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Results on para-Sasakian manifold admitting a quarter symmetric metric connection

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ABSTRACT

In this paper we have studied pseudosymmetric, Ricci-pseudosymmetric and projectively pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection and constructed examples of 3-dimensional and 5-dimensional para-Sasakian manifold admitting a quarter-symmetric metric connection to verify our results.

RESUMEN

En este artículo hemos estudiado variedades para-Sasakianas seudosimétricas, Ricci-seudosimétricas y proyectivamente seudosimétricas que admiten una conexión métrica cuarto-simétrica, y construimos ejemplos de variedades para-Sasakianas 3-dimensional y 5-dimensional que admiten una conexión métrica cuarto-simétrica para verificar nuestros resultados.

Keywords and Phrases: Para-Sasakian manifold, pseudosymmetric, Ricci-pseudosymmetric, projectively pseudosymmetric, quarter-symmetric metric connection.

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1 Introduction

One of the most important geometric property of a space is symmetry. Spaces admitting some sense of symmetry play an important role in differential geometry and general relativity. Cartan [5] introduced locally symmetric spaces, i.e., the Riemannian manifold (M, g) for which $\nabla R = 0$, where ∇ denotes the Levi-Civita connection of the metric. The integrability condition of $\nabla R = 0$ is $R \cdot R = 0$. Thus, every locally symmetric space satisfies $R \cdot R = 0$, whereby the first R stands for the curvature operator of (M, g), i.e., for tangent vector fields X and Y one has $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$, which acts as a derivation on the second R which stands for the Riemann-Christoffel curvature tensor. The converse however does not hold in general. The spaces for which $R \cdot R = 0$ holds at every point were called semi-symmetric spaces and which were classified by Szabo [19].

Semisymmetric manifolds form a subclass of the class of pseudosymmetric manifolds. In some spaces $R \cdot R$ is not identically zero, these turn out to be the pseudo-symmetric spaces of Deszcz [9, 10, 11], which were characterized by the condition $R \cdot R = L\, Q(g, R)$, where L is a real function on M and $Q(g, R)$ is the Tachibana tensor of M.

If at every point of M the curvature tensor satisfies the condition

$$R(X, Y) \cdot \mathcal{J} = L_{\mathcal{J}}[(X \wedge_g Y) \cdot \mathcal{J}], \tag{1.1}$$

then a Riemannian manifold M is called pseudosymmetric (resp., Ricci-pseudosymmetric, projectively pseudosymmetric) when $\mathcal{J} = R(resp., S, P)$. Here $(X \wedge_g Y)$ is an endomorphism and is defined by $(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y$ and $L_{\mathcal{J}}$ is some function on $U_{\mathcal{J}} = \{x \in M : \mathcal{J} \neq 0\}$ at x. A geometric interpretation of the notion of pseudosymmetry is given in [13]. It is also easy to see that every pseudosymmetric manifold is Ricci-pseudosymmetric, but the converse is not true.

An analogue to the almost contact structure, the notion of almost paracontact structure was introduced by Sato [18]. An almost contact manifold is always odd-dimensional but an almost paracontact manifold could be of even dimension as well. Kaneyuki and Williams [14] studied the almost paracontact structure on a pseudo-Riemannian manifold. Recently, almost paracontact geometry in particular, para-Sasakian geometry has taking interest, because of its interplay with the theory of para-Kahler manifolds and its role in pseudo-Riemannian geometry and mathematical physics ([4, 7, 8], etc.,).

As a generalization of semi-symmetric connection, quarter-symmetric connection was introduced. Quarter-symmetric connection on a differentiable manifold with affine connection was defined and studied by Golab [12]. From thereafter many geometers studied this connection on different manifolds.

Para-Sasakian manifold with respect to quarter-symmetric metric connection was studied by

De et.al., [16, 1], Pradeep Kumar et.al., [17] and Bisht and Shanker [15].

Motivated by the above studies in this article we study properties of projective curvature tensor on para-Sasakian manifold admitting a quarter-symmetric metric connection. The organization of the paper is as follows: In Section 2, we present some basic notions of para-Sasakian manifold and quarter-symmetric metric connection on it. Section 3 and 4 are respectively devoted to study the pseudosymmetric and Ricci-pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection. Here we prove that if a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is Pseudosymmetric (resp., Ricci pseudosymmetric) then M^n is an Einstein manifold with respect to quarter-symmetric metric connection or it satisfies $L_{\tilde{R}} = -2$ (resp., $L_{\tilde{S}} = -2$). Section 5 and 6 are concerned with projectively flat and projectively pseudosymmetric para-Sasakian manifold M^n admitting a quarter-symmetric metric connection. Finally, we construct examples of 3-dimensional and 5-dimensional para-Sasakian manifold admitting a quarter-symmetric metric connection and we find some of its geometric characteristics.

2 Preliminaries

A differential manifold M^n is said to admit an almost paracontact Riemannian structure (ϕ, ξ, η, g), where ϕ is a tensor field of type (1, 1), ξ is a vector field, η is a 1-form and g is a Riemannian metric on M^n such that

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for all vector fields $X, Y \in \chi(M^n)$. If (ϕ, ξ, η, g) on M^n satisfies the following equations

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.3}$$

$$d\eta = 0 \quad and \quad \nabla_X \xi = \phi X, \tag{2.4}$$

then M^n is called para-Sasakian manifold [3].

In a para-Sasakian manifold, the following relations hold [6]:

$$(\nabla_X \eta)Y = -g(X, Y) + \eta(X)\eta(Y), \tag{2.5}$$

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X), \tag{2.6}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.7}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.8}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.9}$$

for every vector fields X, Y, Z on M^n. Here ∇ denotes the Levi-Civita connection, R denotes the Riemannian curvature tensor and S denotes the Ricci curvature tensor.

Here we consider a quarter-symmetric metric connection $\tilde{\nabla}$ on a para-Sasakian manifold [16] given by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi X - g(\phi X, Y)\xi. \tag{2.10}$$

The relation between curvature tensor $\tilde{R}(X,Y)Z$ of M^n with respect to quarter-symmetric metric connection $\tilde{\nabla}$ and the curvature tensor $R(X,Y)Z$ with respect to the Levi-Civita connection ∇ is given by

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + 3g(\phi X, Z)\phi Y - 3g(\phi Y, Z)\phi X \\ &+\{\eta(X)Y - \eta(Y)X\}\eta(Z) - [g(Y,Z)\eta(X) - \eta(Y)g(X,Z)]\xi.\end{aligned} \tag{2.11}$$

Also from (2.11) we obtain

$$\tilde{S}(Y,Z) = S(Y,Z) + 2g(Y,Z) - (n+1)\eta(Y)\eta(Z) - 3trace\phi\, g(\phi Y, Z), \tag{2.12}$$

where $\tilde{S}$ and S are Ricci tensors of connections $\tilde{\nabla}$ and ∇ respectively.

3 Pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection

A para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is said to be pseudosymmetric if

$$\tilde{R}(X,Y)\cdot\tilde{R} = L_{\tilde{R}}[(X \wedge_g Y)\cdot\tilde{R}], \tag{3.1}$$

holds on the set $U_{\tilde{R}} = \{x \in M^n : \tilde{R} \neq 0 \ at\ x\}$, where $L_{\tilde{R}}$ is some function on $U_{\tilde{R}}$.

Suppose that M^n be pseudosymmetric, then in view of (3.1) we have

$$\begin{aligned}&\tilde{R}(\xi,Y)\tilde{R}(U,V)W - \tilde{R}(\tilde{R}(\xi,Y)U,V)W - \tilde{R}(U,\tilde{R}(\xi,Y)V)W \\ &-\tilde{R}(U,V)\tilde{R}(\xi,Y)W = L_{\tilde{R}}[(\xi \wedge_g Y)\tilde{R}(U,V)W - \tilde{R}((\xi\wedge_g Y)U,V)W \\ &-\tilde{R}(U,(\xi\wedge_g Y)V)W - \tilde{R}(U,V)(\xi\wedge_g Y)W].\end{aligned} \tag{3.2}$$

By virtue of (2.7) and (2.11), (3.2) takes the form

$$\begin{aligned}&(L_{\tilde{R}}+2)[\eta(\tilde{R}(U,V)W)Y - g(Y,\tilde{R}(U,V)W)\xi - \eta(U)\tilde{R}(Y,V)W + g(Y,U)\tilde{R}(\xi,V)W \\ &-\eta(V)\tilde{R}(U,Y)W + g(Y,V)\tilde{R}(U,\xi)W - \eta(W)\tilde{R}(U,V)Y + g(Y,W)\tilde{R}(U,V)\xi] = 0.\end{aligned} \tag{3.3}$$

Taking inner product of (3.3) with ξ and using (2.6) and (2.11), we get

$$\begin{aligned}&(L_{\tilde{R}}+2)[g(Y,R(U,V)W) + 3g(\phi U,W)g(\phi V,Y) - 3g(\phi V,W)g(\phi U,Y) \\ &+\eta(W)\{\eta(U)g(V,Y) - \eta(V)g(U,Y)\} - \{g(V,W)\eta(U) - \eta(V)g(U,W)\}\eta(Y) \\ &+2\{g(V,W)g(Y,U) - g(V,Y)g(U,W)\}] = 0.\end{aligned} \tag{3.4}$$

Assuming that $L_{\tilde{R}} + 2 \neq 0$, the above equation becomes

$$\begin{aligned} g(Y, R(U,V)W) + 3g(\phi U, W)g(\phi V, Y) - 3g(\phi V, W)g(\phi U, Y) \\ +\eta(W)\{\eta(U)g(V,Y) - \eta(V)g(U,Y)\} - [g(V,W)\eta(U) - \eta(V)g(U,W)]\eta(Y) \\ +2[g(V,W)g(Y,U) - g((V,Y)g(U,W)] = 0. \end{aligned} \tag{3.5}$$

Putting $V = W = e_i$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over i, $i = 1, 2, 3, \cdots, n$, we get

$$\tilde{S}(Y,U) = -2(n-1)g(Y,U). \tag{3.6}$$

Hence, we can state the following:

Theorem 1. *If a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is pseudosymmetric then M^n is an Einstein manifold with respect to quarter-symmetric metric connection or it satisfies $L_{\tilde{R}} = -2$.*

4 Ricci-pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection

A para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is said to be Ricci-pseudosymmetric if the following condition is satisfied

$$\tilde{R}(X,Y) \cdot \tilde{S} = L_{\tilde{S}}[(X \wedge_g Y) \cdot \tilde{S}], \tag{4.1}$$

on $U_{\tilde{S}}$.

Let para-Sasakian manifold M^n admitting a quarter-symmetric metric connection be Ricci-pseudosymmetric. Then we have

$$\tilde{S}(\tilde{R}(X,Y)Z,W) + \tilde{S}(Z,\tilde{R}(X,Y)W) = L_{\tilde{S}}[\tilde{S}((X \wedge_g Y)Z,W) + \tilde{S}(Z,(X \wedge_g Y)W)]. \tag{4.2}$$

By taking $Y = W = \xi$ and making use of (2.7), (2.8) and (2.11), the above equation turns into

$$(L_{\tilde{S}} + 2)[\tilde{S}(X,Z) + 2(n-1)g(X,Z)] = 0 \tag{4.3}$$

Thus, we have the following assertion:

Theorem 2. *If a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is Ricci-pseudosymmetric then M^n is an Einstein manifold with respect to quarter-symmetric metric connection or it satisfies $L_{\tilde{S}} = -2$.*

5 Projectively flat para-Sasakian manifold admitting a quarter-symmetric metric connection

The projective curvature tensor on a Riemannian manifold is defined by [2]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - S(X,Z)Y]. \tag{5.1}$$

For an n-dimensional para-Sasakian manifold M^n admitting a quarter-symmetric metric connection, the projective curvature tensor is given by

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{(n-1)}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y]. \tag{5.2}$$

Theorem 3. *A projectively flat para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is an Einstein manifold with respect to quarter-symmetric metric connection.*

Proof. Consider a projectively flat para-Sasakian manifold admitting a quarter-symmetric metric connection. Then from (5.2) we have

$$g(\tilde{R}(X,Y)Z,W) = \frac{1}{(n-1)}[\tilde{S}(Y,Z)g(X,W) - \tilde{S}(X,Z)g(Y,W)]. \tag{5.3}$$

Setting $X = W = \xi$ in (5.3) and using (2.7), (2.8), (2.11) and (2.12), we get

$$\tilde{S}(X,Z) = -2(n-1)g(X,Z). \tag{5.4}$$

Hence, the proof is completed. □

6 Projectively pseudosymmetric para-Sasakian manifold admitting a quarter-symmetric metric connection

A para-Sasakian manifold admitting a quarter-symmetric metric connection is said to be projectively pseudosymmetric if

$$\tilde{R}(X,Y)\cdot\tilde{P} = L_{\tilde{P}}[(X\wedge_g Y)\cdot\tilde{P}], \tag{6.1}$$

holds on the set $U_{\tilde{P}} = \{x \in M^n : \tilde{P} \neq 0 \text{ at } x\}$, where $L_{\tilde{P}}$ is some function on $U_{\tilde{P}}$.

Let M^n be projectively pseudosymmetric, then we have

$$\begin{aligned}\tilde{R}(X,\xi)\tilde{P}(U,V)\xi - \tilde{P}(\tilde{R}(X,\xi)U,V)\xi - \tilde{P}(U,\tilde{R}(X,\xi)V)\xi \\ -\tilde{P}(U,V)\tilde{R}(X,\xi)\xi = L_{\tilde{P}}[(X\wedge_g\xi)\tilde{P}(U,V)\xi - \tilde{P}((X\wedge_g\xi)U,V)\xi \\ -\tilde{P}(U,(X\wedge_g\xi)V)\xi - \tilde{P}(U,V)(X\wedge_g\xi)\xi].\end{aligned} \tag{6.2}$$

By virtue of (2.11), (2.12) and (5.2), (6.2) becomes

$$(L_{\tilde{P}} + 2)\tilde{P}(U,V)X = 0. \tag{6.3}$$

So, one can state that:

Theorem 4. *If a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is projectively pseudosymmetric then M^n is projectively flat with respect to quarter-symmetric metric connection or $L_{\tilde{P}} = -2$.*

In view of theorem 3, one can state the above theorem as

Theorem 5. *If a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is projectively pseudosymmetric then M^n is an Einstein manifold with respect to quarter-symmetric metric connection or $L_{\tilde{P}} = -2$.*

7 Examples

7.1 Example

We consider a 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3 : z \neq 0\}$, where (x,y,z) are standard coordinates in $\mathbb{R}^3$. Let $\{E_1, E_2, E_3\}$ be a linearly independent global frame field on M given by

$$E_1 = e^z \frac{\partial}{\partial y}, \qquad E_2 = e^z(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}), \qquad E_3 = \frac{\partial}{\partial z},$$

If g is a Riemannian metric defined by

$$g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for $1 \leq i, j \leq 3$, and if η is the 1-form defined by $\eta(Z) = g(Z, E_3)$ for any vector field $Z \in \chi(M)$. We define the (1, 1)-tensor field ϕ as

$$\phi(E_1) = E_1, \quad \phi(E_2) = -E_2, \quad \phi(E_3) = 0.$$

The linearity property of ϕ and g yields that

$$\eta(E_3) = 1,$$
$$\phi^2 U = U - \eta(U)E_3,$$
$$g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V),$$

for any $U, V \in \chi(M)$.

Now we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = E_2.$$

The Riemannian connection ∇ of the metric g known as Koszul's formula and is given by

$$\begin{aligned}2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ -g(Y, [X, Z]) + g(Z, [X, Y]).\end{aligned}$$

Using Koszul's formula we get the followings in matrix form

$$\begin{pmatrix} \nabla_{E_1} E_1 & \nabla_{E_1} E_2 & \nabla_{E_1} E_3 \\ \nabla_{E_2} E_1 & \nabla_{E_2} E_2 & \nabla_{E_2} E_3 \\ \nabla_{E_3} E_1 & \nabla_{E_3} E_2 & \nabla_{E_3} E_3 \end{pmatrix} = \begin{pmatrix} -E_3 & 0 & E_1 \\ 0 & -E_3 & E_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly (ϕ, ξ, η, g) is a para-Sasakian structure on M. Thus $M(\phi, \xi, \eta, g)$ is a 3-dimensional para-Sasakian manifold.

Using (2.10) and the above equation, one can easily obtain the following:

$$\begin{pmatrix} \tilde{\nabla}_{E_1} E_1 & \tilde{\nabla}_{E_1} E_2 & \tilde{\nabla}_{E_1} E_3 \\ \tilde{\nabla}_{E_2} E_1 & \tilde{\nabla}_{E_2} E_2 & \tilde{\nabla}_{E_2} E_3 \\ \tilde{\nabla}_{E_3} E_1 & \tilde{\nabla}_{E_3} E_2 & \tilde{\nabla}_{E_3} E_3 \end{pmatrix} = \begin{pmatrix} -2E_3 & 0 & 2E_1 \\ 0 & -2E_3 & 2E_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

With the help of the above results it can be easily verified that

$$\begin{aligned}
&R(E_1, E_2)E_3 = 0, && R(E_2, E_3)E_3 = -E_2, && R(E_1, E_3)E_3 = -E_1,\\
&R(E_1, E_2)E_2 = -E_1, && R(E_2, E_3)E_2 = E_3, && R(E_1, E_3)E_2 = 0,\\
&R(E_1, E_2)E_1 = E_2, && R(E_2, E_3)E_1 = 0, && R(E_1, E_3)E_1 = E_3.
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{R}(E_1, E_2)E_3 = 0, && \tilde{R}(E_2, E_3)E_3 = -2E_2, && \tilde{R}(E_1, E_3)E_3 = -2E_1,\\
&\tilde{R}(E_1, E_2)E_2 = -4E_1, && \tilde{R}(E_2, E_3)E_2 = 2E_3, && \tilde{R}(E_1, E_3)E_2 = 0,\\
&\tilde{R}(E_1, E_2)E_1 = 4E_2, && \tilde{R}(E_2, E_3)E_1 = 0, && \tilde{R}(E_1, E_3)E_1 = 2E_3.
\end{aligned} \tag{7.1}$$

Since E_1, E_2, E_3 forms a basis, any vector field $X, Y, Z \in \chi(M)$ can be written as $X = a_1E_1 + b_1E_2 + c_1E_3$, $Y = a_2E_1 + b_2E_2 + c_2E_3$, $Z = a_3E_1 + b_3E_2 + c_3E_3$, where $a_i, b_i, c_i \in \mathbb{R}$, $i = 1, 2, 3$. Using the expressions of the curvature tensors, we find values of Riemannian curvature and Ricci curvature with respect to quarter-symmetric metric connection as;

$$\begin{aligned}\tilde{R}(X,Y)Z &= [-4\{a_1b_2-b_1a_2\}b_3+2\{c_1a_2-a_1c_2\}c_3]E_1 \\ &+ [-4\{b_1a_2-a_1b_2\}a_3+2\{c_1b_2-b_1c_2\}c_3]E_2 \\ &+ [-2\{c_1a_2-a_1c_2\}a_3-2\{c_1b_2-b_1c_2\}b_3]E_3, \qquad (7.2)\\ \tilde{S}(E_1,E_1) &= \tilde{S}(E_2,E_2)=-6,\ \tilde{S}(E_3,E_3)=-4. \qquad (7.3)\end{aligned}$$

Using (7.1), (7.3) and the expression of the endomorphism $(X\wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y$, one can easily verify that

$$\tilde{S}(\tilde{R}(X,E_3)Y,E_3)+\tilde{S}(Y,\tilde{R}(X,E_3)E_3) = -2[\tilde{S}((X\wedge_g E_3)Y,E_3)+\tilde{S}(Y,(X\wedge_g E_3)E_3)], \qquad (7.4)$$

here $L_{\tilde{S}}=-2$. Thus, the above equation verify one part of the Theorem 2 of section 4.

Moreover, the manifold under consideration satisfies

$$\tilde{R}(X,Y)Z = -\tilde{R}(Y,X)Z,$$
$$\tilde{R}(X,Y)Z+\tilde{R}(Y,Z)X+\tilde{R}(Z,X)Y=0.$$

Hence, from the above equations one can say that this example verifies the condition (c) of Theorem 3.1 in [1] and first Bianchi identity.

7.2 Example

We consider a 5-dimensional manifold $M=\{(x_1,x_2,x_3,x_4,x_5)\in\mathbb{R}^5\}$, where (x_1,x_2,x_3,x_4,x_5) are standard coordinates in $\mathbb{R}^5$. We choose the vector fields

$$E_1=\frac{\partial}{\partial x_1},\quad E_2=\frac{\partial}{\partial x_2},\quad E_3=\frac{\partial}{\partial x_3},\quad E_4=\frac{\partial}{\partial x_4},\quad E_5=x_1\frac{\partial}{\partial x_1}+x_2\frac{\partial}{\partial x_2}+x_3\frac{\partial}{\partial x_3}+x_4\frac{\partial}{\partial x_4}+\frac{\partial}{\partial x_5},$$

which are linearly independent at each point of M.

Let g be a Riemannian metric defined by

$$g(E_i,E_j)=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}$$

for $1\le i,j\le 5$, and if η is the 1-form defined by $\eta(Z)=g(Z,E_5)$ for any vector field $Z\in\chi(M)$. Let ϕ be the (1, 1)-tensor field defined by

$$\phi(E_1)=E_1,\quad \phi(E_2)=E_2,\quad \phi(E_3)=E_3,\quad \phi(E_4)=E_4,\quad \phi(E_5)=0.$$

The linearity property of ϕ and g yields that

$$\eta(E_5) = 1,$$
$$\phi^2 U = U - \eta(U)E_5,$$
$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V),$$

for any $U, V \in \chi(M)$.

Now we have

$$[E_1, E_2] = 0, \quad [E_1, E_3] = 0, \quad [E_1, E_4] = 0, \quad [E_1, E_5] = E_1,$$
$$[E_2, E_3] = 0, \quad [E_2, E_4] = 0, \quad [E_2, E_5] = E_2,$$
$$[E_3, E_4] = 0, \quad [E_3, E_5] = E3, \quad [E_4, E_5] = E_4.$$

By virtue of Koszul's formula we get the followings in matrix form

$$\begin{pmatrix} \nabla_{E_1}E_1 & \nabla_{E_1}E_2 & \nabla_{E_1}E_3 & \nabla_{E_1}E_4 & \nabla_{E_1}E_5 \\ \nabla_{E_2}E_1 & \nabla_{E_2}E_2 & \nabla_{E_2}E_3 & \nabla_{E_2}E_4 & \nabla_{E_2}E_5 \\ \nabla_{E_3}E_1 & \nabla_{E_3}E_2 & \nabla_{E_3}E_3 & \nabla_{E_3}E_4 & \nabla_{E_3}E_5 \\ \nabla_{E_4}E_1 & \nabla_{E_4}E_2 & \nabla_{E_4}E_3 & \nabla_{E_4}E_4 & \nabla_{E_4}E_5 \\ \nabla_{E_5}E_1 & \nabla_{E_5}E_2 & \nabla_{E_5}E_3 & \nabla_{E_5}E_4 & \nabla_{E_5}E_5 \end{pmatrix} = \begin{pmatrix} -E_5 & 0 & 0 & 0 & E_1 \\ 0 & -E_5 & 0 & 0 & E_2 \\ 0 & 0 & -E_5 & 0 & E_3 \\ 0 & 0 & 0 & -E_5 & E_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Above expressions satisfies all the properties of para-Sasakian manifold. Thus $M(\phi, \xi, \eta, g)$ is a 5-dimensional para-Sasakian manifold.

From the above expressions and the relation of quarter symmetric metric connection and Riemannian connection, one can easily obtain the following:

$$\begin{pmatrix} \tilde{\nabla}_{E_1}E_1 & \tilde{\nabla}_{E_1}E_2 & \tilde{\nabla}_{E_1}E_3 & \tilde{\nabla}_{E_1}E_4 & \tilde{\nabla}_{E_1}E_5 \\ \tilde{\nabla}_{E_2}E_1 & \tilde{\nabla}_{E_2}E_2 & \tilde{\nabla}_{E_2}E_3 & \tilde{\nabla}_{E_2}E_4 & \tilde{\nabla}_{E_2}E_5 \\ \tilde{\nabla}_{E_3}E_1 & \tilde{\nabla}_{E_3}E_2 & \tilde{\nabla}_{E_3}E_3 & \tilde{\nabla}_{E_3}E_4 & \tilde{\nabla}_{E_3}E_5 \\ \tilde{\nabla}_{E_4}E_1 & \tilde{\nabla}_{E_4}E_2 & \tilde{\nabla}_{E_4}E_3 & \tilde{\nabla}_{E_4}E_4 & \tilde{\nabla}_{E_4}E_5 \\ \tilde{\nabla}_{E_5}E_1 & \tilde{\nabla}_{E_5}E_2 & \tilde{\nabla}_{E_5}E_3 & \tilde{\nabla}_{E_5}E_4 & \tilde{\nabla}_{E_5}E_5 \end{pmatrix} = \begin{pmatrix} -2E_5 & 0 & 0 & 0 & 2E_1 \\ 0 & -2E_5 & 0 & 0 & 2E_2 \\ 0 & 0 & -2E_5 & 0 & 2E_3 \\ 0 & 0 & 0 & -2E_5 & 2E_4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

With the help of the above results it can be easily obtain the non-zero components of curvature tensors as

$$R(E_1, E_2)E_1 = E_2, \quad R(E_1, E_2)E_2 = -E_1, \quad R(E_1, E_3)E_1 = E_3, \quad R(E_1, E_3)E_3 = -E_1,$$
$$R(E_1, E_4)E_1 = E_4, \quad R(E_1, E_4)E_4 = -E_1, \quad R(E_1, E_5)E_1 = E_5, \quad R(E_1, E_5)E_5 = -E_1,$$
$$R(E_2, E_3)E_2 = E_3, \quad R(E_2, E_3)E_3 = -E_2, \quad R(E_2, E_4)E_2 = E_4, \quad R(E_2, E_4)E_4 = -E_2,$$
$$R(E_2, E_5)E_2 = E_5, \quad R(E_2, E_5)E_5 = -E_2, \quad R(E_3, E_4)E_3 = E_4, \quad R(E_3, E_4)E_4 = -E_3,$$
$$R(E_3, E_5)E_3 = E_5, \quad R(E_3, E_5)E_5 = -E_3, \quad R(E_4, E_5)E_4 = E_5, \quad R(E_4, E_5)E_5 = -E_4,$$

and

$$\tilde{R}(E_1,E_2)E_1 = 4E_2,\quad \tilde{R}(E_1,E_2)E_2 = -4E_1,\quad \tilde{R}(E_1,E_3)E_1 = 4E_3,\quad \tilde{R}(E_1,E_3)E_3 = -4E_1,$$
$$\tilde{R}(E_1,E_4)E_1 = 4E_4,\quad \tilde{R}(E_1,E_4)E_4 = -4E_1,\quad \tilde{R}(E_1,E_5)E_1 = 2E_5,\quad \tilde{R}(E_1,E_5)E_5 = -2E_1,$$
$$\tilde{R}(E_2,E_3)E_2 = 4E_3,\quad \tilde{R}(E_2,E_3)E_3 = -4E_2,\quad \tilde{R}(E_2,E_4)E_2 = 4E_4,\quad \tilde{R}(E_2,E_4)E_4 = -4E_2,$$
$$\tilde{R}(E_2,E_5)E_2 = 2E_5,\quad \tilde{R}(E_2,E_5)E_5 = -2E_2,\quad \tilde{R}(E_3,E_4)E_3 = 4E_4,\quad \tilde{R}(E_3,E_4)E_4 = -4E_3,$$
$$\tilde{R}(E_3,E_5)E_3 = 2E_5,\ \tilde{R}(E_3,E_5)E_5 = -2E_3,\ \tilde{R}(E_4,E_5)E_4 = 2E_5,\ \tilde{R}(E_4,E_5)E_5 = -2E_4. \quad (7.5)$$

Since E_1, E_2, E_3, E_4, E_5 forms a basis, any vector field $X, Y, Z \in \chi(M)$ can be written as $X = a_1E_1 + b_1E_2 + c_1E_3 + d_1E_4 + f_1E_5$, $Y = a_2E_1 + b_2E_2 + c_2E_3 + d_2E_4 + f_2E_5$, $Z = a_3E_1 + b_3E_2 + c_3E_3 + d_3E_4 + f_3E_5$, where $a_i, b_i, c_i, d_i, f_i \in \mathbb{R}$, $i = 1, 2, 3, 4, 5$. Using the expressions of the curvature tensors, we find values of Riemannian curvature and Ricci curvature with respect to quarter-symmetric metric connection as;

$$\begin{aligned}
\tilde{R}(X,Y)Z &= [-4\{a_1(b_2b_3 + c_2c_3 + d_2d_3) - a_2(b_1b_3 + c_1c_3 + d_1d_3)\} - 2(a_1f_2 - f_1a_2)f_3]E_1 \\
&+ [-4\{b_1(a_2a_3 + c_2c_3 + d_2d_3) - b_2(a_1a_3 + c_1c_3 + d_1d_3)\} - 2(b_1f_2 - f_1b_2)f_3]E_2 \\
&+ [-4\{c_1(a_2a_3 + b_2b_3 + d_2d_3) - c_2(a_1a_3 + b_1b_3 + d_1d_3)\} - 2(c_1f_2 - f_1c_2)f_3]E_3 \\
&+ [-4\{d_1(a_2a_3 + b_2b_3 + c_2c_3) - d_2(a_1a_3 + b_1b_3 + c_1c_3)\} - 2(d_1f_2 - f_1d_2)f_3]E_4 \\
&+ [2\{(a_1f_2 - f_1a_2)a_3 + (b_1f_2 - f_1b_2)b_3 + (c_1f_2 - f_1c_2)c_3 + (d_1f_2 - f_1d_2)d_3\}]E_5, \\
\tilde{S}(E_1,E_1) &= \tilde{S}(E_2,E_2) = \tilde{S}(E_3,E_3) = \tilde{S}(E_4,E_4) = -14,\ \tilde{S}(E_5,E_5) = -8. \qquad (7.6)
\end{aligned}$$

In view of (7.5), (7.6) and the expression of the endomorphism one can easily verify the equation (7.4) and hence the Theorem 2 of section 4 is verified. This example also verifies the condition (c) of Theorem 3.1 in [1] and first Bianchi identity.

Above two examples verifies the one part of the Theorem 2, that is, if a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is Ricci pseudosymmetric then M^n satisfies $L_{\tilde{S}} = -2$ (M^n is not Einstein manifold with respect to quarter-symmetric metric connection). Another part of the theorem is that, if a para-Sasakian manifold M^n admitting a quarter-symmetric metric connection is Ricci pseudosymmetric then M^n is an Einstein manifold with respect to quarter-symmetric metric connection ($L_{\tilde{S}} \neq -2$). Now, the second part of the Theorem 2 can be verified by using the proper example.

7.3 Example

We consider a 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are standard coordinates in $\mathbb{R}^5$. Let $\{E_1, E_2, E_3, E_4, E_5\}$ be a linearly independent global frame field on M given by

$$E_1 = \frac{\partial}{\partial x}, \quad E_2 = e^{-x}\frac{\partial}{\partial y}, \quad E_3 = e^{-x}\frac{\partial}{\partial z}, \quad E_4 = e^{-x}\frac{\partial}{\partial u}, \quad E_5 = e^{-x}\frac{\partial}{\partial v}.$$

Let g be a Riemannian metric defined by

$$g(E_i, E_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

for $1 \leq i, j \leq 5$, and if η is the 1-form defined by $\eta(Z) = g(Z, E_1)$ for any vector field $Z \in \chi(M)$. Let the (1, 1)-tensor field ϕ be defined by

$$\phi(E_1) = 0, \quad \phi(E_2) = E_2, \quad \phi(E_3) = E_3, \quad \phi(E_4) = E_4, \quad \phi(E_5) = E_5.$$

With the help of linearity property of ϕ and g, we have

$$\begin{aligned} &\eta(E_1) = 1, \\ &\phi^2 V = V - \eta(V)E_1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \end{aligned}$$

for any $X, Y \in \chi(M)$.

Now we have

$$\begin{aligned} &[E_1, E_2] = -E_2, \quad [E_1, E_3] = -E_3, \quad [E_1, E_4] = -E_4, \quad [E_1, E_5] = -E_5, \\ &[E_2, E_3] = [E_2, E_4] = [E_2, E_5] = [E_3, E_4] = [E_3, E_5] = E_4, E_5] = 0. \end{aligned}$$

With the help of Koszul's formula we get the followings in matrix form

$$\begin{pmatrix} \nabla_{E_1}E_1 & \nabla_{E_1}E_2 & \nabla_{E_1}E_3 & \nabla_{E_1}E_4 & \nabla_{E_1}E_5 \\ \nabla_{E_2}E_1 & \nabla_{E_2}E_2 & \nabla_{E_2}E_3 & \nabla_{E_2}E_4 & \nabla_{E_2}E_5 \\ \nabla_{E_3}E_1 & \nabla_{E_3}E_2 & \nabla_{E_3}E_3 & \nabla_{E_3}E_4 & \nabla_{E_3}E_5 \\ \nabla_{E_4}E_1 & \nabla_{E_4}E_2 & \nabla_{E_4}E_3 & \nabla_{E_4}E_4 & \nabla_{E_4}E_5 \\ \nabla_{E_5}E_1 & \nabla_{E_5}E_2 & \nabla_{E_5}E_3 & \nabla_{E_5}E_4 & \nabla_{E_5}E_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ E_2 & -E_1 & 0 & 0 & 0 \\ E_3 & 0 & -E_1 & 0 & 0 \\ E_4 & 0 & 0 & -E_1 & 0 \\ E_5 & 0 & 0 & 0 & -E_1 \end{pmatrix}.$$

In this case, (ϕ, ξ, η, g) is a para-Sasakian structure on M and hence $M(\phi, \xi, \eta, g)$ is a 5-dimensional para-Sasakian manifold.

Using (2.10) and the above equation, one can easily obtain the following:

$$\begin{pmatrix} \tilde{\nabla}_{E_1}E_1 & \tilde{\nabla}_{E_1}E_2 & \tilde{\nabla}_{E_1}E_3 & \tilde{\nabla}_{E_1}E_4 & \tilde{\nabla}_{E_1}E_5 \\ \tilde{\nabla}_{E_2}E_1 & \tilde{\nabla}_{E_2}E_2 & \tilde{\nabla}_{E_2}E_3 & \tilde{\nabla}_{E_2}E_4 & \tilde{\nabla}_{E_2}E_5 \\ \tilde{\nabla}_{E_3}E_1 & \tilde{\nabla}_{E_3}E_2 & \tilde{\nabla}_{E_3}E_3 & \tilde{\nabla}_{E_3}E_4 & \tilde{\nabla}_{E_3}E_5 \\ \tilde{\nabla}_{E_4}E_1 & \tilde{\nabla}_{E_4}E_2 & \tilde{\nabla}_{E_4}E_3 & \tilde{\nabla}_{E_4}E_4 & \tilde{\nabla}_{E_4}E_5 \\ \tilde{\nabla}_{E_5}E_1 & \tilde{\nabla}_{E_5}E_2 & \tilde{\nabla}_{E_5}E_3 & \tilde{\nabla}_{E_5}E_4 & \tilde{\nabla}_{E_5}E_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2E_2 & -2E_1 & 0 & 0 & 0 \\ 2E_3 & 0 & -2E_1 & 0 & 0 \\ 2E_4 & 0 & 0 & -2E_1 & 0 \\ 2E_5 & 0 & 0 & 0 & -2E_1 \end{pmatrix}.$$

From above results it can be easily obtain the non-zero components of Riemannian curvature and Ricci curvature tensors as

$$\begin{aligned}
&R(E_1,E_2)E_1 = E_2, \quad R(E_1,E_2)E_2 = -E_1, \quad R(E_1,E_3)E_1 = E_3, \quad R(E_1,E_3)E_3 = -E_1,\\
&R(E_1,E_4)E_1 = E_4, \quad R(E_1,E_4)E_4 = -E_1, \quad R(E_1,E_5)E_1 = E_5, \quad R(E_1,E_5)E_5 = -E_1,\\
&R(E_2,E_3)E_2 = E_3, \quad R(E_2,E_3)E_3 = -E_2, \quad R(E_2,E_4)E_2 = E_4, \quad R(E_2,E_4)E_4 = -E_2,\\
&R(E_2,E_5)E_2 = E_5, \quad R(E_2,E_5)E_5 = -E_2, \quad R(E_3,E_4)E_3 = E_4, \quad R(E_3,E_4)E_4 = -E_3,\\
&R(E_3,E_5)E_3 = E_5, \quad R(E_3,E_5)E_5 = -E_3, \quad R(E_4,E_5)E_4 = E_5, \quad R(E_4,E_5)E_5 = -E_4,
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{R}(E_1,E_2)E_1 = 2E_2, \quad \tilde{R}(E_1,E_2)E_2 = -2E_1, \quad \tilde{R}(E_1,E_3)E_1 = 2E_3, \quad \tilde{R}(E_1,E_3)E_3 = -2E_1,\\
&\tilde{R}(E_1,E_4)E_1 = 2E_4, \quad \tilde{R}(E_1,E_4)E_4 = -2E_1, \quad \tilde{R}(E_1,E_5)E_1 = 2E_5, \quad \tilde{R}(E_1,E_5)E_5 = -2E_1,\\
&\tilde{R}(E_2,E_3)E_2 = 2E_3, \quad \tilde{R}(E_2,E_3)E_3 = -2E_2, \quad \tilde{R}(E_2,E_4)E_2 = 2E_4, \quad \tilde{R}(E_2,E_4)E_4 = -2E_2,\\
&\tilde{R}(E_2,E_5)E_2 = 2E_5, \quad \tilde{R}(E_2,E_5)E_5 = -2E_2, \quad \tilde{R}(E_3,E_4)E_3 = 2E_4, \quad \tilde{R}(E_3,E_4)E_4 = -2E_3,\\
&\tilde{R}(E_3,E_5)E_3 = 2E_5,\ \tilde{R}(E_3,E_5)E_5 = -2E_3,\ \tilde{R}(E_4,E_5)E_4 = 2E_5,\ \tilde{R}(E_4,E_5)E_5 = -2E_4, \quad (7.7)\\
&\tilde{S}(E_1,E_1) = \tilde{S}(E_2,E_2) = \tilde{S}(E_3,E_3) = \tilde{S}(E_4,E_4) = \tilde{S}(E_5,E_5) = -8, \quad (7.8)\\
&\tilde{S}(X,Y) = -2(5-1)g(X,Y) = -8g(X,Y),
\end{aligned}$$

where $X = a_1E_1 + b_1E_2 + c_1E_3 + d_1E_4 + f_1E_5$ and $Y = a_2E_1 + b_2E_2 + c_2E_3 + d_2E_4 + f_2E_5$.

From (7.7), (7.8) and the expression of the endomorphism one can easily substantiate, the equation (7.4) and hence second part of the Theorem 2 (for $L_{\tilde{S}} \neq -2$).

References

[1] Abul Kalam Mondal and U.C. De, Quarter-symmetric nonmetric Connection on P-Sasakian manifolds, ISRN Geometry, (2012), 1–14.

[2] G. Soos, Über die geodätischen Abbildungen von Riemannaschen Räumen auf projektiv-symmetrische Riemannsche Räume, Acta. Math. Acad. Sci. Hungar., 9, (1958), 359–361

[3] A. Barman, Concircular curvature tensor on a P-Sasakian manifold admitting a quarter-symmetric metric connection, Kragujevac J. Math. **42** (2018), 2, 275–285.

[4] D. V. Alekseevsky et al., Cones over pseudo-Riemannian manifolds and their holonomy, J. Reine Angew. Math. **635** (2009), 23–69.

[5] E. Cartan, Surune classe remarquable d'espaces de Riema, Soc. Math., France, **54** (1926), 214–264.

[6] Cihan Ozgur, On A class of para-Sakakian manifolds, Turk J Math., **29** (2005), 249–257.

[7] V. Cortés et al., Special geometry of Euclidean supersymmetry. I. Vector multiplets, J. High Energy Phys., **03**, (2004), 1–64.

[8] V. Cortés, M.-A. Lawn and L. Schäfer, Affine hyperspheres associated to special para-Kähler manifolds, Int. J. Geom. Methods Mod. Phys. **3** (2006), 5-6, 995–1009.

[9] R. Deszcz, *On pseudosymmetric spaces,* Acta Math., Hungarica, **53** (1992), 185–190.

[10] R. Deszcz and S. Yaprak, Curvature properties of certain pseudosymmetric manifolds, Publ. Math. Debrecen **45** (1994), 3-4, 333–345.

[11] R. Deszcz et al., On some curvature conditions of pseudosymmetry type, Period. Math. Hungar. **70** (2015), 2, 153–170.

[12] S. Golab, On semi-symmetric and quarter-symmetric linear connections, Tensor (N.S.) **29** (1975), 3, 249–254.

[13] S. Haesen and L. Verstraelen, Properties of a scalar curvature invariant depending on two planes, Manuscripta Math. **122** (2007), 1, 59–72.

[14] S. Kaneyuki and F. L. Williams, Almost paracontact and parahodge structures on manifolds, Nagoya Math. J. **99** (1985), 173–187.

[15] Lata Bisht and Sandhana Shanker, Curvature tensor on para-Sasakian manifold admitting quarter symmetric metric connection, IOSR Journal of Mathematics, 11(5), (2015), 22–28.

[16] K. Mandal and U. C. De, Quarter-symmetric metric connection in a P-Sasakian manifold, An. Univ. Vest Timiş. Ser. Mat.-Inform. **53** (2015), 1, 137–150.

[17] K.T. Pradeep Kumar, Venkatesha and C.S. Bagewadi, On ϕ-recurrent para-Sasakian manifold admitting quarter-symmetric metric connection, ISRN Geometry, (2012), 1-10.

[18] I. Sato, On a structure similar to the almost contact structure, Tensor (N.S.) **30** (1976), 3, 219–224.

[19] Z. I. Szabó, Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$. I. The local version, J. Differential Geometry **17** (1982), 4, 531–582.