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Basic asymptotic estimates for powers of Wallis' ratios

Vito Lampret[1]

[1] *University of Ljubljana, Ljubljana, 386 Slovenia, EU.*
vito.lampret@guest.arnes.si

ABSTRACT

For any $a \in \mathbb{R}$, for every $n \in \mathbb{N}$, and for n-th Wallis' ratio $w_n := \prod_{k=1}^{n} \frac{2k-1}{2k}$, the relative error $r_0(a,n) := \big(v_0(a,n) - w_n^a\big)/w_n^a$ of the approximation $w_n^a \approx v_0(a,n) := (\pi n)^{-a/2}$ is estimated as $\left|r_0(a,n)\right| < \frac{1}{4n}$. The improvement $w_n^a \approx v(a,n) := (\pi n)^{-a/2}\left(1 - \frac{a}{8n} + \frac{a^2}{128n^2}\right)$ is also studied.

RESUMEN

Para cualquier $a \in \mathbb{R}$, para todo $n \in \mathbb{N}$, y para el n-ésimo cociente de Wallis $w_n := \prod_{k=1}^{n} \frac{2k-1}{2k}$, el error relativo $r_0(a,n) := \big(v_0(a,n) - w_n^a\big)/w_n^a$ de la aproximación $w_n^a \approx v_0(a,n) := (\pi n)^{-a/2}$ se estima como $\left|r_0(a,n)\right| < \frac{1}{4n}$. También se estima la mejora $w_n^a \approx v(a,n) := (\pi n)^{-a/2}\left(1 - \frac{a}{8n} + \frac{a^2}{128n^2}\right)$.

Keywords and Phrases: approximation, asymptotic, estimate, inequality, power, Wallis' ratio.

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1 Introduction

The sequence of Wallis[1] ratios

$$w_n := \prod_{k=1}^{n} \frac{2k-1}{2k} = \frac{(2n-1)!!}{(2n)!!} = 4^{-n}\binom{2n}{n} \tag{1.1}$$

is often encountered in pure and applied mathematics, in physics, and in several other exact sciences too. For example, the perimeter $P(a,b)$ of an ellipse having semi-axes of length a and $b \leq a$ is given as

$$P(a,b) = 4a\int_0^{\pi/2} \sqrt{1-\varepsilon^2\sin^2(\tau)}\ \mathrm{d}\tau = 2a\pi\left(1-\sum_{k=1}^{\infty}\frac{w_k^2}{2k-1}\varepsilon^{2k}\right) \tag{1.2}$$

[20], where $\varepsilon = \sqrt{1-\frac{b^2}{a^2}}$, the eccentricity of an ellipse.

Similarly, the period T of a simple pendulum, located in the gravitational field with the acceleration g and having the length L and the amplitude of the oscillation $\alpha \in (0,\pi)$, is determined by the formula

$$T = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{\mathrm{d}\tau}{\sqrt{1-\varepsilon^2\sin^2(\tau)}} = 2\pi\sqrt{\frac{L}{g}}\left(1+\sum_{k=1}^{\infty} w_k^2\varepsilon^{2k}\right) \tag{1.3}$$

[21, p. 26], where $\varepsilon = \sin(\alpha/2)$. Not only in mechanics, but also in other parts of physics, the Wallis ratio has several interesting roles, see for example [9] and [12].

In mathematics, the sequence of the Landau constants G_n, important in the theory of analytic functions, see [1], is also defined by the Wallis ratios as

$$G_n := \sum_{k=1}^{n} w_k^2 \qquad (n \in \mathbb{N}). \tag{1.4}$$

The Wallis ratio attracts mathematicians also because of its direct connections with Catalan numbers $c_n := \frac{1}{n+1}\binom{2n}{n}$, also important objects for pure and applied mathematics [15, 29]. In fact, the Wallis ratio, i.e. the sequence $n \mapsto w_n$, was investigated by many researches, see for example the papers [2, 3, 4, 5, 6, 7, 8, 11, 14, 16, 22, 23, 26, 27, 28, 29, 31, 33].

In 2007 was presented [33] aesthetically pleasing double inequality

$$\frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n}} < w_n \leq \frac{1}{\sqrt{e\pi n}}\left(1+\frac{1}{2n}\right)^{n-\frac{1}{12n+16}}, \tag{1.5}$$

true for all $n \in \mathbb{N}$.

In 2013 was demonstrated [10] the estimate

$$\sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n} < w_n \leq \frac{4}{3}\left(1-\frac{1}{2n}\right)^n\frac{\sqrt{n-1}}{n}, \tag{1.6}$$

[1] John Wallis, 1616 – 1703

true for $n \geq 2$.

In 2015 was derived [11] the inequalities

$$\left(\frac{2}{3}\right)^{3/2}\left(1-\frac{1}{2n}\right)^{n+1/2}\left(n-\frac{3}{2}\right)^{-1/2} \leq w_n < \sqrt{\frac{e}{\pi}}\left(1-\frac{1}{2n}\right)^{n+1/2}\left(n-\frac{3}{2}\right)^{-1/2}, \tag{1.7}$$

valid for $n \geq 2$. At the same time, in [28, Theorems 4.2 and 5.2] were presented the estimates

$$w_n > \sqrt{\frac{e}{\pi n}}\left(1-\frac{1}{2n}\right)^n \exp\left(\frac{1}{24n^2}+\frac{1}{48n^3}+\frac{1}{160n^4}+\frac{1}{960n^5}\right) \tag{1.8}$$

$$w_n > \sqrt{\frac{e}{\pi n}}\left(1-\frac{1}{2(n+1/3)}\right)^{n+1/3} \tag{1.9}$$

and

$$w_n < \sqrt{\frac{e}{\pi n}}\left(1-\frac{1}{2(n+1/3)}\right)^{n+1/3} \exp\left(\frac{1}{144n^3}\right), \tag{1.10}$$

all true for $n \geq 1$.

In the mentioned formulas for the perimeter of an ellipse and the period of a simple pendulum, as well as for the Landau sequence, see (1.2)–(1.4), we met the second powers of the Wallis ratios. This fact initiated our desire to approximate any power of the Wallis ratio. But, all the inequalities (1.5)–(1.10) are less suitable for estimating the power w_n^a for $a \in \mathbb{R}$. Fortunately, the approximation formula for the Wallis ratio, presented in [19], is more convenient for this task. In this contribution we shall show the first two steps how to approximate simply and accurately the powers of the Wallis ratios having real exponents.

2 Basic discussion

The sequence of Wallis' ratios was estimated recently [19] as

$$w_n = \frac{1}{\sqrt{\pi\, n}} \exp\left(-\widetilde{s}_r(n)+\delta_r(n)\right) \qquad (n \in \mathbb{N}), \tag{2.1}$$

where

$$\widetilde{s}_r(n) = \sum_{i=1}^{r} \frac{(1-4^{-i})B_{2i}}{i(2i-1)n^{2i-1}} \qquad (n, r \in \mathbb{N}) \tag{2.2}$$

and, for any $n, r \in \mathbb{N}$, the error $\delta_r(n)$ is estimated as

$$-\frac{\left|B_{2r+2}\right|}{(r+1)(2r+1)n^{2r+1}} < (-1)^r \delta_r(n) < \frac{\left|B_{2r+2}\right|}{2(r+1)(2r+1)(2n)^{2r+1}}\,. \tag{2.3}$$

Here $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $\ldots$ are the Bernoulli numbers, defined by the identity $\frac{x}{e^x-1} \equiv \sum_{j=0}^{\infty} B_j \frac{x^j}{j!}$ $(\,|x| < 2\pi\,)$.

We obtain the basic approximation by using $r = 1$,

$$w_n^a = (\pi n)^{-a/2} \exp\left(-a\widetilde{s}_1(n)+a\delta_1(n)\right) \qquad (a \in \mathbb{R},\, n \in \mathbb{N}), \tag{2.4}$$

with, for $n \in \mathbb{N}$,

$$\widetilde{s}_1(n) := \frac{1}{8n} > 0 \tag{2.5}$$

and

$$-\frac{1}{180\,n^3} < -\frac{1}{2880\,n^3} < \delta_1(n) < \frac{1}{180\,n^3}\,. \tag{2.6}$$

Thus, due to (2.5),

$$a\widetilde{s}_1(n) = \frac{a}{8n} \qquad (a \in \mathbb{R},\, n \in \mathbb{N})\,. \tag{2.7}$$

Moreover, thanks to (2.5)–(2.6), we estimate, for $n \in \mathbb{N}$,

$$-\widetilde{s}_1(n) \pm \left|\delta_1(n)\right| \geq -\widetilde{s}_1(n) - \left|\delta_1(n)\right| > -\frac{1}{8n} - \frac{1}{180n^3} > -\frac{1}{7n} \tag{2.8}$$

and

$$-\widetilde{s}_1(n) \pm \left|\delta_1(n)\right| \leq -\widetilde{s}_1(n) + \left|\delta_1(n)\right| < -\frac{1}{8n} + \frac{1}{180n^3} < -\frac{1}{9n}\,. \tag{2.9}$$

Therefore, $-\frac{a}{7n} < a\big(-\widetilde{s}_1(n) \pm \delta_1(n)\big) < -\frac{a}{9n}$, for $a > 0$ and $-\frac{a}{7n} > a\big(-\widetilde{s}_1(n) \pm \delta_1(n)\big) > -\frac{a}{9n}$, for $a < 0$. Thus,

$$\min\left\{-\tfrac{a}{7n}, -\tfrac{a}{9n}\right\} < a\big(-\widetilde{s}_1(n) \pm \left|\delta_1(n)\right|\big) < \max\left\{-\tfrac{a}{7n}, -\tfrac{a}{9n}\right\} \qquad (a \neq 0,\, n \in \mathbb{N}). \tag{2.10}$$

Hence, considering (2.4), together with the equality $\min(-S) = -\max(S)$, for every $S \subseteq \mathbb{R}$, we derive the following theorem.

Theorem 2.1. *For $a \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{N}$, the following double inequality holds:*

$$(\pi n)^{-a/2} \exp\big(-\max\left\{\tfrac{a}{7n}, \tfrac{a}{9n}\right\}\big) < w_n^a < (\pi n)^{-a/2} \exp\big(-\min\left\{\tfrac{a}{7n}, \tfrac{a}{9n}\right\}\big). \tag{2.11}$$

Figure 1 shows[2] the graphs of the function $a \mapsto w_2^a$ and its approximation (dashed line) $a \mapsto (\pi \cdot 2)^{-a/2}$.

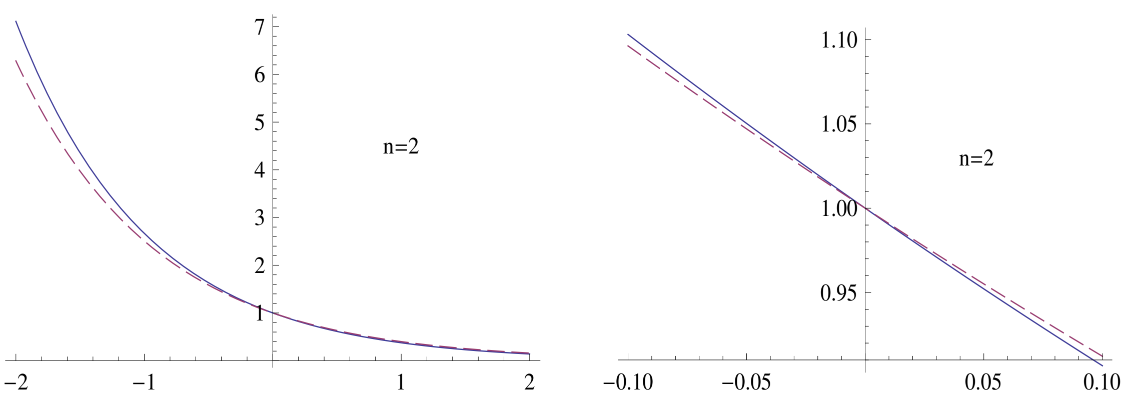

Figure 1: The graphs of the function $a \mapsto w_2^a$ and its approximation (dashed line) $a \mapsto (\pi \cdot 2)^{-a/2}$.

[2]All graphics and calculations in this paper are made using the Mathematica [32] computer system.

Example 2.2. *For any $n \in \mathbb{N}$ we have*

$$(\pi n)^{-50} \exp\big(-\tfrac{100}{7n}\big) < w_n^{100} < (\pi n)^{-50} \exp\big(-\tfrac{100}{9n}\big),$$
$$(\pi n)^{50} \exp\big(\tfrac{100}{9n}\big) < w_n^{-100} < (\pi n)^{50} \exp\big(\tfrac{100}{7n}\big).$$

From Theorem 2.1 there follows the next corollary.

Corollary 2.3. *For every $a \in \mathbb{R} \setminus \{0\}$ and for any positive integer $n \geq a$ we have*

$$w_n^a > \frac{6}{7}(\pi n)^{-a/2}. \tag{2.12}$$

Proof. For $k \geq a > 0$, using (2.11), we obtain[3] $w_k^a > (\pi k)^{-a/2} \exp\big(-\frac{a}{7k}\big) > (\pi k)^{-a/2}(1 - \frac{a}{7k}) \geq \big(\pi k)^{-a/2}(1 - \frac{1}{7}\big) = \frac{6}{7}(\pi k)^{-a/2}$. Furthermore, for $a < 0$, due to (2.11), we estimate $w_k^a > (\pi k)^{-a/2} \exp\big(-\frac{a}{9k}\big) = (\pi k)^{-a/2} \exp\big(\frac{|a|}{9k}\big) > (\pi k)^{-a/2} \cdot 1$. $\square$

Lemma 2.4. *Let real numbers α, β, v and w satisfy the inequalities $\alpha\beta \geq 0$, $\beta \leq \frac{1}{2}$, $v > 0$ and $e^{\alpha}v < w < e^{\beta}v$. Then we have $|v - w| < \frac{3}{2}v \cdot \max\{|\alpha|, |\beta|\}$.*

Proof. Supposing that all conditions of Lemma 2.4 are satisfied, we have only two possibilities $\alpha < \beta \leq 0$ or $0 \leq \alpha < \beta$, together with the estimate

$$(e^{\alpha} - 1)v < w - v < (e^{\beta} - 1)v.$$

Therefore, in case $\alpha \leq 0$, we have $(1 - e^{\alpha})v > v - w > (1 - e^{\beta})v \geq 0$. Thus, see Footnote 3, $|v - w| < -\alpha = |\alpha|$. Additionally, using the first order Taylor's formula and the estimate $0 \leq \beta \leq \frac{1}{2}$, in case $\alpha \geq 0$, we obtain, $0 \leq (e^{\alpha} - 1)v < -(v - w) < (e^{\beta} - 1)v < \beta + \frac{1}{2}e^{\beta}\beta^2 \leq \beta + \frac{1}{2}e^{1/2}\frac{1}{2}\beta < \frac{3}{2}\beta$. Hence, in both cases we have $|v - w| < v \cdot \max\{|\alpha|, \frac{3}{2}|\beta|\}$. $\square$

Corollary 2.5 (relative error)**.** *For every $a \in \mathbb{R} \setminus \{0\}$ and for any positive integer $n \geq a$ the relative error $r_0(a, n) := \big(w_n^a - v_0(a, n)\big)/w_n^a$ of the approximation $w_n^a \approx v_0(a, n) := (\pi n)^{-a/2}$ is roughly estimated as*

$$\big|r_0(a, n)\big| < \frac{1}{4n}.$$

Proof. Thanks to Theorem 2.1 and Lemma 2.4, using the notations $\alpha = -\max\big\{\frac{a}{7n}, \frac{a}{9n}\big\}$, $\beta = -\min\big\{\frac{a}{7n}, \frac{a}{9n}\big\}$, $v = v_0(a, n) = (\pi n)^{-a/2}$ and $w = w_n^a$, we obtain

$$\big|v_0(a, n) - w_n^a\big| < \frac{3}{2}(\pi n)^{-a/2} \cdot \max\Big\{\, \big|\max\big\{\tfrac{a}{7n}, \tfrac{a}{9n}\big\}\big|, \big|\min\big\{\tfrac{a}{7n}, \tfrac{a}{9n}\big\}\big| \,\Big\}.$$

Thus, according to the identity $\max\big\{\, |\max\{x, y\}|, |\min\{x, y\}| \,\big\} = \max\big\{|x|, |y|\big\}$, we get

$$\big|v_0(a, n) - w_n^a\big| < \frac{3}{2}(\pi n)^{-a/2} \cdot \frac{|a|}{7n}.$$

[3] considering the well-known estimate $e^x > 1 + x$, true for $x \in \mathbb{R} \setminus \{0\}$.

Hence, using Corollary 2.3,

$$\frac{\left|v_0(a,n)-w_n^a\right|}{w_n^a} < \frac{3}{2}(\pi n)^{-a/2}\frac{|a|}{7n}\cdot\frac{7}{6}(\pi n)^{a/2} = \frac{|a|}{4n}\,. \qquad \square$$

Figure 2 shows, on the left – the graph of the actual relative error function $a \mapsto r_0(a,n)$ and on the right – the graphs of the functions $a \mapsto r_0(a,n)$ and $a \mapsto \frac{|a|}{4\times 1000}$ (dashed line).

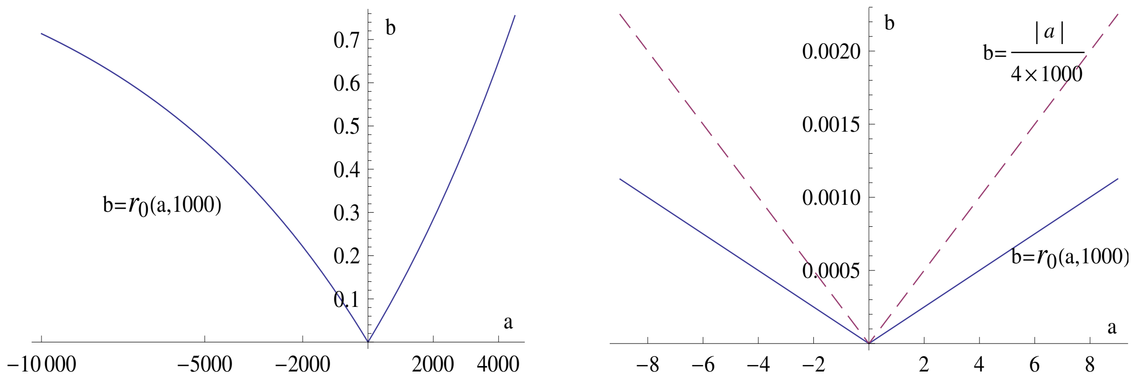

Figure 2: Left – the graph of the actual relative error function $a \mapsto r_0(a,1000)$; Right – the graphs of the actual relative error $a \mapsto r_0(a,1000)$ and its approximation (dashed line) $a \mapsto \frac{|a|}{4\times 1000}$.

3 Improvement

The relations (2.4)–(2.6) can be exploited more accurately to derive the next theorem.

Theorem 3.1. *For any $a \in \mathbb{R}$ and every integer $n \ge \frac{|a|}{8}$, we have*

$$w_n^a = v(a,n) + \varepsilon(a,n), \tag{3.1}$$

where

$$v(a,n) := (\pi n)^{-a/2}\left(1 - \frac{a}{8n} + \frac{a^2}{128n^2}\right), \tag{3.2}$$

and the error $\varepsilon(a,n)$ is estimated as

$$\left|\varepsilon(a,n)\right| \le \varepsilon^*(a,n) := (\pi n)^{-a/2}\left[\frac{a^2}{100} + \frac{1}{18}\exp\left(-\min\left\{\frac{a}{7n},\frac{a}{9n}\right\}\right)\right]\frac{|a|}{10n^3} \tag{3.3}$$

$$\le (\pi n)^{-a/2}\left(\frac{a^2}{100} + \frac{1}{18}\exp\left(\frac{|a|}{7n}\right)\right)\frac{|a|}{10n^3}$$

$$\le \varepsilon^{**}(a,n) := (\pi n)^{-a/2}\left(a^2 + \frac{35}{2}\right)\frac{|a|}{(10n)^3}\,. \tag{3.4}$$

Proof. Using the second order Taylor's formula, we have

$$\exp\left(-a\widetilde{s}_1(n)\right) = \exp\left(-\frac{a}{8n}\right) = 1 - \frac{a}{8n} + \frac{1}{2}\left(-\frac{a}{8n}\right)^2 + R_2(a,n) \tag{3.5}$$

with

$$R_2(a,n) = \frac{1}{6}\exp\left(-\vartheta\cdot\frac{a}{8n}\right)\left(-\frac{a}{8n}\right)^3, \qquad \text{for some } \vartheta = \vartheta(a,n)\in(0,1).$$

Therefore, for $a\in\mathbb{R}$ and $n\geq\frac{|a|}{8}$,

$$\left|R_2(a,n)\right| \leq \frac{1}{6}\exp\left(\frac{|a|}{8n}\right)\cdot\left(\frac{|a|}{8n}\right)^3 \leq \frac{e}{6}\cdot\frac{|a|^3}{512n^3} \leq \frac{|a|^3}{1000n^3}. \tag{3.6}$$

Similarly,

$$\exp\big(a\delta_1(n)\big) = 1+\exp\big(\vartheta\cdot a\delta_1(n)\big)\cdot a\delta_1(n), \tag{3.7}$$

for some $\vartheta=\vartheta(a,n)\in(0,1)$.

Thanks to (3.7), (2.10) and (2.6), we estimate, using some $\theta=\theta(a,n)\in(0,1)$,

$$\begin{aligned}
\Big|\underbrace{\exp\big(-a\widetilde{s}_1(n)+a\delta_1(n)\big)-\exp\big(-a\widetilde{s}_1(n)\big)}_{=\Delta(a,n)}\Big| &= \exp\big(-a\widetilde{s}_1(n)\big)\cdot\big|\exp\big(\theta\cdot a\delta_1(n)\big)\cdot a\delta_1(n)\big| \\
&\leq \exp\big(-a\widetilde{s}_1(n)\big)\cdot\exp\big(|a\delta_1(n)|\big)\cdot\big|a\delta_1(n)\big| \\
&= \exp\Big(a\big(-\widetilde{s}_1(n)\pm|\delta_1(n)|\big)\Big)|a|\big|\delta_1(n)\big| \\
&\underset{(2.6)}{\overset{(2.10)}{\leq}} \exp\left(\max\left\{-\frac{a}{7n},-\frac{a}{9n}\right\}\right)\cdot\frac{|a|}{180n^3}.
\end{aligned} \tag{3.8}$$

Consequently, according to (2.4) and (3.5), we obtain

$$w_n^a \overset{(2.4)}{=} (\pi k)^{-a/2}\Big(\exp\big(-a\widetilde{s}_1(n)+a\delta_1(n)\big)\Big) \overset{(3.5)}{=} (\pi n)^{-a/2}\Big(\underbrace{1-\frac{a}{8n}+\frac{a^2}{128n^2}+R_2(a,n)}_{=\exp(-a\widetilde{s}_1(n))}+\Delta(a,n)\Big),$$

where, considering (3.6) and (3.8), for $a\in\mathbb{R}$ and $n\geq\frac{|a|}{8}$, we estimate the error $\varepsilon(a,n):=(\pi n)^{-a/2}\big(R_2(a,n)+\Delta(a,n)\big)$ as

$$\begin{aligned}
\big|\varepsilon(a,n)\big| &\leq (\pi n)^{-a/2}\left[\frac{|a|^3}{1000n^3}+\exp\left(-\min\left\{\frac{a}{7n},\frac{a}{9n}\right\}\right)\frac{|a|}{180n^3}\right] \\
&= (\pi n)^{-a/2}\left[\frac{a^2}{100}+\frac{1}{18}\exp\left(-\min\left\{\frac{a}{7n},\frac{a}{9n}\right\}\right)\right]\frac{|a|}{10n^3} \\
&\leq (\pi n)^{-a/2}\left[\frac{a^2}{100}+\frac{1}{18}\exp\left(\frac{|a|}{8n}\cdot\frac{8}{7}\right)\right]\frac{|a|}{10n^3} \\
&\leq (\pi n)^{-a/2}\left(\frac{a^2}{100}+\frac{1}{18}\exp\left(1\cdot\frac{8}{7}\right)\right)\frac{|a|}{10n^3} \\
&\leq (\pi n)^{-a/2}\left(\frac{a^2}{100}+\frac{7}{40}\right)\frac{|a|}{10n^3}.
\end{aligned}$$

□

Remark 3.2. *The sequence* $n\mapsto W_n := \frac{1}{2n+1}\left(\prod_{k=1}^n\frac{2k}{2k-1}\right)^2$*, called the Wallis sequence, is closely connected to the sequence of the Wallis ratios* w_n *by the identity* $W_n = w_n^{-2}/(2n+1)$*. So,* W_n *can be estimated easily using Theorem 3.1, e.g. its consequence* (3.14).

Remark 3.3. *According to Theorem 3.1, the constant π can be easily approximated using certain rational functions $R_{\mp}(n)$. For example, from (3.14) we get, for any $n \in \mathbb{N}$,*

$$\frac{1}{n}\left(w_n^{-2} - \frac{1}{5n^2}\right)\left(1 + \frac{1}{4n} + \frac{1}{32n^2}\right)^{-1} < \pi < \frac{1}{n}\left(w_n^{-2} + \frac{1}{5n^2}\right)\left(1 + \frac{1}{4n} + \frac{1}{32n^2}\right)^{-1}.$$

Directly from Theorem 3.1 and Corollary 2.3, from (3.4) and (2.12), we read the next corollary.

Corollary 3.4 (relative error)**.** *For every $a \in \mathbb{R}$ and for any positive integer $n \geq |a|$ the relative error of the approximation $w_n^a \approx v(a,n)$,*

$$r(a,n) := \frac{w_n^a - v(a,n)}{w_n^a}, \tag{3.9}$$

is a priori estimated as

$$\big|r(a,n)\big| \leq r^*(a,n) := \left(a^2 + \frac{13}{2}\right)\frac{7|a|}{6(10n)^3}. \tag{3.10}$$

For any $a \in \mathbb{R}$ and all integers $n \geq |a|$ the rough estimate $r^*(a,n) < 8.2$ ‰ holds true.

Figure 3 shows the graphs of the actual relative error functions $a \mapsto r(a,n)$, for $n \in \{10,\ 100\}$.

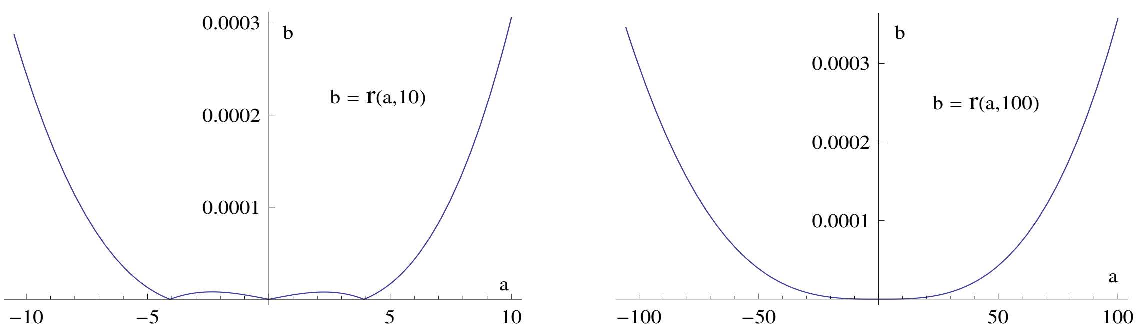

Figure 3: The graphs of the actual relative error functions $a \mapsto r(a,n)$ for $n \in \{10,\ 100\}$.

Figures 4–5 compare the actual relative error functions $a \mapsto r(a,n)$ and their approximations $a \mapsto r^*(a,n)$, for $n \in \{1,\ 3,\ 10,\ 100\}$.

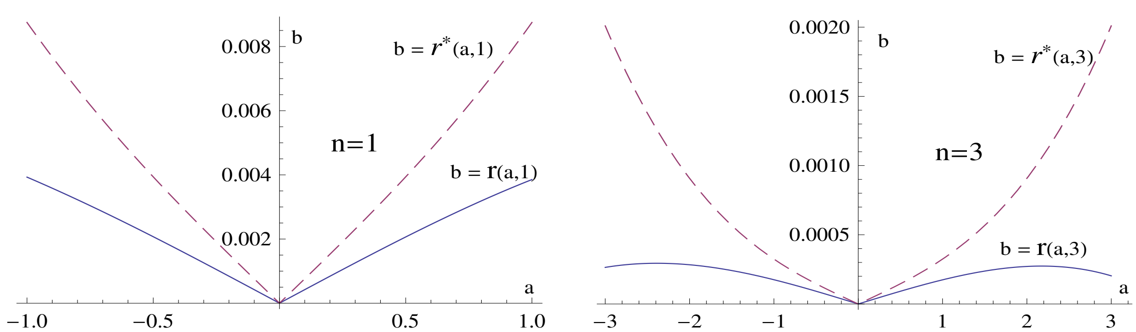

Figure 4: The graphs of the actual relative error functions $a \mapsto r(a,n)$ and their approximations $a \mapsto r^*(a,n)$, for $n \in \{1,\ 3\}$.

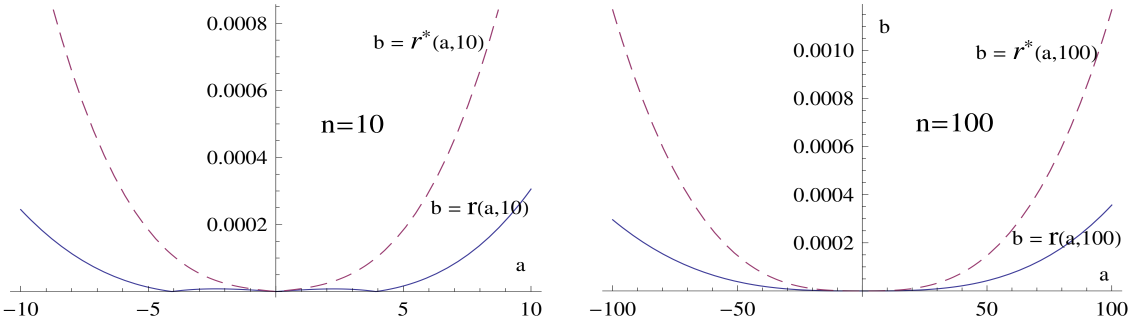

Figure 5: The graphs of the actual relative error functions $a \mapsto r(a,n)$ and their approximations $a \mapsto r^*(a,n)$, for $n \in \{10,\ 100\}$.

Using $a \in \{1,\ -1,\ 2,\ -2,\ \frac{1}{2},\ \pi,\ -2\pi\}$ in Theorem 3.1, considering (3.1) and (3.4), we obtain several inequalities for Wallis' ratios, presented in the next corollary.

Corollary 3.5. *For every*[4] *positive integer n we have*

$$\frac{1}{\sqrt{\pi\, n}}\left(1-\tfrac{1}{8n}+\tfrac{1}{128n^2}\right)-\tfrac{1}{95\,n^{7/2}}<w_n<\frac{1}{\sqrt{\pi\, n}}\left(1-\tfrac{1}{8n}+\tfrac{1}{128n^2}\right)+\tfrac{1}{95\,n^{7/2}}\,, \tag{3.11}$$

$$\sqrt{\pi\, n}\left(1+\tfrac{1}{8n}+\tfrac{1}{128n^2}\right)-\tfrac{1}{30\,n^{5/2}}<\frac{1}{w_n}<\sqrt{\pi\, n}\left(1+\tfrac{1}{8n}+\tfrac{1}{128n^2}\right)+\tfrac{1}{30\,n^{5/2}}\,, \tag{3.12}$$

$$\frac{1}{\pi\, n}\left(1-\tfrac{1}{4n}+\tfrac{1}{32n^2}\right)-\tfrac{1}{73\,n^4}<w_n^2<\frac{1}{\pi\, n}\left(1-\tfrac{1}{4n}+\tfrac{1}{32n^2}\right)+\tfrac{1}{73\,n^4}\,, \tag{3.13}$$

$$(\pi\, n)\left(1+\tfrac{1}{4n}+\tfrac{1}{32n^2}\right)-\tfrac{1}{7\,n^2}<\frac{1}{w_n^2}<(\pi\, n)\left(1+\tfrac{1}{4n}+\tfrac{1}{32n^2}\right)+\tfrac{1}{7\,n^2}\,, \tag{3.14}$$

$$\frac{1}{\sqrt[4]{\pi\, n}}\left(1-\tfrac{1}{16n}+\tfrac{1}{512n^2}\right)-\tfrac{1}{150n^{13/4}}<\sqrt{w_n}<\frac{1}{\sqrt[4]{\pi\, n}}\left(1-\tfrac{1}{16n}+\tfrac{1}{512n^2}\right)+\tfrac{1}{150n^{13/4}}\,, \tag{3.15}$$

$$\frac{1}{(\pi\, n)^{\pi/2}}\left(1-\tfrac{\pi}{8n}+\tfrac{\pi^2}{128n^2}\right)-\tfrac{1}{70n^{3+\pi/2}}<w_n^{\pi}<\frac{1}{(\pi\, n)^{\pi/2}}\left(1-\tfrac{\pi}{8n}+\tfrac{\pi^2}{128n^2}\right)+\tfrac{1}{70n^{3+\pi/2}}\,, \tag{3.16}$$

$$(\pi\, n)^{\pi}\left(1+\tfrac{\pi}{4n}+\tfrac{\pi^2}{32n^2}\right)-14\,n^{\pi-3}<w_n^{-2\pi}<(\pi\, n)^{\pi}\left(1+\tfrac{\pi}{4n}+\tfrac{\pi^2}{32n^2}\right)+14\,n^{\pi-3}. \tag{3.17}$$

Remark 3.6. *In case $a>0$, the inequalities in Corollary 3.5 can be slightly improved using* (3.3) *instead of* (3.4)*. For example, due to* (3.3)*, we have, for $a \in \{1,\ 2\}$,*

$$|\varepsilon(1,n)| \le \varepsilon^*(1,n) = (\pi\, n)^{-1/2}\left(\tfrac{1}{100}+\tfrac{1}{18}\cdot 1\right)\tfrac{1}{10n^3} < \tfrac{1}{270n^{7/2}}$$

$$|\varepsilon(2,n)| \le \varepsilon^*(2,n) = (\pi\, n)^{-1}\left(\tfrac{1}{25}+\tfrac{1}{18}\cdot 1\right)\tfrac{2}{10n^3} < \tfrac{1}{164n^4}\,.$$

[4]For $1 \le n < |a|$ the inequalities are approved directly.

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