Z) = 0. \quad (4.12)$$

If we assume that $\kappa = 0$ in (4.11) then either $\mu = 0$ or $g(Z, hY) = 0$. Further, if $\kappa = 0 = \mu$ in (2.9), then we get $R(X, Y)\xi = 0$ for all X, Y and in the light of Lemma 2.1, the manifold under consideration is locally isometric to the Riemannian product $E^{n+1} \times S^n(4)$.

So from Lemma 2.1, we can state the theorem as follows:

Theorem 4.2. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a conharmonically locally ϕ -symmetric (κ, μ) -contact metric manifold. Then the manifold is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$.*

5 h -Conharmonically semisymmetric non-Sasakian (κ, μ) -contact metric manifolds

Definition 5.1. *A Riemannian manifold (M^{2n+1}, g) is said to be h -conharmonically semisymmetric if it satisfies*

$$N(X, Y) \cdot h = 0. \quad (5.1)$$

The following lemma which was proved in [3] is helpful to state our theorem.

Lemma 5.1. [3]: Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a contact metric manifold with ξ belonging to the (κ, μ) -nullity distribution. Then for any vector fields X, Y, Z ,

$$\begin{aligned} R(X, Y)hZ - hR(X, Y)Z &= \{\kappa[g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)] \\ &+ \mu(\kappa - 1)[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\}\xi \\ &+ \kappa\{g(Y, \phi Z)\phi hX - g(X, \phi Z)\phi hY + g(Z, \phi hY)\phi X \\ &- g(Z, \phi hX)\phi Y + \eta(Z)[\eta(X)hY - \eta(Y)hX]\} \\ &- \mu\{\eta(Y)[(1 - \kappa)\eta(Z)X + \mu\eta(X)hZ] \\ &- \eta(X)[(1 - \kappa)\eta(Z)Y + \mu\eta(Y)hZ] + 2g(X, \phi Y)\phi hZ\}. \end{aligned} \tag{5.2}$$

Let M^{2n+1} be h -conharmonically semisymmetric non-Sasakian (κ, μ) -contact metric manifold. The condition $N(X, Y) \cdot h = 0$ can be expressed as follows,

$$(N(X, Y) \cdot h)Z = N(X, Y)hZ - hN(X, Y)Z = 0, \tag{5.3}$$

for any vector fields X, Y, Z .

With the help of (1.1) and (5.2), (5.3) can be written as

$$\begin{aligned} &[\kappa\{g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)\} + \mu(\kappa - 1)\{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\}]\xi \\ &+ \kappa\{g(Y, \phi Z)\phi hX - g(X, \phi Z)\phi hY + g(Z, \phi hY)\phi X - g(Z, \phi hX)\phi Y + \eta(Z)[\eta(X)hY \\ &- \eta(Y)hX]\} - \mu\{\eta(Y)[(1 - \kappa)\eta(Z)X + \mu\eta(X)hZ] - \eta(X)[(1 - \kappa)\eta(Z)Y + \mu\eta(Y)hZ] \\ &+ 2g(X, \phi Y)\phi hZ\} - \frac{1}{2n - 1}[S(Y, hZ)X - S(X, hZ)Y + g(Y, hZ)QX - g(X, hZ)QY \\ &- S(Y, Z)hX + S(X, Z)hY - g(Y, Z)QhX + g(X, Z)QhY] = 0. \end{aligned} \tag{5.4}$$

By taking inner product of (5.4) with T , we get

$$\begin{aligned} &[\kappa\{g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)\} + \mu(\kappa - 1)\{g(X, Z)\eta(Y) - g(Y, Z)\eta(X)\}]\eta(T) \\ &+ \kappa\{g(Y, \phi Z)g(\phi hX, T) - g(X, \phi Z)g(\phi hY, T) + g(Z, \phi hY)g(\phi X, T) \\ &- g(Z, \phi hX)g(\phi Y, T) + \eta(Z)[\eta(X)g(hY, T) - \eta(Y)g(hX, T)]\} \\ &- \mu\{\eta(Y)[(1 - \kappa)\eta(Z)g(X, T) + \mu\eta(X)g(hZ, T)] - \eta(X)[(1 - \kappa)\eta(Z)g(Y, T) \\ &+ \mu\eta(Y)g(hZ, T)] + 2g(X, \phi Y)g(\phi hZ, T)\} - \frac{1}{2n - 1}[S(Y, hZ)g(X, T) \\ &- S(X, hZ)g(Y, T) + g(Y, hZ)S(X, T) - g(X, hZ)S(Y, T) - S(Y, Z)g(hX, T) \\ &+ S(X, Z)g(hY, T) - g(Y, Z)S(hX, T) + g(X, Z)S(hY, T)] = 0. \end{aligned} \tag{5.5}$$

Setting $Y = T = \xi$ in (5.5) and using (2.2) and (2.5), we get

$$\begin{aligned} \frac{1}{2n - 1}S(X, hZ) &= -\mu(1 - \kappa)g(X, Z) + [2(1 - \mu) + (1 - \kappa)]\eta(X)\eta(Z) \\ &+ [\kappa - \frac{2(2n + 1)\kappa}{2(n - 1)}]g(X, hZ). \end{aligned} \tag{5.6}$$

Replacing X by hX in the above equation and using (2.10), we have

$$S(X, Z) = -\kappa g(X, Z) + \kappa \eta(X)\eta(Z) - 2\mu(n-1)g(hX, Z). \quad (5.7)$$

If we consider $\mu = 0$ in (5.7) then it is an η -Einstein manifold.

Using (2.14) in (5.7) and simplifying, we finally obtain

$$S(X, Z) = n_1 g(X, Z) + n_2 \eta(X)\eta(Z), \quad (5.8)$$

where $n_1 = \frac{-\kappa[2(n-1)+\mu]+\mu(2n-1)[2(n-1)+n\mu]}{[2(n-1)+\mu]+\mu(2n-1)}$

and

$n_2 = \frac{\kappa[2(n-1)+\mu]+\mu(2n-1)[2(1-n)+n(2\kappa+\mu)]}{[2(n-1)+\mu]+\mu(2n-1)}$.

Thus from (5.8), we can conclude the following theorem:

Theorem 5.2. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a non-Sasakian (κ, μ) -contact metric manifold. If M is h -conharmonically semisymmetric, then the manifold is an η -Einstein manifold with constant coefficients.*

From Proposition 2.2 and Theorem 5.5 we can state the following:

Corollary 2. *If M^{2n+1} is a h -conharmonically semisymmetric (κ, μ) -contact metric manifold then the Ricci operator Q commutes with ϕ i.e., $Q\phi = \phi Q$.*

6 ϕ -Conharmonically semisymmetric non-Sasakian (κ, μ) -contact metric manifolds

Definition 6.1. *A Riemannian manifold (M^{2n+1}, g) is said to be ϕ -conharmonically semisymmetric if*

$$N(X, Y) \cdot \phi = 0. \quad (6.1)$$

Now we need the following lemma:

Lemma 6.1. [3]: *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be a contact metric manifold with ξ belonging to the (κ, μ) -nullity distribution. Then for any vector fields X, Y, Z ,*

$$\begin{aligned} R(X, Y)\phi Z - \phi R(X, Y)Z &= \{(1 - \kappa)[g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y)] \\ &+ (1 - \mu)[g(\phi hY, Z)\eta(X) - g(\phi hX, Z)\eta(Y)]\}\xi \\ &- g(Y + hY, Z)(\phi X + \phi hX) + g(X + hX, Z)(\phi Y \\ &+ \phi hY) - g(\phi Y + \phi hY, Z)(X + hX) + g(\phi X \\ &+ \phi hX, Z)(Y + hY) - \eta(Z)\{(1 - \kappa)[\eta(X)\phi Y \\ &- \eta(Y)\phi X] + (1 - \mu)[\eta(X)\phi hY - \eta(Y)\phi hX]\}. \end{aligned} \quad (6.2)$$

Let M^{2n+1} be a $(2n + 1)$ -dimensional ϕ -conharmonically semisymmetric non-Sasakian (κ, μ) -contact metric manifold. The condition $N(X, Y) \cdot \phi = 0$ turns into,

$$(N(X, Y) \cdot \phi)Z = N(X, Y)\phi Z - \phi N(X, Y)Z = 0, \tag{6.3}$$

for any vector fields X, Y, Z .

In view of (1.1) and (6.2), (6.3) becomes

$$\begin{aligned} & \{(1 - \kappa)[g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y)] + (1 - \mu)[g(\phi hY, Z)\eta(X) - g(\phi hX, Z)\eta(Y)]\}\xi \tag{6.4} \\ & -g(Y + hY, Z)(\phi X + \phi hX) + g(X + hX, Z)(\phi Y + \phi hY) - g(\phi Y + \phi hY, Z)(X + hX) \\ & +g(\phi X + \phi hX, Z)(Y + hY) - \eta(Z)\{(1 - \kappa)[\eta(X)\phi Y - \eta(Y)\phi X] + (1 - \mu)[\eta(X)\phi hY \\ & - \eta(Y)\phi hX]\} - \frac{1}{2n - 1}[S(Y, \phi Z)X - S(X, \phi Z)Y + g(Y, \phi Z)QX - g(X, \phi Z)QY \\ & - S(Y, Z)\phi X + S(X, Z)\phi Y - g(Y, Z)Q\phi X + g(X, Z)Q\phi Y] = 0. \end{aligned}$$

Taking inner product of (6.4) with T , we get

$$\begin{aligned} & \{(1 - \kappa)[g(\phi Y, Z)\eta(X) - g(\phi X, Z)\eta(Y)] + (1 - \mu)[g(\phi hY, Z)\eta(X) \tag{6.5} \\ & -g(\phi hX, Z)\eta(Y)]\}\eta(T) - g(Y, Z)g(\phi X, T) - g(hY, Z)g(\phi X, T) - g(Y, Z)g(\phi hX, T) \\ & -g(hY, Z)g(\phi hX, T) + g(X, Z)g(\phi Y, T) + g(hX, Z)g(\phi Y, T) + g(X, Z)g(\phi hY, T) \\ & +g(hX, Z)g(\phi hY, T) - g(\phi Y, Z)g(X, T) - g(\phi Y, Z)g(hX, T) - g(\phi hY, Z)g(X, T) \\ & -g(\phi hY, Z)g(hX, T) + g(\phi X, Z)g(Y, T) + g(\phi hX, Z)g(Y, T) + g(\phi X, Z)g(hY, T) \\ & +g(\phi hX, Z)g(hY, T) - \eta(Z)\{(1 - \kappa)[\eta(X)g(\phi Y, T) - \eta(Y)g(\phi X, T)] \\ & + (1 - \mu)[\eta(X)g(\phi hY, T) - \eta(Y)g(\phi hX, T)]\} - \frac{1}{2n - 1}[S(Y, \phi Z)g(X, T) \\ & - S(X, \phi Z)g(Y, T) + g(Y, \phi Z)g(QX, T) - g(X, \phi Z)g(QY, T) - S(Y, Z)g(\phi X, T) \\ & + S(X, Z)g(\phi Y, T) - g(Y, Z)g(Q\phi X, T) + g(X, Z)g(Q\phi Y, T)] = 0. \end{aligned}$$

Treating $Y = T = \xi$ in (6.5) and using (2.1), (2.2), (2.4), (2.5) and (2.13), we have

$$\frac{1}{2n - 1}S(X, \phi Z) = \{(\kappa - 2) + \frac{2(2n + 1)\kappa}{2n - 1}\}g(X, \phi Z) - \mu g(\phi X, hZ). \tag{6.6}$$

Substituting X by ϕX in (6.6) and using (2.1), (2.2) and (2.16), one can get

$$\begin{aligned} S(X, Z) &= [(\kappa - 2)(2n - 1) + 2n\kappa]g(X, Z) - [(\kappa - 2)(2n - 1)]\eta(X)\eta(Z) \tag{6.7} \\ &+ [\mu(\kappa - 1)(2n - 1) + 2\{2(n - 1) + \mu\}]g(hX, Z). \end{aligned}$$

Making use of (2.14), (6.7) yields

$$S(X, Z) = n_3g(X, Z) + n_4\eta(X)\eta(Z), \tag{6.8}$$

where $n_3 = \frac{\{(\kappa-2)(2n-1)+2n\kappa\}\{2(n-1)+\mu\}-\{\mu(\kappa-1)(2n-1)+2[2(n-1)+\mu]\}\{2(n-1)-n\mu\}}{[2(n-1)+\mu]-[\mu(\kappa-1)(2n-1)+2\{2(n-1)+\mu\}]}$ and

$$n_4 = \frac{[(2-\kappa)(2n-1)][2(n-1)+\mu]-\{\mu(\kappa-1)(2n-1)+2[2(n-1)+\mu]\}[2(1-n)+2n(2\kappa+\mu)]}{[2(n-1)+\mu]-[\mu(\kappa-1)(2n-1)+2\{2(n-1)+\mu\}]}$$

Hence from (6.8), the theorem can be stated as follows:

Theorem 6.2. *If a $(2n + 1)$ -dimensional non-Sasakian (κ, μ) -contact metric manifold M^{2n+1} is ϕ -conharmonically semisymmetric then the manifold is an η -Einstein manifold with constant coefficients.*

Similarly, from Proposition 2.2 and Theorem 6.6, we get the following statement:

Corollary 3. *If M^{2n+1} is a ϕ -conharmonically semisymmetric (κ, μ) -contact metric manifold then the Ricci operator Q commutes with ϕ i.e., $Q\phi = \phi Q$.*

7 (κ, μ) -contact metric manifold with divergent free conharmonic curvature tensor

In this section, we study divergent free conharmonic curvature tensor on (κ, μ) -contact metric manifold.

Let $M^{2n+1}(\phi, \xi, \eta, g)$ ($n > 1$) be a (κ, μ) -contact metric manifold satisfying the following condition

$$(\text{Div}N)(X, Y)Z = 0. \quad (7.1)$$

In view of (7.1), (1.1) leads to

$$(\text{Div}R)(X, Y)Z = \frac{1}{2n-1}[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) + g(Y, Z)\text{dr}(X) - g(X, Z)\text{dr}(Y)]. \quad (7.2)$$

The above equation simplifies to,

$$\frac{2(n-1)}{(2n-1)}[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)] - \frac{1}{(2n-1)}[g(Y, Z)\text{dr}(X) - g(X, Z)\text{dr}(Y)] = 0. \quad (7.3)$$

On contracting and taking summation over i , $1 \leq i \leq n$ in (7.3), we get

$$2(3n-1)\text{dr}(Y) = 0, \quad (7.4)$$

which implies

$$\text{dr}(Y) = 0, \quad (7.5)$$

since $2(3n - 1) \neq 0$.

Further, considering (7.5) in (7.3), we obtain

$$(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) = 0, \tag{7.6}$$

which gives

$$(\nabla_X Q)Y = (\nabla_Y Q)X. \tag{7.7}$$

Thus, we can state:

Theorem 7.1. *Let $M^{2n+1}(\phi, \xi, \eta, g)$ ($n > 1$) be a (κ, μ) -contact metric manifold. If the manifold has divergent free conharmonic curvature tensor then the Ricci tensor S is a Codazzi tensor.*

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