CUBO, A Mathematical Journal
Vol.22, N°02, (215–231). August 2020
http://dx.doi.org/10.4067/S0719-06462020000200215
Received: 12 March, 2019 | Accepted: 14 July, 2020

D-metric Spaces and Composition Operators Between Hyperbolic Weighted Family of Function Spaces

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ABSTRACT

The aim of this paper is to introduce new hyperbolic classes of functions, which will be called $\mathcal{B}^*_{\alpha,\,\log}$ and $F^*_{\log}(p,q,s)$ classes. Furthermore, we introduce D-metrics space in the hyperbolic type classes $\mathcal{B}^*_{\alpha,\,\log}$ and $F^*_{\log}(p,q,s)$. These classes are shown to be complete metric spaces with respect to the corresponding metrics. Moreover, necessary and sufficient conditions are given for the composition operator C_ϕ to be bounded and compact from $\mathcal{B}^*_{\alpha,\,\log}$ to $F^*_{\log}(p,q,s)$ spaces.

RESUMEN

El objetivo de este artículo es introducir nuevas clases hiperbólicas de funciones, que serán llamadas clases $\mathcal{B}^*_{\alpha,\,\log}$ y $F^*_{\log}(p,q,s)$. A continuación, introducimos D-espacios métricos en las clases de tipo hiperbólicas $\mathcal{B}^*_{\alpha,\,\log}$ y $F^*_{\log}(p,q,s)$. Mostramos que estas clases son espacios métricos completos con respecto a las métricas correspondientes. Más aún, damos condiciones necesarias y suficientes para que el operador composición C_ϕ sea acotado y compacto desde el espacio $\mathcal{B}^*_{\alpha,\,\log}$ a $F^*_{\log}(p,q,s)$.

Keywords and Phrases: D-metric spaces, Logarithmic hyperbolic classes, Composition operators.

2020 AMS Mathematics Subject Classification: 47B38, 46E15.

1 Introduction

Let ϕ be an analytic self-map of the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ in the complex plane $\mathbb{C}$ and let $\partial\mathbb{D}$ be its boundary. Let $H(\mathbb{D})$ denote the space of all analytic functions in $\mathbb{D}$ and let $B(\mathbb{D})$ be the subset of $H(\mathbb{D})$ consisting of those $f \in H(\mathbb{D})$ for which $|f(z)| < 1$ for all $z \in \mathbb{D}$.

Let the Green's function of $\mathbb{D}$ be defined as $g(z, a) = \log \frac{1}{|\varphi_a(z)|}$, where $\varphi_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Möbius transformation related to the point $a \in \mathbb{D}$.

A linear composition operator C_ϕ is defined by $C_\phi(f) = (f \circ \phi)$ for f in the set $H(\mathbb{D})$ of analyticfunctions on $\mathbb{D}$ (see [9]). A function $f \in B(\mathbb{D})$ belongs to α-Bloch space $\mathcal{B}_\alpha, 0 < \alpha < \infty$, if

$$||f||_{\mathcal{B}_\alpha} = \sup_{z\in\mathbb{D}} (1 - |z|)^\alpha |f'(z)| < \infty.$$

The little α-Bloch space $\mathcal{B}_{\alpha,\,0}$ consisting of all $f \in \mathcal{B}_\alpha$ so that

$$\lim_{|z|\to 1^-} (1 - |z|^2)|f'(z)| = 0.$$

Definition 1. *[15] For an analytic function f on $\mathbb{D}$ and $0 < \alpha < \infty$, if*

$$||f||_{\mathcal{B}^\alpha_{\log}} = \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| \left(\log \frac{2}{1 - |z|^2}\right) < \infty,$$

then, f belongs to the weighted α-Bloch spaces $\mathcal{B}^\alpha_{\log}$.

If $\alpha = 1$, the weighted Bloch space $\mathcal{B}_{\log}$ is the set for all analytic functions f in $\mathbb{D}$ for which $||f||_{\mathcal{B}_{\log}} < \infty$.

The expression $||f||_{\mathcal{B}_{\log}}$ defines a seminorm while the norm is defined by

$$||f||_{\mathcal{B}_{\log}} = |f(0)| + ||f||_{\mathcal{B}_{\log}}.$$

Definition 2. *[14] For $0 < p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$, a function $f \in H(\mathbb{D})$ is in $F(p, q, s)$, if*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) < \infty.$$

Moreover, if

$$\lim_{|a|\to 1^-} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q g^s(z, a) dA(z) = 0,$$

then $f \in F_0(p, q, s)$.

El-Sayed and Bakhit [5] gave the following definition:

Definition 3. *For $0 < p, s < \infty$, $-2 < q < \infty$ and $q + s > -1$, a function $f \in H(\mathbb{D})$ is said to belong to $F_{\log}(p, q, s)$, if*

$$\sup_{I \subset \partial\mathbb{D}} \frac{\left(\log \frac{2}{|I|}\right)^p}{|I|^s} \int_{S(I)} |f'(z)|^p (1 - |z|^2)^q \left(\log \frac{1}{|z|}\right)^s dA(z) < \infty.$$

Where $|I|$ denotes the arc length of $I \subset \partial\mathbb{D}$ and $S(I)$ is the Carleson box defined by (see [8, 6])

$$S(I) = \{z \in \mathbb{D} : 1 - |I| < |z| < 1, \frac{z}{|z|} \in |I|\}.$$

The interest in the $F_{\log}(p, q, s)$-spaces rises from the fact that they cover some well known function spaces. It is immediate that $F_{\log}(2, 0, 1) = BMOA_{\log}$ and $F_{\log}(2, 0, p) = Q^p_{\log}$, where $0 < p < \infty$.

2 Preliminaries

Definition 4. *[11] The hyperbolic Bloch space $\mathcal{B}^*_\alpha$ is defined as*

$$\mathcal{B}^*_\alpha = \{f : f \in B(\mathbb{D}) \text{ and } \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha f^*(z) < \infty\}.$$

Denoting $f^*(z) = \frac{|f'(z)|}{1 - |f(z)|^2}$, the hyperbolic derivative of $f \in B(\mathbb{D})$. [7]

The little hyperbolic Bloch space $\mathcal{B}^*_{\alpha,\,0}$ is a subspace of $\mathcal{B}^*_\alpha$ consisting of all $f \in \mathcal{B}^*_\alpha$ so that

$$\lim_{|z| \to 1^-} (1 - |z|^2)^\alpha f^*(z) = 0.$$

The space $\mathcal{B}^*_\alpha$ is Banach space with the norm defined as

$$||f||_{\mathcal{B}^*_\alpha} = |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|)^\alpha |f^*(z)|.$$

Definition 5. *For $0 < p, s < \infty$, $-2 < q < \infty$, $\alpha = \frac{q+2}{p}$ and $q + s > -1$, a function $f \in H(\mathbb{D})$ is said to belong to $F^*(p, q, s)$, if*

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (f^*(z))^p (1 - |z|^2)^{\alpha p - 2} g^s(z, a) dA(z) < \infty.$$

Definition 6. *For $f \in B(\mathbb{D})$ and $0 < \alpha < \infty$, if*

$$||f||_{\mathcal{B}^*_{\alpha,\,\log}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha (f^*(z)) \left(\log \frac{2}{1 - |z|^2}\right) < \infty,$$

then f belongs to the $\mathcal{B}^*_{\alpha,\,\log}$.

We must consider the following lemmas in our study:

Lemma 2.1. *[12] Let* $0 < r \leq t \leq 1$, *then*

$$\log\frac{1}{t} \leq \frac{1}{r}(1-t^2)$$

Lemma 2.2. *[12] Let* $0 \leq k_1 < \infty,\ 0 \leq k_2 < \infty$, *and* $k_1 - k_2 > -1$, *then*

$$C(k_1,k_2) = \int_{\mathbb{D}}\left(\log\frac{1}{|z|}\right)^{k_1}(1-|z|^2)^{-k_2}dA(z) < \infty.$$

To study composition operators on $\mathcal{B}^*_{\alpha,\ \log}$ and $F^*_{\log}(p,q,s)$ spaces, we need to prove the following result:

Theorem 1. *If* $0 < p < \infty$, $1 < s < \infty$ *and* $\alpha = \frac{q+2}{p}$ *with* $q + s > -1$. *Then the following are equivalent:*

(A) $f \in \mathcal{B}^*_{\alpha,\ \log}$.

(B) $f \in F^*_{\log}(p,q,s)$.

(C) $\sup_{a\in\mathbb{D}}\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} (f^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi(z)|^2)^s dA(z) < \infty$,

(D) $\sup_{a\in\mathbb{D}}\left(\log\frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} (f^*(z))^p(1-|z|^2|^{\alpha p-2}g^s(z,a)dA(z) < \infty$.

Proof. Let $0 < p < \infty,\ -2 < q < \infty,\ 1 < s < \infty$ and $0 < r < 1$. By subharmonicity we have for an analytic function $g \in \mathbb{D}$ that

$$|g(0)|^p \leq \frac{1}{\pi r^2}\int_{\mathbb{D}(0,r)}|g(w)|^p dA(w).$$

For $a \in \mathbb{D}$, the substitution $z = \varphi_a(z)$ results in Jacobian change in measure given by

$$dA(w) = |\varphi_a'(z)|^2\ dA(z).$$

For a Lebesgue integrable or a non-negative Lebesgue measurable function f on $\mathbb{D}$, we thus have the following change of variable formula:

$$\int_{\mathbb{D}(0,r)} f(\varphi_a(w))dA(w) = \int_{\mathbb{D}(a,r)} f(z)|\varphi_a'(z)|^2 dA(z).$$

Let $g = \frac{f'\circ\varphi_a}{1-|f\circ\varphi_a|^2}$ then we have

$$\left(\frac{|f'(a)|}{1-|f(a)|^2}\right)^p = (f^*(a))^p \leq \frac{1}{\pi r^2}\int_{\mathbb{D}(0,r)}\left(\frac{|f'(\varphi_a(w))|}{1-|f(\varphi_a(w))|^2}\right)^p dA(w)$$
$$= \frac{1}{\pi r^2}\int_{\mathbb{D}(a,r)}(f^*(z))^p|\varphi_a'(z)|^2 dA(z).$$

Since

$$|\varphi_a'(z)| = \frac{1-|\varphi_a(z)|^2}{1-|z|^2},$$

and

$$\frac{1-|\varphi_a(z)|^2}{1-|z|^2} \leq \frac{4}{1-|a|^2} \qquad a, z \in \mathbb{D}.$$

So we obtain that

$$(f^*(a))^p \leq \frac{16}{\pi r^2(1-|a|^2)^2}\int_{\mathbb{D}(a,r)} (f^*(z))^p dA(z).$$

Again $f \in \mathcal{B}^*_{\alpha,\,\log}$, and $(1-|z|^2)^2 \approx (1-|a|^2)^2 \approx \mathbb{D}(a,r)$, for $z \in \mathbb{D}(a,r)$. Thus, we have

$$\left(\log \tfrac{2}{1-|a|^2}\right)^p (f^*(a))^p(1-|a|^2)^{\alpha p}$$

$$\begin{aligned}
&\leq \frac{16}{\pi r^2(1-|a|^2)^{2-\alpha p}} \times \left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p dA(z)\\
&\leq \frac{16}{\pi r^2} \times \left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p (1-|z|^2)^{\alpha p-2} dA(z)\\
&\leq \frac{16}{\pi r^2} \times \left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p (1-|z|^2)^{\alpha p-2} \times \left(\frac{1-|\varphi_a(z)|^2}{1-|\varphi_a(z)|^2}\right)^s dA(z)\\
&\leq \frac{16}{\pi r^2(1-r^2)^s} \times \left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\
&\leq M(r) \times \left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a'(z)|^2)^s dA(z).
\end{aligned}$$

Where $M(r)$ is a constant depending on r. Thus, the quantity (A) is less than or equal to constant times the quantity (C).

From the fact

$$(1-|\varphi_a(z)|^2) \leq 2\log\frac{1}{|\varphi_a(z)|} = 2g(z,a) \qquad for\ a,\ z \in \mathbb{D},$$

we have

$$\begin{aligned}
&\left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\
\leq\ &\left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}(a,r)} (f^*(z))^p (1-|z|^2)^{\alpha p-2} g^s(z,a) dA(z).
\end{aligned}$$

Hence, the quantity (C) is less than or equal to a constant times (D). By taking $\alpha = \frac{q+2}{p}$, it follows $f \in F^*_{\log}(p,q,s)$. Thus, the quantity (C) is less than or equal to a constant times the quantity (B).

Finally, from the following inequality, let $z = \varphi_a(w)$ then $w = \varphi_a(z)$. Hence,

$$\begin{aligned}
&\left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} (f^*(\varphi_a(w)))^p (1-|\varphi_a(w)|^2)^{\alpha p-2} \left(\log \frac{1}{|w|}\right)^s |\varphi_a'(w)|^2 dA(w) \\
= &\left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} (f^*(\varphi_a(w)))^p (1-|\varphi_a(w)|^2)^{\alpha p} \left(\log \frac{1}{|w|}\right)^s \frac{|\varphi_a'(w)|^2}{(1-|\varphi_a(w)|^2)^2} dA(w) \\
= &\left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} (f^*(\varphi_a(w)))^p (1-|\varphi_a(w)|^2)^{\alpha p} \left(\log \frac{1}{|w|}\right)^s \frac{1}{(1-|w|^2)^2} dA(w) \\
\leq\ & ||f||^p_{\mathcal{B}^*_{\alpha,\log}} \left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} \left(\log \frac{1}{|w|}\right)^s (1-|w|^2)^{-2} dA(w) \\
= &\ C(s,2)||f||^p_{\mathcal{B}^*_{\alpha,\log}}.
\end{aligned}$$

By lemma 2.2, $C(s,2) = \int_{\mathbb{D}} \left(\log \frac{1}{|w|}\right)^s (1-|w|^2)^{-2} dA(w) < \infty, \quad for\ 1 < s < \infty.$

Thus, the quantity (D) is less than or equal to a constant times the quantity (A). Hence, it is proved.

Let us we give the following equivalent definition for $F^*_{\log}(p,q,s)$.

Definition 7. *For $0 < p, s < \infty$, $-2 < q < \infty$, $\alpha = \frac{q+2}{p}$ and $q + s > -1$, a function $f \in H(\mathbb{D})$ is said to belong to $F^*_{\log}(p,q,s)$, if*

$$\sup_{a\in\mathbb{D}} \left(\log \frac{2}{1-|a|^2}\right)^p \int_{\mathbb{D}} (f^*(z))^p (1-|z|^2)^{\alpha p-2} (1-|\varphi_a(z)|^2)^s dA(z) < \infty.$$

Definition 8. *A composition operator $C_\phi : \mathcal{B}^*_{\alpha,\log} \to F^*_{\log}(p,q,s)$ is said to be bounded if there is a positive constant C so that $||C_\phi f||_{F^*_{\log}}(p,q,s) \leq C||f||_{\mathcal{B}^*_{\alpha,\log}}$ for all $f \in \mathcal{B}^*_{p,\alpha}$.*

Definition 9. *A composition operator $C_\phi : \mathcal{B}^*_{\alpha,\log} \to F^*_{\log}(p,q,s)$ is said to be compact if it maps any ball in $\mathcal{B}^*_{p,\alpha}$ onto a precompact set in $F^*(p,q,s)$.*

The following lemma follows by standard arguments similar to those outline in [13]. Hence, we omit the proof.

Lemma 2.3. *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself. Let $0 < p, s, \alpha < \infty, -2 < q < \infty$, then $C_\phi : \mathcal{B}^*_{\alpha,\log} \to F^*_{\log}(p,q,s)$ is compact if and only if for any bounded sequence $\{f_n\}_{n\in N} \in \mathcal{B}^*_{\alpha,\log}$ which converges to zero uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ we have* $\lim_{n\to\infty} ||C_\phi f_n||_{F^*_{\log}(p,q,s)} = 0$.

3 D-metric space

Topological properties of generalized metric space called D- metric space was introduced in [1], see for example, ([2] and [3]). This structure of D-metric space is quite different from a 2-metric space and natural generalization of an ordinary metric space in some sense.

Definition 10. *[4] Let X denote a nonempty set and $\mathbb{R}$ the set of real numbers. A function $D : X \times X \times X \to \mathbb{R}$ is said to be a D-metric on X if it satisfies the following properties:*

(i) $D(x, y, z) \geq 0$ for all $x, y, z \in X$ and equality holds if and only if $x = y = z$ (nonnegativity),

(ii) $D(x, y, z) = D(x, z, y) = \cdots$ (symmetry),

(iii) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$ for all $x, y, z, a \in X$
(tetrahedral inequality).

A nonempty set X together with a D-metric D is called a D-metric space and is represented by (X, D). The generalization of a D-metric space with D-metric as a function of n variables is provided in Dhage [2].

Example1.1: [4] Let (X, d) be an ordinary metric space and define a function D_1 on X^3 by

$$D_1(x, y, z) = max\{d(x, y), d(y, z), d(z, x)\},$$

for all $x, y, z \in X$. Then, the function D_1 is a D-metric on X and (X, D_1) is a D-metric space.

Example1.2: [4] Let (X, d) be an ordinary metric space and define a function D_2 on X^3 by

$$D_2(x, y, z) = d(x, y) + d(y, z) + d(z, x)$$

for $x, y, z \in X$. Then, D_2 is a metric on X and (X, D_2) is a D-metric space.

Remark 1. *Geometrically, the D-metric D_1 represents the diameter of a set consisting of three points x, y and z in X and the D-metric $D_2(x, y, z)$ represents the perimeter of a triangle formed by three points x, y, z in X as its vertices.*

Definition 11. *(Cauchy sequence , completeness)[10] For every $m, n > N$. A sequence (x_n) in a metric space $X = (X, d)$ is said to be-Cauchy if for every $\varepsilon > 0$ there is an $N = N(\varepsilon)$ such that*

$$d(x_m, x_n) < \varepsilon.$$

The space X is said to be complete if every Cauchy sequence in X converges (that is, has a limit which is an element of X).

The following theorem can be found in [4]:

Theorem 2. *[4] Let d be an ordinary metric on X and let D_1 and D_2 be corresponding associated D-metrics on X. Then, (X, D_1) and (X, D_2) are complete if and only if (X, d) is complete.*

4 D-metrics in $\mathcal{B}^*_{\alpha,\,\log}$ and $F^*_{\log}(p,q,s)$

In this section, we introduce a D-metric on $\mathcal{B}^*_{\alpha,\,\log}$ and $F^*_{\log}(p,q,s)$.
Let $0 < p, s < \infty, -2 < q < \infty$, and $0 < \alpha < 1$. First, we can find a D-metric in $\mathcal{B}^*_{\alpha,\,\log}$, for $f,g,h \in \mathcal{B}^*_{\alpha,\,\log}$ by defining

$$D(f,g,h;\mathcal{B}^*_{\alpha,\,\log}) := D_{\mathcal{B}^*_{\alpha,\,\log}}(f,g,h) + ||f-g||_{\mathcal{B}_{\alpha,\,\log}} + ||g-h||_{\mathcal{B}_{\alpha,\,\log}} + ||h-f||_{\mathcal{B}_{\alpha,\,\log}}$$
$$+|f(0)-g(0)| + |g(0)-h(0)| + |h(0)-f(0)|,$$

where

$$D_{\mathcal{B}^*_{\alpha,\,\log}}(f,g,h) := d_{\mathcal{B}^*_{\alpha,\,\log}}(f,g) + d_{\mathcal{B}^*_{\alpha,\,\log}}(g,h) + d_{\mathcal{B}^*_{\alpha,\,\log}}(h,f)$$

and

$$D_{\mathcal{B}^*_{\alpha,\,\log}}(f,g,h) := \left(\sup_{z\in\mathbb{D}} |f^*(z)-g^*(z)| + \sup_{z\in\mathbb{D}} |g^*(z)-h^*(z)| + \sup_{z\in\mathbb{D}} |h^*(z)-f^*(z)|\right)$$
$$\times\left((1-|z|^2)^{\alpha}\left(\log\frac{2}{1-|z|^2}\right)\right).$$

Also, for $f,g,h \in F^*_{\log}(p,q,s)$ we introduce a D-metric on $F^*_{\log}(p,q,s)$ by defining

$$D(f,g,h;F^*_{\log}(p,q,s)) := D_{F^*_{\log}(p,q,s)}(f,g,h) + ||f-g||_{F_{\log}(p,q,s)} + ||g-h||_{F_{\log}(p,q,s)} +$$
$$||h-f||_{F_{\log}(p,q,s)} + |f(0)-g(0)| + |g(0)-h(0)| + |h(0)-f(0)|,$$

where

$$D_{F^*_{\log}(p,q,s)}(f,g,h) := d_{F^*_{\log}(p,q,s)}(f,g) + d_{F^*_{\log}(p,q,s)}(g,h) + d_{F^*_{\log}(p,q,s)}(h,f)$$

and

$$d_{F^*_{\log}(p,q,s)}(f,g) := \left(\sup_{z\in\mathbb{D}} \ell^p(a)\int_{\mathbb{D}} |f^*(z)-g^*(z)|^p(1-|z|^2)^q(1-|\varphi(z)|^2)^s dA(z)\right)^{\frac{1}{p}}.$$

Proposition 1. *The class $\mathcal{B}^*_{\alpha,\,\log}$ equipped with the D-metric $D(.,.;\mathcal{B}^*_{\alpha,\,\log})$ is a complete metric space. Moreover, $\mathcal{B}^*_{\alpha,\,\log,\,0}$ is a closed (and therefore complete) subspace of $\mathcal{B}^*_{\alpha,\,\log}$.*

Proof. Let $f,g,h,a \in \mathcal{B}^*_{\alpha,\,\log}$. Then, clearly

(i) $D(f,g,h;\mathcal{B}^*_{\alpha,\,\log}) \geq 0$, for all $f,g,h \in \mathcal{B}^*_{\alpha,\,\log}$.

(ii)$D(f,g,h;\mathcal{B}^*_{\alpha,\,\log}) = D(f,h,g;\mathcal{B}^*_{\alpha,\,\log}) = D(g,h,f;\mathcal{B}^*_{\alpha,\,\log})$.

(iii)$D(f,g,h;\mathcal{B}^*_{\alpha,\,\log}) \leq D(f,g,a;\mathcal{B}^*_{\alpha,\,\log}) + D(f,a,h;\mathcal{B}^*_{\alpha,\,\log}) + D(a,g,h;\mathcal{B}^*_{\alpha,\,\log})$

for all $f,g,h,a \in \mathcal{B}^*_{\alpha,\,\log}$.

(iv)$D(f,g,h;\mathcal{B}^*_{\alpha,\,\log}) = 0$ implies $f = g = h$.

Hence, D is a D-metric on $\mathcal{B}^*_{\alpha,\,\log}$, and $(\mathcal{B}^*_{\alpha,\,\log}, D)$ is D-metric space.

To prove the completeness, we use Theorem 2, let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in the metric space $(\mathcal{B}^*_{\alpha,\,\log}, d)$, that is, for any $\varepsilon > 0$ there is an $N = N(\varepsilon) \in \mathbb{N}$ such that $d(f_n, f_m;\mathcal{B}^*_{\alpha,\,\log}) < \varepsilon$, for all $n,m > N$. Since $(f_n) \subset B(\mathbb{D})$, the family (f_n) is uniformly bounded and hence normal in $\mathbb{D}$. Therefore, there exists $f \in B(\mathbb{D})$ and a subsequence $(f_{n_j})_{j=1}^{\infty}$ such that f_{n_j} converges to f uniformly on compact subsets of $\mathbb{D}$. It follows that f_n also converges to f uniformly on compact subsets, and by the Cauchy formula, the same also holds for the derivatives. Now let $m > N$. Then, the uniform convergence yields

$$\begin{aligned} & \Big| f^*(z) - f^*_m(z)\Big|(1-|z|^2)^{\alpha}\left(\log\frac{2}{1-|z|^2}\right) \\ = \ & \lim_{n\to\infty}\Big| f^*_n(z) - f^*_m(z)\Big|(1-|z|^2)^{\alpha}\left(\log\frac{2}{1-|z|^2}\right) \\ \leq \ & \lim_{n\to\infty} d(f_n, f_m;\mathcal{B}^*_{\alpha,\,\log}) \leq \varepsilon \end{aligned}$$

for all $z \in \mathbb{D}$, and it follows that $||f||_{\mathcal{B}^*_{\alpha,\,\log}} \leq ||f_m||_{\mathcal{B}^*_{\alpha,\,\log}} + \varepsilon$. Thus $f \in \mathcal{B}^*_{\alpha,\,\log}$ as desired. Moreover, the above inequality and the compactness of the usual $\mathcal{B}^*_{\alpha,\,\log}$ space imply that $(f_n)_{n=1}^{\infty}$ converges to f with respect to the metric d, and $(\mathcal{B}^*_{\alpha,\,\log}, D)$ is complete D-metric space.
Since $\lim_{n\to\infty} d(f_n, f_m;\mathcal{B}^*_{\alpha,\,\log}) \leq \varepsilon$, the second part of the assertion follows.

Next we give characterization of the complete D-metric space $D(.,.;F^*_{\log}(p,q,s))$.

Proposition 2. *The class $F^*_{\log}(p,q,s)$ equipped with the D-metric $D(.,.;F^*_{\log}(p,q,s))$ is a complete metric space. Moreover, $F^*_{\log,\,0}(p,q,s)$ is a closed (and therefore complete) subspace of $F^*_{\log}(p,q,s)$.*

Proof. Let $f,g,h,a \in F^*_{\log}(p,q,s)$. Then clearly

(i) $D(f,g,h;F^*_{\log}(p,q,s)) \geq 0$, for all $f,g,h \in F^*_{\log}(p,q,s)$.

(ii)$D(f,g,h;F^*_{\log}(p,q,s)) = D(f,h,g;F^*_{\log}(p,q,s)) = D(g,h,f;F^*_{\log}(p,q,s))$.

(iii)$D(f,g,h;F^*_{\log}(p,q,s)) \le D(f,g,a;F^*_{\log}(p,q,s)) + D(f,a,h;F^*_{\log}(p,q,s))$

$$+D(a,g,h;F^*_{\log}(p,q,s))$$

for all $f,g,h,a \in F^*_{\log}(p,q,s)$.

(iv)$D(f,g,h;F^*_{\log}(p,q,s)) = 0$ implies $f=g=h$.

Hence, D is a D-metric on $F^*_{\log}(p,q,s)$, and $(F^*_{\log}(p,q,s),D)$ is D-metric space.

For the complete proof, by using Theorem 2, let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in the metric space $(F^*_{\log}(p,q,s),d)$, that is, for any $\varepsilon>0$ there is an $N=N(\varepsilon)\in\mathbb{N}$ so that $d(f_n,f_m;F^*_{\log}(p,q,s))<\varepsilon$, for all $n,m>N$. Since $(f_n)\subset B(\mathbb{D})$, such that f_{n_j} converges to f uniformly on compact subsets of $\mathbb{D}$. It follows that f_n also converges to f uniformly on compact subsets, now let $m>N$, and $0<r<1$. Then, the Fatou's yields

$$\begin{aligned}
&\int_{\mathbb{D}(0,r)}\Big|f^*(z)-f^*_m(z)\Big|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)\\
=\;&\int_{\mathbb{D}(0,r)}\lim_{n\to\infty}\Big|f^*_n(z)-f^*_m(z)\Big|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)\\
\le\;&\lim_{n\to\infty}\int_{\mathbb{D}(0,r)}\Big|f^*(z)-f^*_m(z)\Big|^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)\le\varepsilon^p,
\end{aligned}$$

and by taking $r\to 1^-$, it follows that,

$$\begin{aligned}
&\int_{\mathbb{D}}(f^*(z))^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z)\\
&\le 2^p\varepsilon^p+2^p\int_{\mathbb{D}}(f^*_m(z))^p(1-|z|^2)^q(1-|\varphi_a(z)|^2)^s dA(z).
\end{aligned}$$

This yields

$$||f||^p_{F^*_{\log}(p,q,s)}\le 2^p||f_m||^p_{F^*_{\log}(p,q,s)}+2^p\varepsilon^p.$$

And thus $f\in F^*_{\log}(p,q,s)$. We also find that $f_n\to f$ with respect to the metric of $(F^*_{\log}(p,q,s),D)$ and $(F^*_{\log}(p,q,s),D)$ is complete D-metric space. The second part of the assertion follows.

5 Composition operators of $C_\phi:\mathcal{B}^*_{\alpha,\log}\to F^*_{\log}(p,q,s)$

In this section, we study boundedness and compactness of composition operators on $\mathcal{B}^*_{\alpha,\log}$ and $F^*_{\log}(p,q,s)$ spaces. We need the following notation:

$$\Phi_\phi(\alpha,p,s;a)=\ell^p(a)\int_{\mathbb{D}}|\phi'(z)|^p\frac{(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s}{(1-|\phi(z)|^2)^{\alpha p}\Big(\log\frac{2}{(1-|\phi(z)|^2)}\Big)^p}\,dA(z),$$

where $\ell^p(a) = \left(\log \frac{2}{1-|a|^2}\right)^p$.

For $0 < \alpha < 1$, we suppose there exist two functions $f, g \in \mathcal{B}^*_{\alpha,\,\log}$ such that for some constant C,

$$(|f^*(z)| + |g^*(z)|) \geq \frac{C}{(1-|z|^2)^{\alpha}\left(\log \frac{2}{1-|a|^2}\right)^p} > 0, \qquad for\ each\ z\ \in\ \mathbb{D}.$$

Now, we provide the following theorem:

Theorem 3. *Assume ϕ is a holomorphic mapping from $\mathbb{D}$ into itself and let $0 < p, 1 < s < \infty, 0 < \alpha \leq 1$. Then the induced composition operator C_ϕ maps $\mathcal{B}^*_{\alpha,\,\log}$ into $F^*_{\log}(p, \alpha p - 2, s)$ is bounded if and only if,*

$$\sup_{z\in\mathbb{D}} \Phi_\phi(\alpha, p, s; a) < \infty. \tag{5.1}$$

Proof. First assume that $\sup_{z\in\mathbb{D}} \Phi_\phi(\alpha, p, s; a) < \infty$ is held, and $f \in \mathcal{B}^*_{\alpha,\,\log}$ with $||f||_{\mathcal{B}_{\alpha,\,\log}} \leq 1$, we can see that

$$\begin{aligned}
& ||C_\phi f||^p_{F^*_{\log}(p,\alpha p-2,s)} \\
= \quad & \sup_{a\in\mathbb{D}} \ell^p(a) \int_{\mathbb{D}} ((f\circ\phi)^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
= \quad & \sup_{a\in\mathbb{D}} \ell^p(a) \int_{\mathbb{D}} (f^*(\phi(z)))^p |\phi'(z)|^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
\leq \quad & ||f||^p_{\mathcal{B}^*_{\alpha,\,\log}} \sup_{a\in\mathbb{D}} \ell^p(a) \int_{\mathbb{D}} \frac{|\phi'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s}{(1-|\phi(z)^2|)^{p\alpha}(\log \frac{2}{1-|z|^2})} dA(z) \\
= \quad & ||f||^p_{\mathcal{B}^*_{\alpha,\,\log}} \Phi_\phi(\alpha, p, s; a) < \infty.
\end{aligned}$$

For the other direction, we use the fact that for each function $f \in \mathcal{B}^*_{\alpha,\,\log}$, the analytic function

$C_\phi(f) \in F^*_{\log}(p, \alpha p-2, s)$. Then, using the functions of lemma 1.2

$$\begin{aligned}
& 2^p\Big\{||C_\phi f_1||^p_{F^*_{\log}(p,\alpha p-2,s)} + ||C_\phi f_2||^p_{F^*_{\log}(p,\alpha p-2,s)}\Big\} \\
= \; & 2^p\Big\{\sup_{a\in\mathbb{D}} \ell^p(a)\int_{\mathbb{D}}\Big[((f_1\circ\phi)^*(z))^p + ((f_2\circ\phi)^*(z))^p\Big] \\
& \times(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\Big\} \\
\geq \; & \Big\{\sup_{a\in\mathbb{D}} \ell^p(a)\int_{\mathbb{D}}\Big[(f_1\circ\phi)^*(z) + (f_2\circ\phi)^*(z)\Big]^p \\
& \times(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\Big\} \\
\geq \; & \Big\{\sup_{a\in\mathbb{D}} \ell^p(a)\int_{\mathbb{D}}\Big[(f_1^*(\phi))(z) + (f_2^*(\phi))(z)\Big]^p \\
& \times|\phi'(z)|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\Big\} \\
\geq \; & C\Big\{\sup_{a\in\mathbb{D}} \ell^p(a)\int_{\mathbb{D}}|\phi'(z)|^p\frac{(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s}{(1-|\phi(z)|^2)^{\alpha p}\Big(\log\frac{2}{(1-|\phi(z)|^2)}\Big)^p}dA(z)\Big\} \\
\geq \; & C\sup_{a\in\mathbb{D}}\Phi_\phi(\alpha,p,s;a).
\end{aligned}$$

Hence C_ϕ is bounded, the proof is completed.

The composition operator $C_\phi : \mathcal{B}^*_{\alpha,\,\log} \to F^*_{\log}(p, \alpha p-2, s)$ is compact if and only if for every sequence $f_n \in \mathbb{N} \subset F^*_{\log}(p, \alpha p-2, s)$ is bounded in $F^*_{\log}(p, \alpha p-2, s)$ norm and$f_n \to 0$, $n \to \infty$, uniformly on compact subset of the unit disk (where $\mathbb{N}$ be the set of all natural numbers), hence,

$$||C_\phi(f_n)||_{F^*_{\log}(p,\alpha p-2,s)} \to 0, n \to \infty.$$

Now, we describe compactness in the following result:

Theorem 4. *Let $0 < p, 1 < s < \infty, \alpha < \infty$. If ϕ is an analytic self-map of the unit disk, then the induced composition operator $C_\phi : \mathcal{B}^*_{\alpha,\,\log} \to F^*_{\log}(p, \alpha p-2, s)$ is compact if and only if $\phi \in F^*_{\log}(p, \alpha p-2, s)$, and*

$$\lim_{r\to 1}\sup_{a\in\mathbb{D}}\Phi_\phi(\alpha,p,s;a) \to 0. \tag{5.2}$$

Proof. Let $C_\phi : \mathcal{B}^*_{\alpha,\,\log} \to F^*_{\log}(p, \alpha p-2, s)$ be compact. This means that
$\phi \in F^*_{\log}(p, \alpha p-2, s)$.

Let

$$U^1_r = \{z : |\phi(z)| > r, r \in (0,1)\},$$

and

$$U_r^2 = \{z : |\phi(z)| \le r, r \in (0,1)\}.$$

Let $f_n(z) = \frac{z^n}{n}$ if $\alpha \in [0,\infty)$ or $f_n(z) = \frac{z^n}{n^{1-\alpha}}$ if $\alpha \in (0,1)$. Without loss of generality, we only consider $\alpha \in (0,1)$. Since $||f_n||_{\mathcal{B}^*_{\alpha,\,\log}} \le M$ and $f_n(z) \to 0$ as $n \to \infty$, locally uniformly on the unit disk, then $||C_\phi(f_n)||_{F^*_{\log}(p,\alpha p-2,s)}$, $n \to \infty$. This means that for each $r \in (0,1)$ and for all $\varepsilon > 0$, there exist $N \in \mathbb{N}$ so that if $n \ge N$, then

$$\frac{N^{\alpha p}}{r^{p(1-N)}} \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} |\phi'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon.$$

If we choose r so that $\frac{N^{\alpha p}}{r^{p(1-N)}} = 1$, then

$$\sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} |\phi'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon. \tag{5.3}$$

Let now f be with $||f||_{\mathcal{B}^*_{\alpha,\,\log}} \le 1$. We consider the functions $f_t(z) = f(tz)$, $t \in (0,1)$. $f_t \to f$ uniformly on compact subset of the unit disk as $t \to 1$ and the family (f_t) is bounded on $\mathcal{B}^*_{\alpha,\,\log}$, thus

$$||(f_t \circ \phi\ ) - (f \circ \phi\ )|| \to 0.$$

Due to compactness of C_ϕ, we get that for $\varepsilon > 0$ there is $t \in (0,1)$ so that

$$\sup_{a\in\mathbb{D}} \ell^p(a) \int_{\mathbb{D}} |F_t(\phi(z))|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) < \varepsilon,$$

where

$$F_t(\phi(z)) = \big[(f \circ \phi\ )^* - (f_t \circ \phi\ )^*\big].$$

Thus, if we fix t, then

$$\begin{aligned}
& \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} ((f\circ\phi)^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
\le\ & 2^p \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} |F_t(\phi(z))|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
& + 2^p \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} ((f_t\circ\phi)^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
\le\ & 2^p\varepsilon + ||f_t^*||^p_{H^\infty} \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} |\phi'(z)|^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\
\le\ & 2^p\varepsilon + 2^p\varepsilon ||f_t^*||^p_{H^\infty}.
\end{aligned}$$

i.e,

$$\begin{aligned}&\sup_{a\in\mathbb{D}}\ell^p(a)\int_{U_r^1}((f\circ\phi)^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\ \leq\ & 2^p\varepsilon(1+||f_t^*||^p_{H^\infty}),\end{aligned}\tag{5.4}$$

where we have used (4). On the other hand, for each $||f||_{\mathcal{B}^*_{\alpha,\ \log}}\leq 1$ and $\varepsilon>0$, there exists a δ depending on f and ε, so that for $r\in[\delta,1)$,

$$\sup_{a\in\mathbb{D}}\ell^p(a)\int_{U_r^1}((f\circ\phi)^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)<\varepsilon.\tag{5.5}$$

Since C_ϕ is compact, then it maps the unit ball of $\mathcal{B}^*_{\alpha,\ \log}$ to a relatively compact subset of $F^*_{\log}(p,q,s)$. Thus, for each $\varepsilon>0$, there exists a finite collection of functions $f_1,f_2,...,f_n$ in the unit ball of $\mathcal{B}^*_{\alpha,\ \log}$ so that for each $||f||_{\mathcal{B}^*_{\alpha,\ \log}}$, there is $k\in\{1,2,3,...,n\}$ so that

$$\sup_{a\in\mathbb{D}}\ell^p(a)\int_{U_r^1}|F_k(\phi(z))|^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)<\varepsilon,$$

where

$$F_k(\phi(z))=\big[(f\circ\phi\ )^*-(f_k\circ\phi\ )^*\big].$$

Also, using (5), we get for $\delta=max_{1\leq k\leq n}\delta(f_k,\varepsilon)$ and $r\in[\delta,1)$, that

$$\sup_{a\in\mathbb{D}}\ell^p(a)\int_{U_r^1}((f_k\circ\phi)^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)<\varepsilon.$$

Hence, for any $f,||f||_{\mathcal{B}^*_{\alpha,\ \log}}\leq 1$, combining the two relations as above, we get the following

$$\sup_{a\in\mathbb{D}}\ell^p(a)\int_{U_r^1}((f\circ\phi)^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\leq 2^p\varepsilon.$$

Therefore, we get that (2) holds. For the sufficiency, we use that $\phi\in F^*_{\log}(p,\alpha p-2,s)$ and (2) holds.

Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of functions in the unit ball of $\mathcal{B}^*_{\alpha,\ \log}$ so that $f_n\to 0$ as $n\to\infty$, uniformly on the compact subsets of the unit disk. Let also $r\in(0,1)$. Then,

$$\begin{aligned}&||f_n\circ\phi||^p_{F^*_{\log}(p,\alpha p-2,s)}\leq 2^p|f_n(\phi(0))|\\ &+2^p\sup_{a\in\mathbb{D}}\ell^p(a)\int_{U_r^2}((f_n\circ\phi)^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\ &+2^p\sup_{a\in\mathbb{D}}\ell^p(a)\int_{U_r^1}((f_n\circ\phi)^*(z))^p(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z)\\ =\ & 2^p(I_1+I_2+I_3).\end{aligned}$$

Since $f_n \to 0$ as $n \to \infty$, locally uniformly on the unit disk, then $I_1 = |f_n(\phi(0))|$ goes to zero as $n \to \infty$ and for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ so that for each $n > N$,

$$\begin{aligned} I_2 &= \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^2} ((f_n \circ \phi)^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\ &\leq \varepsilon ||\phi||^p_{F^*_{\log}(p,\alpha p-2,s)}. \end{aligned}$$

We also observe that

$$\begin{aligned} I_3 &= \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} ((f_n \circ \phi)^*(z))^p (1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s dA(z) \\ &\leq ||f||^p_{\mathcal{B}^*_{\alpha,\,\log}} \\ &\quad \times \sup_{a\in\mathbb{D}} \ell^p(a) \int_{U_r^1} |\phi'(z)|^p \frac{(1-|z|^2)^{\alpha p-2}(1-|\varphi_a(z)|^2)^s}{(1-|\phi(z)|^2)^{\alpha p}\left(\log \frac{2}{(1-|\phi(z)|^2)}\right)^p} dA(z). \end{aligned}$$

Under the assumption that (2) holds, then for every $n > N$ and for every $\varepsilon > 0$, there exists r_1 so that for every $r > r_1$, $I_3 < \varepsilon$.
Thus, if $\phi(z) \in F^*_{\log}(p, \alpha p - 2, s)$, we get

$$||f_n \circ \phi||^p_{F^*_{\log}(p,\alpha p-2,s)} \leq 2^p \left\{ 0 + \varepsilon ||\phi||^p_{F^*_{\log}(p,\alpha p-2,s)} + \varepsilon \right\} \leq C\varepsilon.$$

Combining the above, we get $||C_\phi(f_n)||^p_{F^*_{\log}(p,\alpha p-2,s)} \to 0$ as $n \to \infty$ which proves compactness. Thus, the theorem we presented is proved.

6 Conclusions

We have obtained some essential and important D-metric spaces. Moreover, the important properties for D-metric on $\mathcal{B}^*_{\alpha,\,\log}$ and $F^*_{\log}(p, q, s)$ are investigated in Section 4. Finally, we introduced composition operators in hyperbolic weighted family of function spaces.

7 Acknowledgements

The authors would like to thank the referees for their useful comments, which improved the original manuscript.

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