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Entropy solution for a nonlinear parabolic problem with homogeneous Neumann boundary condition involving variable exponents

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ABSTRACT

In this paper we prove the existence and uniqueness of an entropy solution for a non-linear parabolic equation with homogeneous Neumann boundary condition and initial data in L^1. By a time discretization technique we analyze the existence, uniqueness and stability questions. The functional setting involves Lebesgue and Sobolev spaces with variable exponents.

RESUMEN

En este artículo probamos la existencia y unicidad de una solución de entropía para una ecuación parabólica no lineal con condiciones de borde Neumann homogéneas y data inicial en L^1. Usando una técnica de discretización del tiempo, analizamos las preguntas de existencia, unicidad y estabilidad. El contexto funcional involucra espacios de Lebesgue y Sobolev con exponentes variables.

Keywords and Phrases: Nonlinear parabolic problem, variable exponents, entropy solution, Neumann-type boundary conditions, semi-discretization.

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1 Introduction and main result

Let Ω be a smooth bounded open domain of $\mathbb{R}^d$, $(d \geq 3)$ with Lipschitz boundary $\partial\Omega$, T is a fixed positive number, in this paper we study the existence and uniqueness of an entropy solution for the following nonlinear parabolic problem

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - \text{div} a\left(x, \nabla u\right) + b\left(u\right) = f \text{ in } Q_T =]0, T[\times\Omega, \\ \\ a\left(x, \nabla u\right).\eta = 0 \text{ on } \sum_T =]0, T[\times\partial\Omega, \\ \\ u(0, .) = u_0 \text{ in } \Omega, \end{cases}$$

where $f \in L^1(Q_T)$, $b : \mathbb{R} \to \mathbb{R}$, $a(x, \xi) : \Omega \times \mathbb{R}^d \to \mathbb{R}$ is Carathéodory function and verifying some assumptions which will be given later, η denotes the unit vector normal on $\partial\Omega$.

The usual weak formulations of parabolic problems in the case where the initial data are in L^1 do not ensure existence and uniqueness of solutions. For this reason, new formulations and types of solutions are given in order to obtain existence and uniqueness. For that, three notions of solution have been adopted: solutions named SOLA (Solution Obtained as the Limit of Approximations) defined by A. Dall'Aglio (see [10]); renormalized solutions defined by R. DiPerna and P.-L. Lions (see [12]); and entropy solutions defined by Ph. Bénilan *et al.* in [8]. In this paper, we will be interested in the entropy formulation.

The stationary version of the problem for the problem (P) has been already studied by Bonzi *et al.* (cf. [9]), where they proved the existence and uniqueness of an entropy solution for the initial data in L^1.

The study of parabolic equations with variable exponents is a very active field (see [1, 2, 20, 21, 23, 27, 29]), in these papers, the authors consider the homogeneous Dirichlet boundary conditions, which permit them to use many results in the generalized Sobolev space $W^{1,p(.)}(\Omega)$ and the many results concerned the differential equation in the literature to achieve there works. In particular in the case of $p(x)$-Laplace, where $b \equiv 0$, Bendahmane *et al.* (see [6]) have proved the existence and uniqueness of renormalized solution. We can also point out that the well-posedness of triply nonlinear degenerate elliptic- parabolic-hyperbolic problems: $b(u)_t - \text{div}a(x, \nabla\phi(u)) + \psi(u) = f$ in a bounded domain with homogeneous Dirichlet boundary conditions by K. H. Karlsen *et al.* in [3].

Unfortunately, in this paper, due to the Neumann boundary condition, we cannot use the ideas developed in these papers and also some functional analysis results which play and important role in the a priori estimation, in particular the famous Poincaré inequality.

To overcome these difficulties we apply a time discretization of given continuous problem by the Euler forward scheme. Let's recall that this method has been used in the literature for the study of some nonlinear parabolic problems, we refer for example to [7, 13, 16, 17] for some details. This scheme is usually used to prove existence of solutions as well as to compute numerical approximations.

In this paper, our assumptions are the following:

$$\begin{cases} p(.) : \overline{\Omega} \quad \to \quad \mathbb{R} \quad \text{is a continuous function such that} \\ 1 < p_- \quad \leq \quad p_+ < +\infty, \end{cases} \tag{1.1}$$

where $p_- := ess \inf_{x\in\Omega} p(x)$ and $p_+ := ess \sup_{x\in\Omega} p(x)$ and

$$b : \overline{\Omega} \to \mathbb{R} \quad \text{is a continuous, nondecreasing function, surjective such that } b(0) = 0. \tag{1.2}$$

Also, we assume that $a(x,\xi) : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ is Carathéodory such that:

- there exists a positive constant C_1 with

$$|a(x,\xi)| \leq C_1 \left(j(x) + |\xi|^{p(x)-1} \right) \tag{1.3}$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, where j is a nonnegative function in $L^{p'(.)}(\Omega)$ with $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$;

- there exists a positive constant C_2 such that for every $x \in \Omega$ and every ξ_1, $\xi_2 \in \mathbb{R}^d$ with $\xi_1 \neq \xi_2$, the following two inequalities hold

$$(a(x,\xi_1) - a(x,\xi_2)) \,.\, (\xi_1 - \xi_2) > 0 \tag{1.4}$$

$$a(x,\xi) \,.\, \xi \geq C_2 |\xi|^{p(x)}. \tag{1.5}$$

The rest of the paper is organized as follows: after some preliminary results in Section 2, we introduce the Euler forward scheme associated with the problem (P) in Section 3. We analyze the stability of the discretized problem and we study the existence of an entropy solution to the parabolic problem (P) in the Section 4.

2 Preliminaries

We define the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ (see [11]) as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u|^{p(x)}\, dx$$

is finite.

If the exponent is bounded, *i.e.*, if $p_+ < +\infty$, then the expression

$$\|u\|_{p(.)} := \inf \left\{ \lambda > 0 : \rho_{p(.)}(u/\lambda) \leq 1 \right\}$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxembourg norm.
The space $\left(L^{p(.)}(\Omega), \|.\|_{p(.)}\right)$ is a separable Banach space. Moreover, if $1 < p_- \le p_+ < +\infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1.$
Finally, we have the Hölder type inequality

$$\left|\int_\Omega uvdx\right| \le \left(\frac{1}{p_-} + \frac{1}{(p_-)'}\right) \|u\|_{p(.)} \|v\|_{p'(.)}, \tag{2.1}$$

for all $u \in L^{p(.)}(\Omega)$ and $v \in L^{p'(.)}(\Omega)$.
Let

$$W^{1,p(.)}(\Omega) := \left\{u \in L^{p(.)}(\Omega) : |\nabla u| \in L^{p(.)}(\Omega)\right\},$$

which is Banach space equipped with the following norm

$$\|u\|_{1,p(.)} := \|u\|_{p(.)} + \|\nabla u\|_{p(.)}.$$

The space $\left(W^{1,p(.)}(\Omega), \|.\|_{1,p(.)}\right)$ is a separable and reflexive Banach space.
An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(.)}$ of the space $L^{p(.)}(\Omega)$. We have the following result.

Proposition 2.1 (see [14, 28])**.** *If $u_n, u \in L^{p(.)}(\Omega)$ and $p_+ < \infty$, the following properties hold true:*

(i) $\|u\|_{p(.)} > 1 \Rightarrow \|u\|_{p(.)}^{p_-} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p_+}$;

(ii) $\|u\|_{p(.)} < 1 \Rightarrow \|u\|_{p(.)}^{p_+} < \rho_{p(.)}(u) < \|u\|_{p(.)}^{p_-}$;

(iii) $\|u\|_{p(.)} < 1\ (respectively = 1; > 1) \Leftrightarrow \rho_{p(.)}(u) < 1\ (respectively = 1; > 1)$;

(iv) $\|u_n\|_{p(.)} \to 0\ (respectively \to +\infty) \Leftrightarrow \rho_{p(.)}(u_n) < 1\ (respectively \to +\infty)$;

(v) $\rho_{p(.)}\left(u/\|u\|_{p(.)}\right) = 1.$

For a measurable function $u : \Omega \to \mathbb{R}$ we introduce the following notation:

$$\rho_{1,p(.)}(u) = \int_\Omega |u|^{p(x)} dx + \int_\Omega |\nabla u|^{p(x)} dx.$$

Proposition 2.2 (see [25, 26])**.** *If $u \in W^{1,p(.)}(\Omega)$, the following properties hold true:*

(i) $\|u\|_{1,p(.)} > 1 \Rightarrow \|u\|_{1,p(.)}^{p_-} < \rho_{1,p(.)}(u) < \|u\|_{1,p(.)}^{p_+}$;

(ii) $\|u\|_{1,p(.)} < 1 \Rightarrow \|u\|_{1,p(.)}^{p_+} < \rho_{p(.)}(u) < \|u\|_{1,p(.)}^{p_-}$;

(iii) $\|u\|_{1,p(.)} < 1\ (respectively = 1; > 1) \Leftrightarrow \rho_{1,p(.)}(u) < 1\ (respectively = 1; > 1)$.

Put

$$p^{\partial}(x) := (p(x))^{\partial} = \begin{cases} \dfrac{(N-1)\,p(x)}{N-p(x)}, & \text{if} \quad p(x) < N \\ \infty, & \text{if} \quad p(x) \geq N. \end{cases}$$

Proposition 2.3 (see [26])**.** *Let $p \in C\left(\bar{\Omega}\right)$ and $p_- > 1$. If $q \in C(\partial\Omega)$ satisfies the condition*

$$1 < q(x) < p^{\partial}(x) \quad \forall x \in \partial\Omega,$$

then, there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\partial\Omega)$.
In particular, there is a compact embedding $W^{1,p(.)}(\Omega) \hookrightarrow L^{p(.)}(\partial\Omega)$.

Following [29], we extend a variable exponent $p : \overline{\Omega} \to [1,+\infty)$ to $\overline{Q_T} = [0,T] \times \overline{\Omega}$ by setting $p(t,x) = p(x)$ for all $(t,x) \in \overline{Q_T}$.
We may also consider the generalized Lebesgue space

$$L^{p(.)}(Q) = \left\{ u : Q \to \mathbb{R} \text{ measurable such that } \iint_Q |u(t,x)|^{p(x)}\, d(t,x) < \infty \right\}$$

endowed with the norm

$$\|u\|_{L^{p(.)}(Q_T)} := \inf \left\{ \lambda > 0,\ \iint_{Q_T} \left| \frac{u(t,x)}{\lambda} \right|^{p(x)} d(t,x) < 1 \right\},$$

which share the same properties as $L^{p(.)}(\Omega)$.

For a measurable set U in $\mathbb{R}^d$, meas(U) denotes its measure, C_i and C will denote various positive constants. For a Banach space X and $a < b$, $L^q(a,b;X)$ is the space of measurable functions $u : [a,b] \to X$ such that

$$\left(\int_a^b \|u\|_X^q \, dt \right)^{\frac{1}{q}} := \|u\|_{L^q(a,b;X)} < \infty. \tag{2.2}$$

For a given constant $k > 0$ we define the cut-off function $T_k : \mathbb{R} \to \mathbb{R}$ by

$$T_k(s) := \begin{cases} s & \text{if} \quad |s| \leq k \\ k \text{ sign}(s) & \text{if} \quad |s| > k \end{cases}$$

with

$$\text{sign}(s) := \begin{cases} 1 & \text{if} \quad s > 0 \\ 0 & \text{if} \quad s = 0 \\ -1 & \text{if} \quad s < 0. \end{cases}$$

Let $J_k : \mathbb{R} \to \mathbb{R}^+$ defined by

$$J_k(x) = \int_0^x T_k(s)ds$$

(J_k is a primitive of T_k). We have (see [15])

$$\left\langle \frac{\partial v}{\partial t}, T_k(s) \right\rangle = \frac{d}{dt} \left(\int_\Omega J_k(v)dx \right) \text{ in } L^1(]0,T[)$$

which implies that

$$\int_0^t \left\langle \frac{\partial v}{\partial t}, T_k(s) \right\rangle = \int_\Omega J(v(t))dx - \int_\Omega J(v(0))dx$$

For all $u \in W^{1,p(.)}(\Omega)$ we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense.
In the sequel, we will identify at the boundary, u and $\tau(u)$.
Set

$$\mathcal{T}^{1,p(.)}(\Omega) = \Big\{ u : \Omega \to \mathbb{R}, \text{ measurable such that } T_k(u) \in W^{1,p(.)}(\Omega), \text{ for any } k > 0 \Big\}.$$

Proposition 2.4 (see [8]). *Let $u \in \mathcal{T}^{1,p(.)}(\Omega)$. Then there exists a unique measurable function $v : \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, for all $k > 0$. The function v is denoted by ∇u. Moreover, if $u \in W^{1,p(.)}(\Omega)$ then $v \in \left(L^{p(.)}(\Omega)\right)^N$ and $v = \nabla u$ in the usual sense.*

We denote by $\mathcal{T}_{tr}^{1,p(.)}(\Omega)$ (*cf.* [4, 5, 18, 19]) the set of functions $u \in \mathcal{T}^{1,p(.)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p(.)}(\Omega)$ satisfying the following conditions:

i) $u_n \to u$ a.e. in Ω.

ii) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $\left(L^1(\Omega)\right)^N$ for any $k > 0$.

iii) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [4, 5]. In the sequel, the trace of $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. If $u \in W^{1,p(.)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover $u \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$ and if $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$ then $(u - \varphi) \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$ and $tr(u - \varphi) = tr(u) - tr(\varphi)$.

3 The semi-discrete problem

In this section, we study the Euler forward scheme associated with the problem (P):

$$(P_n) \begin{cases} U^n - \tau \mathrm{div}\, a(x, \nabla U^n) + \tau b(U^n) = \tau f^n + U^{n-1} \text{ in } \Omega \\ \\ a(x, \nabla U^n).\eta = 0 \text{ on } \partial\Omega, \\ \\ U^0 = u_0 \text{ in } \Omega \end{cases}$$

where $N\tau = T$, $0 < \tau < 1$, $1 \le n \le N$ and

$$f_n(.) \quad = \quad \frac{1}{\tau}\int_{(n-1)\tau}^{n\tau} f(s,.)ds \quad \text{in } \Omega.$$

Definition 3.1. *An entropy solution to the discretized problems* (P_n) *is a sequence* $(U^n)_{0\leq n\leq N}$ *such that* $U^0 = u_0 \in L^1(\Omega)$ *and* U^n *is defined by induction as an entropy solution to the problem*

$$\begin{cases} U^n - \tau \operatorname{div} a(x, \nabla U^n) + \tau b(U^n) = \tau f_n + U^{n-1} \text{ in } \Omega \\ \\ a(x, \nabla U^n).\eta = 0 \text{ on } \partial\Omega \end{cases}$$

i.e. $U^n \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$, $b(U^n) \in L^1(\Omega)$, *and for every* $k > 0$

$$\tau \int_\Omega a(x, \nabla U^n).\nabla T_k(U^n - \varphi)dx + \int_\Omega (\tau b(U^n) + U^n) T_k(U^n - \varphi)dx \leq \int_\Omega (\tau f_n + U^{n-1}) T_k(U^n - \varphi)dx \tag{3.1}$$

for all $\varphi \in W^{1,p(.)}(\Omega) \cap L^\infty(\Omega)$.

We have the following result

Lemma 3.2. *Let hypotheses* (1.3) – (1.5) *be satisfied. If* $(U^n)_{0\leq n\leq N}$ *is an entropy solution of problems* (P_n), *then* $U^n \in L^1(\Omega)$ *for all* $n = 1, \ldots, N$.

Proof. For $n = 1$, we take $\varphi = 0$ in (3.1), to get,

$$\tau \int_\Omega a(x, \nabla U^1).\nabla T_k(U^1)dx + \int_\Omega (\tau b(U^1) + U^1) T_k(U^1)dx \leq \int_\Omega (\tau f_1 + u_0) T_k(U^1)dx,$$

which is equivalent to

$$\tau \int_\Omega a(x, \nabla T_k(U^1))\nabla T_k(U^1)dx + \int_\Omega \tau b(U^1) T_k(U^1)dx + \int_\Omega U^1 T_k(U^1)dx \leq \int_\Omega (\tau f_1 + u_0) T_k(U^1)dx, \tag{3.2}$$

By the assumption (1.5) and the properties of the function b, we have

$$\tau \int_\Omega a(x, \nabla T_k(U^1))\nabla T_k(U^1)dx + \int_\Omega \tau b(U^1) T_k(U^1)dx \geq 0,$$

then it follows that

$$\int_\Omega U^1 T_k(U^1)dx \leq k\tau \|f_1\|_1 + k \|u_0\|_1 .$$

Since

$$\sum_{n=1}^N \tau \|f_n\|_1 \leq \|f\|_1 .$$

Then, it follows that

$$\int_\Omega U^1 T_k(U^1)dx \leq k(\|f\|_1 + \|u_0\|_1). \tag{3.3}$$

Since

$$\lim_{k\to 0} U^1 \frac{T_k(U^1)}{k} = |U^1|.$$

Then dividing (3.3) by k and letting $k \to 0$; we deduce by Fatou's lemma that

$$\|U^1\|_1 \leq (\|f\|_1 + \|u_0\|_1) \tag{3.4}$$

□

Theorem 3.3. *Let hypotheses* (1.3) − (1.5) *be satisfied. Then for all* $N \in \mathbb{N}$, *the problems* (P_n) *have unique entropy solution* $U^n \in \mathcal{T}_{tr}^{1,p(.)}(\Omega) \cap L^1(\Omega)$ *for all* $n = 1, \ldots, N$.

Proof. The problem (P_1) can be rewritten in the following form

$$-\tau \text{div} a\,(x, \nabla u) + \overline{b}(u) = F_1 \text{ in } \Omega$$

$$a(x, \nabla u).\eta = 0 \text{ on } \partial\Omega$$

with

$$\overline{b}(s) := \tau b(s) + s, \quad F_1 := \tau f_1 + u_0.$$

From the assumption (H_2), we have $F_1 \in L^1(\Omega)$, and using the properties of b, we obtain $\overline{b}$ is a continuous, nondecreasing function, surjective such that $b(0) = 0$. Hence, using [9, Theorem 4.3], we have the existence of unique entropy solution $U^1 \in \mathcal{T}_{tr}^{1,p(.)}(\Omega)$, $b\left(U^1\right) \in L^1(\Omega)$.

Thanks to Lemma 3.2, by induction, we deduce that for $n = 2, \ldots, N$, the problem

$$\begin{aligned} &u - \tau \text{div} a\,(x, \nabla u) + \tau\alpha\,(u) = \tau f_n + U^{n-1} \text{ in } \Omega \\ &a\,(x, \nabla u)\,.\eta = 0 \text{ on } \partial\Omega, \end{aligned}$$

has an unique entropy solution $U^n \in \mathcal{T}_{tr}^{1,p(.)}(\Omega) \cap L^1(\Omega)$, $b\,(U^n) \in L^1(\Omega)$. □

4 Stability

This section is devoted to the a priori estimates for the discrete entropy solution $(U^n)_{1 \le n \le N}$. These result are essentials for the study of the convergence of the Euler forward scheme.

Theorem 4.1. *Let hypotheses* (1.3)−(1.5) *be satisfied. Then there exist positive constants* $C(u_0, f)$, $C(u_0, f, \Omega)$ *depending on the data but not on* N *such that for all* $n = 1, \ldots, N$, *we have*

1. $\|U^n\|_1 \le C(u_0, f)$
2. $\tau \sum_{i=1}^n \left\|b(U^i)\right\|_1 \le C(u_0, f)$
3. $\sum_{i=1}^n \left\|U^i - U^{i-1}\right\|_1 \le C(, u_0, f)$
4. $\tau \sum_{i=1}^n \rho_{p(.)}(\nabla T_k(U^i)) \le kC(u_0, f)$
5. $\tau \sum_{i=1}^n \int_{\{|U^i| \le k\}} |\nabla U^i|^{p_-} dx \le kC(u_0, f, \Omega)$

Proof. 1 and 2. For $\varphi = 0$ as a test function in (3.1), we have

$$\begin{aligned} &\frac{\tau}{k} \int_\Omega a(x, \nabla T_k(U^i)) \nabla T_k(U^i) dx + \int_\Omega U^i \frac{T_k(U^i)}{k} dx + \int_\Omega \tau b(U^i) \frac{T_k(U^i)}{k} dx \\ &\le \tau \|f_i\|_1 + \left\|U^{i-1}\right\|_1 dx. \end{aligned}$$

Since

$$\int_\Omega a(x, \nabla T_k(U^i))\nabla T_k(U^i) dx \geq 0.$$

Then, it follows that

$$\int_\Omega U^i \frac{T_k(U^i)}{k} dx + \int_\Omega \tau b(U^i) \frac{T_k(U^i)}{k} dx \leq \tau \|f_i\|_1 + \left\|U^{i-1}\right\|_1.$$

Then letting $k \to 0$ and using Fatou's lemma, we deduce that

$$\left\|U^i\right\|_1 + \tau \left\|b(U^i)\right\|_1 \leq \tau \|f_i\|_1 + \left\|U^{i-1}\right\|_1. \tag{4.1}$$

Now, we sum (4.1) from $i = 1$ to n to obtain

$$\|U^n\|_1 + \tau \sum_{i=1}^{n} \left\|b(U^i)\right\|_1 \leq \|f\|_1 + \|u_0\|_1 \tag{4.2}$$

which give, the inequalities 1 and 2.

3. For $k \geq 1$, we take $\varphi = T_h(U^i - \text{sign}(U^i - U^{i-1}))$, $(h > 1)$ as a test function in (3.1), then letting $h \to \infty$, for $k \geq 1$, we obtain,

$$\tau \lim_{h\to\infty} \mathcal{I}(k, h) + \left\|U^i - U^{i-1}\right\|_1 \leq \tau \left(\|f_i\|_1 + \left\|b(U^i)\right\|_1\right)$$

where

$$\begin{aligned} \mathcal{I}(k,h) &:= \int_\Omega a(x, \nabla U^i)\nabla T_k(U^i - T_h(U^i - \text{sign}(U^i - U^{i-1})))dx \\ &= \int_{\Omega_{k,h}\cap\overline{\Omega(k)}} a(x, \nabla U^i)\nabla U^i dx \end{aligned}$$

and

$$\begin{aligned} \Omega_{k,h} &:= \left\{|U^i - T_h(U^i - \text{sign}(U^i - U^{i-1}))| \leq k\right\} \\ \overline{\Omega(k)} &= \left\{|U^i - \text{sign}(U^i - U^{i-1})| > h\right\}. \end{aligned}$$

Then by the hypothesis (1.3) , we have

$$\lim_{h\to\infty} \mathcal{I}(k, h) \geq 0.$$

Then, it follows that

$$\left\|U^i - U^{i-1}\right\|_1 \leq k\tau \left(\|f_i\|_1 + \left\|b(U^i)\right\|_1\right). \tag{4.3}$$

Then, summing (4.3) from $i = 1$ to n and by the stability result 2, we obtain the stability result 3.

4. We take $\varphi = 0$ as a test function in 3.1 to get

$$\tau \left(\int_\Omega |a(x, \nabla T_k(U^i))\nabla T_k(U^i)dx\right) \leq k\tau(\|f_i\|_1 + \left\|b(U^i)\right\|_1) + k \left\|U^i - U^{i-1}\right\|_1.$$

Therefore, using the assumption (1.5) it follows that

$$\tau\rho_{p(x)}(\nabla T_k(U^i)) \leq C_3[k\tau(\|f_i\|_1 + \left\|b(U^i)\right\|_1) + k\left\|U^i - U^{i-1}\right\|_1]. \tag{4.4}$$

Now, summing (4.4) from $i = 1$ to n and using the stability results 1, 2, 3, we get

$$\begin{aligned} \tau\sum_{i=1}^{n}\rho_{p(x)}(\nabla T_k(U^i)) &\leq C_3 k\left[\|f\|_1 + \tau\sum_{i=1}^{n}\left\|b(U^i)\right\|_1 + \sum_{i=1}^{n}\left\|U^i - U^{i-1}\right\|_1\right] \\ &\leq kC(f, u_0). \end{aligned} \tag{4.5}$$

5. According to (4.5), we get from the above estimate

$$\tau\sum_{i=1}^{n}\int_{\{|U^i|\leq k\}}|\nabla U^i|^{p(x)}dx \leq kC(u_0, f). \tag{4.6}$$

Now, note that

$$\begin{aligned} \int_{\{|U^i|\leq k\}}|\nabla U^i|^{p_-}dx &= \int_{\left\{|U^i|\leq k,\ |\nabla U^i|>\frac{1}{N}\right\}}|\nabla U^i|^{p_-}dx + \int_{\left\{|U^i|\leq k\ |\nabla U^i|\leq\frac{1}{N}\right\}}|\nabla U^i|^{p_-}dx \\ &\leq \int_{\left\{|U^i|\leq k,\ |\nabla U^i|>\frac{1}{N}\right\}}|\nabla U^i|^{p_-}dx + \frac{1}{N}\text{meas}(\Omega) \\ &\leq \int_{\{|U^i|\leq k\}}|\nabla U^i|^{p(x)}dx + \frac{1}{N}\text{meas}(\Omega). \end{aligned}$$

By the inequalities above, thanks to (4.6), we obtain

$$\begin{aligned} \tau\sum_{i=1}^{n}\int_{\{|U^i|\leq k\}}|\nabla U^i|^{p_-}dx &\leq kC(u_0, f) + \frac{n}{N}\text{meas}(\Omega) \\ &\leq kC(u_0, f) + \text{meas}(\Omega) \leq k(C(u_0, f) + \text{meas}(\Omega)) \end{aligned} \tag{4.7}$$

for all $k \geq 1$. □

5 Convergence and existence result

In this section, we prove the existence of an entropy solution of problem (P). First of all, we introduce the appropriate spaces for the entropy formulation of the nonlinear parabolic problem (P).

We define the space:

$$V = \left\{v \in L^{p_-}(0, T; W^{1,p(\cdot)}(\Omega)) : \nabla v \in (L^{p(\cdot)}(Q_T))^d\right\},$$

and

$$\begin{aligned} \mathcal{T}^{1,p(\cdot)}(Q_T) = &\Big\{u : \Omega \times (0, T]; \text{measurable} \mid T_k(u) \in L^{p_-}(0, T; W^{1,p(\cdot)}(\Omega)) \\ &\text{with } \nabla T_k(u) \in (L^{p(\cdot)}(Q_T))^d \ \text{ for every } k > 0\Big\}. \end{aligned}$$

Definition 5.1. *An entropy solution to problem (P) is a function $u \in \mathcal{T}^{1,p(\cdot)}(Q_T) \cap C(0,T;L^1(\Omega))$ such that and for all $k > 0$ we have*

$$\begin{aligned}
&\int_0^t \int_\Omega a(x,\nabla u)\nabla T_k(u-\varphi) + \int_0^t \int_\Omega b(u)T_k(u-\varphi) \\
&\leq -\int_0^t \left\langle \frac{\partial \varphi}{\partial s}, T_k(u-\varphi) \right\rangle + \int_\Omega J_k(u(0)-\varphi(0)) - \int_\Omega J_k(u(t)-\varphi(t)) \\
&+ \int_0^t \int_\Omega f T_k(u-\varphi)
\end{aligned}$$

for all $\varphi \in L^\infty(Q) \cap V \cap W^{1,1}(0,T;L^1(\Omega))$ and $t \in [0,T]$.

Our main result is

Theorem 5.2. *Let hypotheses $(H1)-(H3)$ be satisfied. Then the nonlinear parabolic problem (P) has an entropy solution.*

Proof. The proof is divided into two steps

Step 1: The Rothe function. We introduce a piecewise linear extension:

$$\begin{cases} u^N(0) := u_0, \\ u^N(t) := U^{n-1} + (U^n - U^{n-1})\frac{t-t^{n-1}}{\tau} \end{cases} \tag{5.1}$$

for all $t \in]t^{n-1}, t^n]$, $n = 1, \cdots, N$, in Ω and a piecewise constant function

$$\begin{cases} \overline{u}^N(0) := u_0, \\ \overline{u}^N(t) := U^n, \ \forall t \in]t^{n-1}, t^n], \ n = 1, \cdots, N, \text{ in } \Omega, \end{cases} \tag{5.2}$$

where $t^n := n\tau$ and $(U^n)_{1 \leq n \leq N}$ an entropy solution of (P_n).

By Theorem 3.3, for any $N \in \mathbb{N}$; the solution $(U^n)_{N \in \mathbb{N}}$ of problems (P_n) is unique. Thus, u^N and $\overline{u}^N$ are uniquely defined. Consequently, by the Theorem 4.1, we deduce the existence of a constant $C(T,u_0,f)$ not depending on N such that for all $N \in \mathbb{N}$, we have

$$\begin{aligned}
&\left\|\overline{u}^N - u^N\right\|_{L^1(Q_T)} \leq \frac{1}{N} C(T,u_0,f) \\
&\left\|u^N\right\|_{L^1(Q_T)} \leq C(T,u_0,f) \\
&\left\|\overline{u}^N\right\|_{L^1(Q_T)} \leq C(T,u_0,f) \\
&\left\|\frac{\partial u^N}{\partial t}\right\|_{L^1(Q_T)} \leq C(T,u_0,f) \\
&\left\|b(\overline{u}^N)\right\|_{L^1(Q_T)} \leq C(T,u_0,f)
\end{aligned} \tag{5.3}$$

Moreover combining Proposition 2.1 and Young inequality, we get

$$\begin{aligned}
\left\|\nabla T_k(U^N)\right\|_{p(x)}^{p_-} &\leq \max\left\{\rho_{p(x)}(\nabla T_k(U^N)), \rho_{1,p(x)}(\nabla T_k U^N)^{\frac{p_-}{p_+}}\right\} \\
&\leq \rho_{p(x)}(\nabla T_k(U^N)) + \rho_{1,p(x)}(\nabla T_k U^N)^{\frac{p_-}{p_+}} \\
&\leq \rho_{p(x)}(\nabla T_k(U^N)) + \frac{p_-}{p_+}\rho_{p(x)}(\nabla T_k(U^N)) + 1 - \frac{p_-}{p_+} \qquad (5.4)\\
&\leq 2\rho_{p(x)}(\nabla T_k(U^N)) + 1.
\end{aligned}$$

Thanks to Poincaré-Wirtinger inequality, we have

$$\left\|T_k(U^N)\right\|_{p(x)} \leq C\text{meas}(\Omega)\left\|\nabla T_k(U^N)\right\|_{p(x)} + k\left\|1\right\|_{p(x)},$$

which implies that

$$\left\|T_k(U^N)\right\|_{p(x)}^{p_-} \leq 2^{p_- -1}\left((C\text{meas}(\Omega))^{p_-}\left\|\nabla T_k(U^N)\right\|_{p(x)}^{p_-} + k^{p_-}\left\|1\right\|_{p(x)}^{p_-}\right), \qquad (5.5)$$

then it follows that,

$$\begin{aligned}
\left\|T_k(U^N)\right\|_{1,p(x)}^{p_-} &\leq 2^{p_- -1}\left[(C\text{meas}(\Omega))^{p_-}\left(2\rho_{p(x)}(\nabla T_k(U^N)) + 1\right) + k^{p_-}\left\|1\right\|_{p(x)}^{p_-}\right] \qquad (5.6)\\
&\quad + 2\rho_{p(x)}(\nabla T_k(U^N)) + 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^T \left\|T_k(U^N)\right\|_{1,p(.)}^{p_-} dt &\leq 2^{p_- -1}\Bigg[(C\text{meas}(\Omega))^{p_-}\left(2\int_0^T \rho_{p(.)}(\nabla T_k(U^N))dt + T\right) \\
&\quad + Tk^{p_-}\left\|1\right\|_{p(x)}^{p_-}\Bigg] + 2\int_0^T \rho_{p(.)}(\nabla T_k(U^N))dt + T \\
&\leq 2^{p_- -1}\Bigg[(C\text{meas}(\Omega))^{p_-}\left(2\sum_{n=1}^{N}\int_{(n-1)\tau}^{n\tau} \rho_{p(.)}(\nabla T_k(U^N))dt + T\right) \\
&\quad + Tk^{p_-}\left\|1\right\|_{p(.)}^{p_-}\Bigg] + 2\sum_{n=1}^{N}\int_{(n-1)\tau}^{n\tau} \rho_{p(.)}(\nabla T_k(U^N))dt + T \qquad (5.7)\\
&\leq 2^{p_- -1}\Bigg[(C\text{meas}(\Omega))^{p_-}\left(2\sum_{n=1}^{N}\tau\rho_{p(.)}(\nabla T_k(U^n)) + T\right) \\
&\quad + Tk^{p_-}\left\|1\right\|_{p(.)}^{p_-}\Bigg] + 2\sum_{n=1}^{N}\tau\rho_{1,p(.)}(T_k(U^n)) + T.
\end{aligned}$$

Consequently from stability result 4 it follows that

$$\left\|T_k(\overline{u}^N)\right\|_{L^{p_-}(0,T;W^{1,p(x)}(\Omega))} \leq C(T, k, u_0, f, p_-). \qquad (5.8)$$

Lemma 5.3. *Let hypotheses* (1.3) – (1.5) *be satisfied. Then the sequence* $(\overline{u}^N)_{N\in\mathbb{N}}$ *converges in measure and a.e. in* Q_T.

Proof. Let ε, r, k be positive numbers. For $N, M \in \mathbb{N}$, we have the inclusion

$$\begin{aligned}\left\{|\overline{u}^N - \overline{u}^M| > r\right\} &\subset \left\{|\overline{u}^N| > k\right\} \cup \left\{|\overline{u}^M| > k\right\} \\ &\cup \left\{|\overline{u}^N| \le k,\, |\overline{u}^M| \le k, |\overline{u}^N - \overline{u}^M| > r\right\}.\end{aligned}$$

Firstly, we have

$$\text{meas}\left\{|\overline{u}^N| > k\right\} \le \frac{1}{k}\left\|\overline{u}^N\right\|_{L^1(Q_T)} \le \frac{1}{k}C(T, u_0, f). \tag{5.9}$$

Similarly, we have

$$\text{meas}\left\{|\overline{u}^M| > k\right\} \le \frac{1}{k}\left\|\overline{u}^N\right\|_{L^1(Q_T)} \le \frac{1}{k}C(T, u_0, f). \tag{5.10}$$

Therefore, for k large enough, we have

$$\text{meas}(\left\{|\overline{u}^M| > k\right\} \cup \left\{|\overline{u}^M| > k\right\}) \le \frac{\varepsilon}{2}. \tag{5.11}$$

Secondly, by the Proposition 2.1 and Young inequality, we have

$$\begin{aligned}\left\|\nabla T_k(\overline{u}^N)\right\|_{L^{p(.)}(Q_T)} &\le \max\left\{\left(\int_0^T\int_\Omega |\nabla T_k(\overline{u}^N)|^{p(x)}dxdt\right)^{\frac{1}{p_-}}; \left(\int_0^T\int_\Omega |\nabla T_k(\overline{u}^N)|^{p(x)}dxdt\right)^{\frac{1}{p_+}}\right\} \\ &\le \left(\int_0^T\int_\Omega |\nabla T_k(\overline{u}^N)|^{p(x)}dxdt\right)^{\frac{1}{p_-}} + \left(\int_0^T\int_\Omega |\nabla T_k(\overline{u}^N)|^{p(x)}dxdt\right)^{\frac{1}{p_+}}\end{aligned}$$

and also, we have

$$\begin{aligned}\int_0^T\int_\Omega |\nabla T_k(\overline{u}^N)|^{p(x)}dxdt = \int_0^T \rho_{p(.)}(T_k(\nabla\overline{u}^N)) &= \sum_{n=1}^N \int_{(n-1)\tau}^{n\tau} \rho_{p(.)}(\nabla T_k(U^N))dt \\ &\le \sum_{n=1}^N \tau\rho_{p(.)}(\nabla T_k(U^n)).\end{aligned}$$

Therefore, using the stability result 4 and Proposition 2.1, it follows

$$\left\|\nabla T_k(\overline{u}^N)\right\|_{(L^{p(x)}(Q_T))^d} \le (kC(u_0, f))^{\frac{1}{p_-}} + (kC(u_0, f))^{\frac{1}{p_+}}. \tag{5.12}$$

Since by the Poincaré-Wirtinger inequality, we have

$$\left\|T_k(\overline{u}^N)\right\|_{L^{p(x)}(Q_T)} \le C\text{meas}(\Omega)\left\|\nabla T_k(\overline{u}^N)\right\|_{L^{p(x)}(Q_T)} + k\left\|1\right\|_{L^{p(x)}(Q_T)},$$

then by (5.12), we get

$$\left\|T_k(\overline{u}^N)\right\|_{L^{p(x)}(Q_T)} \le C\text{meas}(\Omega)\left((kC(u_0, f))^{\frac{1}{p_-}} + (kC(u_0, f))^{\frac{1}{p_+}}\right) + k\left\|1\right\|\,|_{L^{p(x)}(Q_T)}. \tag{5.13}$$

Hence, the sequences $(T_k(\overline{u}^N))_{N\in\mathbb{N}}$ are bounded in $L^{p(.)}(Q_T)$. Then, there exists a subsequence, still denoted by $(T_k(\overline{u}^N))_{N\in\mathbb{N}}$, that is a Cauchy sequence in $L^{p(.)}(Q_T)$ and in measure. Thus, there exists $N_0 \in \mathbb{N}$ such that for all $N, M \ge N_0$, we have

$$\text{meas}\left(\left\{|\overline{u}^N| \le k,\, |\overline{u}^M| \le k, |\overline{u}^N - \overline{u}^M| > r\right\}\right) < \frac{\varepsilon}{2}. \tag{5.14}$$

Then, by (5.11) and (5.14), $(\overline{u}^N)_{N\in\mathbb{N}}$ converges in measure. Therefore there exists an element $u \in M(Q_T)$ such that

$$\overline{u}^N \to u \ \text{ a.e. in}\, Q_T.$$

□

Now, by (5.12)

$$(\nabla T_k(\overline{u}^N))_{N\in\mathbb{N}} \text{ is uniformly bounded in, } (L^{p(.)}(Q_T))^d. \tag{5.15}$$

Hence there exists a subsequence, still denoted by

$$(\nabla T_k(\overline{u}^N))_{N\in\mathbb{N}} \quad \text{converges weakly to an element } \ V \text{ in } L^{p(.)}(Q_T).$$

Since

$$T_k(\overline{u}^N) \quad \text{converges weakly to } \ T_k(u) \text{ in } L^{p(.)}(Q_T).$$

Then

$$\nabla T_k(\overline{u}^N) \quad \text{converges weakly to } \ \nabla T_k(u) \text{ in } (L^{p(.)}(Q_T))^d. \tag{5.16}$$

and by (5.8) we conclude that

$$T_k(u) \in L^{p_-}(0,T;W^{1,p(.)}(\Omega)) \ \text{ for all } \ k>0.$$

In the sequel, we need the following Lemma (see [22]).

Lemma 5.4. *Let $(v_n)_{n\geq 1}$ be a sequence of measurable functions in Ω. If $(v_n)_{n\geq 1}$ converges in measure to v and is uniformly bounded in $L^{p(.)}(\Omega)$ for some $1 << p(.) \in L^\infty(\Omega)$, then $(v_n)_{n\geq 1} \to v$ strongly in $L^1(\Omega)$.*

Now, we have the following result

Lemma 5.5. *Let hypotheses (1.3) – (1.5) be satisfied. Then*

(i) $(\nabla T_k(\overline{u}^N))_{N\in\mathbb{N}}$ *converges in measure to* $\nabla T_k(u)$;

(ii) $(a(x,T_k(\overline{u}^N)))_{N\in\mathbb{N}}$ *converges strongly to* $a(x,\nabla T_k(u))$ *in* $(L^1(Q_T))^d$ *and weakly in* $(L^{p'(.)}(Q_T))^d$.

Proof. (i) Let $h\geq 1$, from the Hölder type inequality, we have

$$\begin{aligned}
\text{meas}\left\{|\nabla T_k(\overline{u}^N)-\nabla T_k(u)|>h\right\} &\leq \frac{1}{h}\int_{Q_T}|\nabla T_k(\overline{u}^N)-\nabla T_k(u)|dxds \\
&\leq \frac{1}{h}\left(\frac{1}{p_-}+\frac{1}{p_+}\right)\left\|\nabla T_k(\overline{u}^N)-\nabla T_k(u)\right\|_{p(.)}\|1\|_{p'(.)} \qquad (5.17)\\
&\leq \frac{1}{h}\left(\frac{1}{p_-}+\frac{1}{(p_-)'}\right)\left(\left\|\nabla T_k(\overline{u}^N)\right\|_{p(.)}+\|\nabla T_k(u)\|_{p(.)}\right)\|1\|_{p'(.)}.
\end{aligned}$$

So by (5.15), meas $\left\{|\nabla T_k(\overline{u}^N)-\nabla T_k(u)|>h\right\}\to 0$ as $h\to\infty$ for any fixed $k>0$ and the proof of (i) is complete.

As a consequence of (i), up to a subsequence, we can assume that $\nabla T_k(\overline{u}^N)\to\nabla T_k(u)$ a.e in Q_T.

(ii) Since $a(x,\xi)$ is continuous with respect to $\xi\in\mathbb{R}^N$, then by (i) we deduce that

$$(a(x,T_k(\overline{u}^N)))_{N\in\mathbb{N}} \text{ converges in measure to } a(x,\nabla T_k(u)) \text{ and a.e. in } Q_T.$$

Moreover, using the hypotheses (1.3) and (5.12) one shows that $(a(x, \nabla T_k(\overline{u}^N)))_{N\in\mathbb{N}}$ is uniformly bounded in $(L^{p'(.)}(Q_T))^d$.

Consequently, in the one part thanks to Lemma 5.4 it follows that $(a(x, T_k(\overline{u}^N)))_{N\in\mathbb{N}} \to a(x, \nabla T_k(u))$ strongly in $\left(L^1(Q_T)\right)^d$.

On the other part, we can extract a subsequence still denoted by $(a(x, \nabla T_k(\overline{u}^N)))_{N\in\mathbb{N}}$ such that $a(x, \nabla T_k(\overline{u}^N)) \rightharpoonup \zeta_k$ in $(L^{p'(.)}(Q_T))^d$. Since each of the convergence implies the weak L^1-convergence, ζ_k can be identified with $a(x, \nabla T_k(u))$, thus $a(x, \nabla T_k(u)) \in (L^{p'(.)}(Q_T))^d$. This completes the proof. ☐

Lemma 5.6. $(\overline{u}^N)_{N\in\mathbb{N}}$ *converges a.e. in* Σ_T.

Proof. We know that the trace operator is compact from $W^{1,1}(\Omega)$ into $L^1(\partial\Omega)$, then there exists a constant C such that

$$\int_0^T \left\|T_k(\overline{u}^N(t)) - T_k(u(t))\right\|_{L^1(\partial\Omega)} dt \leq C \int_0^T \left\|T_k(\overline{u}^N(t)) - T_k(u(t))\right\|_{W^{1,1}(\Omega)} dt.$$

Since $W^{1,p(.)}(\Omega) \hookrightarrow W^{1,1}(\Omega)$ for all $p(.) \geq 1$, then by the Hölder type inequality, we deduce that

$$T_k(\overline{u}^N(t)) \to T_k(u) \,\text{in}\, L^1(\Sigma_T) \text{ and a.e. on } \Sigma_T.$$

So, there exists $A \subset \Sigma_T$ such that $T_k(\overline{u}^N(t))$ converges to $T_k(u(t))$ on $\Sigma_T \setminus A$ with $\text{meas}(A) = 0$. For every $k > 0$, we set

$$A_k = \{(t,x) \in \Sigma_T : |T_k(u(t))| < k\}, \quad \text{and } B = \Sigma_T \setminus \bigcup_{k=1}^{\infty} A_k.$$

We have, by Hölder's inequality

$$\begin{aligned}
\text{meas}\,(B) &\leq \frac{1}{k}\int_B |T_k(u)|\,d\sigma \\
&\leq \frac{1}{k}\int_0^T \|T_k(u)\|_{L^1(\partial\Omega)}\,dt \\
&\leq \frac{1}{k}\int_0^T \|T_k(u)\|_{W^{1,1}(\Omega)}\,dt \qquad (5.18)\\
&\leq \frac{1}{k}\int_0^T \int_\Omega (|T_k(u)| + |\nabla T_k(u)|) \\
&\leq \frac{1}{k}\left(\frac{1}{p_-} + \frac{1}{(p_-)'}\right)\|1\|_{L^{p'(x)}(Q_T)}\left(\|T_k(u)\|_{L^{p(x)}(Q_T)} + \|\nabla T_k(u)\|_{(L^{p(x)}(Q_T))^d}\right).
\end{aligned}$$

Thanks to (5.12) and (5.13), for all $k > 0$, we have

$$\begin{aligned}
\left\|T_k(\overline{u}^N)\right\|_{L^{p(x)}(Q)} + \left\|\nabla T_k(\overline{u}^N)\right\|_{(L^{p(x)}(Q))^d} &\leq 2\left(k^{\frac{1}{p_-}} + k^{\frac{1}{p_+}}\right) \qquad (5.19)\\
&\quad \times \max\left\{C(u_0, p_+, f, g)^{\frac{1}{p_+}}, C(u_0, p_+, f, g)^{\frac{1}{p_+}}\right\}
\end{aligned}$$

We now use the Fatou's lemma in (5.19) to get

$$\begin{aligned}\|T_k(u)\|_{L^{p(x)}(Q)} + \|\nabla T_k(u)\|_{(L^{p(x)}(Q))^d} \quad &\leq \quad 2\left(k^{\frac{1}{p_-}} + k^{\frac{1}{p_+}}\right) \\ &\quad \times \max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}}, C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\},\end{aligned}$$

and (5.18) becomes

$$\operatorname{meas}(B) \leq 2\left(\frac{1}{k^{1-\frac{1}{p_-}}} + \frac{1}{k^{1-\frac{1}{p_+}}}\right)\max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}}, C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\}. \tag{5.20}$$

Therefore, we get by letting $k \to \infty$ in (5.20) that $\operatorname{meas}(B) = 0$.
Let us now define on $\partial\Omega$, the function v by

$$v(t,x) = T_k(u(t))(x) \ \text{ if } \ (x,t) \in A_k.$$

We take $(x,t) \in \Sigma_T \setminus (A \cup B)$; then there exists $k > 0$ such that $(x,t) \in A_k$ and we have

$$\overline{u}^N(t,x) - v(t,x) = (\overline{u}^N(t,x) - T_k(\overline{u}^N(t))(x)) + (T_k(\overline{u}^N(t))(x) - T_k(u(t))(x)).$$

Since $(x,t) \in A_k$, we have $\left|T_k(\overline{u}^N(t))(x)\right| < k$ from which we deduce that $T_k(\overline{u}^N(t))(x) = \overline{u}^N(t,x)$. Therefore,

$$\overline{u}^N(t,x) - v(t,x) = (T_k(\overline{u}^N(t))(x) - T_k(u(t))(x)) \to 0, \text{ as } N \to \infty.$$

This means that $\left(\overline{u}^N\right)$ converges to v a.e. on Σ_T. □

Lemma 5.7. *The sequence $(\overline{u}^N)_{N\in\mathbb{N}}$ converges to u in $C(0,T;L^1(\Omega))$.*

Proof. Let $(t^n = n\tau_N)_{n=1}^N$ and $(t^m = m\tau_M)_{n=1}^M$ be two partitions of the interval $[0,T]$ and let $(u^N(t), \overline{u}^N(t))$, $(u^M(t); \overline{u}^M(t))$; be the semi-discrete solutions defined by (5.1), (5.2) and corresponding to the respective partitions. Let $\varphi \in L^\infty(\Omega) \cap V \cap W^{1,1}(0,T;L^1(\Omega))$. We rewrite (3.1) in the forms

$$\begin{aligned}&\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_k(\overline{u}^N - \varphi)\right\rangle ds + \int_0^t \int_\Omega a(x,\nabla\overline{u}^N).\nabla T_k(\overline{u}^N - \varphi)dxds \\ &+ \int_0^t \int_\Omega b(\overline{u}^N)T_k(\overline{u}^N - \varphi)dxds \\ &\leq \int_0^t \int_\Omega f_N T_k(\overline{u}^N - \varphi)dxds\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}&\int_0^t \left\langle \frac{\partial u^M}{\partial s}, T_k(\overline{u}^M - \varphi)\right\rangle ds + \int_0^t \int_\Omega a(x,\nabla\overline{u}^M).\nabla T_k(\overline{u}^M - \varphi)dxds \\ &+ \int_0^t \int_\Omega b(\overline{u}^M)T_k(\overline{u}^M - \varphi)dxds \\ &\leq \int_0^t \int_\Omega f_M T_k(\overline{u}^M - \varphi)dxds\end{aligned} \tag{5.22}$$

where

$$f_N(t,x) = f_n(x) \qquad \forall t \in]t^{n-1}, t^n]$$

$$f_M(t,x) = f_m(x) \qquad \forall t \in]t^{m-1}, t^m]$$

Let $h > 1$, in inequality (5.21) we take $\varphi = T_h(\overline{u}^M)$ and in inequality (5.22) we take $\varphi = T_h(\overline{u}^N)$. Summing both inequalities, we get, for $k = 1$,

$$\begin{aligned}
&\int_0^t \left\langle \frac{\partial(u^N - u^M)}{\partial s}, T_1(u^N - u^M) \right\rangle ds + I_{N,M}(h) \\
&+ \int_0^t \int_\Omega b(\overline{u}^N) T_1(\overline{u}^N - T_h(\overline{u}^M)) dx ds \\
&+ \int_0^t \int_\Omega b(\overline{u}^M) T_1(\overline{u}^M - T_h(\overline{u}^N)) dx ds \\
\leq\; & \int_0^t \left\langle \frac{\partial(u^N - u^M)}{\partial s}, T_1(u^N - u^M) \right\rangle - \left\langle \frac{\partial u^N}{\partial s}, T_1(\overline{u}^N - T_h(\overline{u}^M)) \right\rangle ds \qquad (5.23) \\
&- \int_0^t \left\langle \frac{\partial u^M}{\partial s}, T_1(\overline{u}^M - T_h(\overline{u}^N)) \right\rangle ds \\
&+ \int_0^t \int_\Omega [f_N T_1(\overline{u}^N - T_h(\overline{u}^M)) + f_M T_1(\overline{u}^M - T_h(\overline{u}^N))] dx ds
\end{aligned}$$

where

$$\begin{aligned}
I_{N,M}(h) \;=\; & \int_0^t \int_\Omega a(x, \nabla \overline{u}^N).\nabla T_1(\overline{u}^N - T_h(\overline{u}^M)) dx ds \\
&+ \int_0^t \int_\Omega a(x, \nabla \overline{u}^M).\nabla T_1(\overline{u}^M - T_h(\overline{u}^N)) dx ds.
\end{aligned}$$

We have

$$\begin{aligned}
\left| \int_0^t \left\langle \frac{\partial(u^N - u^M)}{\partial s}, T_1(u^N - u^M) \right\rangle ds \right| \;\leq\; & \left\| \frac{\partial(u^N - u^M)}{\partial s} \right\|_{L^1(Q_T)} \left\| T_1(u^N - u^M) \right\|_{L^\infty(Q_T)} \\
\leq\; & 2C(T, f, u_0) \left\| T_1(u^N - u^M) \right\|_{L^\infty(Q_T)}.
\end{aligned}$$

Since

$$\lim_{N,M \to \infty} \left\| T_1(u^N - u^M) \right\|_{L^\infty(Q_T)} = 0.$$

Then it follows that

$$\lim_{h \to \infty} \lim_{N,M \to \infty} \int_0^t \left\langle \frac{\partial(u^N - u^M)}{\partial s}, T_1(u^N - u^M) \right\rangle ds = 0. \qquad (5.24)$$

Similarly, we show that

$$\lim_{h \to \infty} \lim_{N,M \to \infty} \left(\int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_1(\overline{u}^N - T_h(\overline{u}^M)) \right\rangle + \left\langle \frac{\partial u^M}{\partial s}, T_1(\overline{u}^M - T_h(\overline{u}^N)) \right\rangle ds \right) = 0$$

$$\lim_{h \to \infty} \lim_{N,M \to \infty} \int_0^t \int_\Omega [f_N T_1(\overline{u}^N - T_h(\overline{u}^M)) + f_M T_1(\overline{u}^M - T_h(\overline{u}^N))] dx ds = 0$$

and

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_\Omega b(\overline{u}^N)T_1(\overline{u}^N-T_h(\overline{u}^M))dxds+\int_0^t\int_\Omega b(\overline{u}^M)T_1(\overline{u}^M-T_h(\overline{u}^N))dxds=0.$$

Then, letting N, $M\to\infty$ and $h\to\infty$, in (5.23)we get

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\left\langle\frac{\partial(u^N-u^M)}{\partial s},T_1(u^N-u^M)\right\rangle ds+\lim_{h\to\infty}\lim_{N,M\to\infty}I_{N,M}(h)\le 0. \tag{5.25}$$

Since

$$\left\langle\frac{\partial v}{\partial t},T_k(v)\right\rangle=\frac{d}{dt}\int_\Omega J_k(v)\quad\text{in } L^1(]0,T[),$$

inequality (5.25) becomes

$$\lim_{N,M\to\infty}\int_\Omega J_1(u^N(t)-u^M(t))dx+\lim_{h\to\infty}\lim_{N,M\to\infty}I_{N,M}(h)\le 0. \tag{5.26}$$

Now, we show that

$$\lim_{h\to\infty}\lim_{N,M\to\infty}I_{N,M}(h)\ge 0.$$

We consider the decomposition

$$I_{N,M}(h)=\sum_{i=1}^4 L_i(h),$$

where

$$L_i(h)=\int_0^t\int_{\Omega_i(h)}a(x,\nabla\overline{u}^N).\nabla T_1(\overline{u}^N-T_h(\overline{u}^M))dxds$$
$$+\int_0^t\int_{\Omega_i(h)}a(x,\nabla\overline{u}^M).\nabla T_1(\overline{u}^M-T_h(\overline{u}^N))dxds$$

and

$$\Omega_1(h)=\left\{|\overline{u}^N|\le h,\,|\overline{u}^M|\le h\right\}\qquad\Omega_2(h)=\left\{|\overline{u}^N|\le h,\,|\overline{u}^M|>h\right\}$$
$$\Omega_3(h)=\left\{|\overline{u}^N|>h,\,|\overline{u}^M|\le h\right\}\qquad\Omega_4(h)=\left\{|\overline{u}^N|>h,\,|\overline{u}^M|>h\right\}.$$

On the one hand, thanks to assumption (1.4) we have

$$L_1(h)=\int_0^t\int_{\Omega_1^1(h)}[a(x,\nabla\overline{u}^N)-a(x,\nabla\overline{u}^M)].\nabla(\overline{u}^N-\overline{u}^M)dxds\ge 0.$$

Therefore

$$\lim_{h\to\infty}\lim_{N,M\to\infty}L_1(h)\ge 0.$$

On the other hand, we have

$$\begin{aligned}L_2(h)\quad&=\quad\int_0^t\int_{\Omega_2^1(h)}a(x,\nabla\overline{u}^N).\nabla\overline{u}^N dxds\\&\qquad+\int_0^t\int_{\Omega_2^2(h)}a(x,\nabla\overline{u}^M).\nabla(\overline{u}^M-\overline{u}^N)dxds\\&\ge\quad-\int_0^t\int_{\Omega_2^2(h)}a(x,\nabla\overline{u}^M).\nabla\overline{u}^N dxds,\end{aligned}$$

where

$$\Omega_2^1(h) = \left\{|\overline{u}^N| \le h,\, |\overline{u}^M| > h, |\overline{u}^N - h\text{sign}(\overline{u}^M)| \le 1\right\},$$
$$\Omega_2^2(h) = \left\{|\overline{u}^N| \le h,\, |\overline{u}^M| > h, |\overline{u}^N - \overline{u}^M| \le 1\right\}.$$

Now, taking $\varphi = T_h(\overline{u}^N)$ in (5.21), we deduce that

$$\lim_{h\to\infty}\lim_{N\to\infty}\int_0^t\int_{\left\{h\le|\overline{u}^N|\le h+k\right\}} a(x,\nabla\overline{u}^N).\nabla\overline{u}^N = 0.$$

This implies

$$\lim_{h\to\infty}\lim_{N\to\infty}\int_0^t\int_{\left\{h\le|\overline{u}^N|\le h+k\right\}} |\nabla\overline{u}^N|^{p(x)} = 0, \quad k>0. \tag{5.27}$$

By the Young inequality, we have

$$\begin{aligned}
&\left|\int_0^t\int_{\Omega_2^2(h)} a(x,\nabla\overline{u}^M).\nabla\overline{u}^N dxds\right| \\
\le\ & \int_0^t\int_{\Omega_2^2(h)} |\nabla\overline{u}^M|^{p(x)-1}|\nabla\overline{u}^N|dxds \\
\le\ & \int_0^t\int_{\left\{h\le|\overline{u}^M|\le h+1\right\}} \frac{1}{p'(x)}|\nabla\overline{u}^M|^{p(x)}dxds + \int_0^t\int_{\left\{h-1\le|\overline{u}^N|\le h\right\}} \frac{1}{p(x)}|\nabla\overline{u}^M|^{p(x)}dxds \\
\le\ & \int_0^t\int_{\left\{h\le|\overline{u}^M|\le h+1\right\}} \frac{1}{p'_-}|\nabla\overline{u}^M|^{p(x)}dxds + \int_0^t\int_{\left\{h-1\le|\overline{u}^N|\le h\right\}} \frac{1}{p_-}|\nabla\overline{u}^M|^{p(x)}dxds.
\end{aligned}$$

Thus (5.27) gives

$$\lim_{N,M\to\infty}\int_0^t\int_0^t\int_{\Omega_2^2(h)} a(x,\nabla\overline{u}^M).\nabla\overline{u}^N dxds = 0,$$

which implies that

$$\lim_{h\to\infty}\lim_{N,M\to\infty} L_2(h) \ge 0.$$

Similarly, we show that

$$\lim_{h\to\infty}\lim_{N,M\to\infty}(L_3(h)+L_4(h)) \ge 0.$$

Therefore

$$\lim_{h\to\infty}\lim_{N,M\to\infty} I_{N,M}(h) \ge 0.$$

Thus (5.26) becomes

$$\lim_{N,M\to\infty}\int_\Omega J_1(u^N(t)-u^M(t))dx = 0. \tag{5.28}$$

Since

$$\frac{1}{2}\int_{\{|u^N-u^M|\le 1\}} |u^N(t)-u^M(t)|^2dx + \int_{\{|u^N-u^M|\ge 1\}} |u^N(t)-u^M(t)|dx \le \int_\Omega J_1(u^N(t)-u^M(t));$$

we have

$$\begin{aligned}
& \int_{\{|u^N-u^M|\geq 1\}} |u^N(t)-u^M(t)|dx \\
= & \int_{\{|u^N-u^M|\leq 1\}} |u^N(t)-u^M(t)|dx + \int_{\{|u^N-u^M|\geq 1\}} |u^N(t)-u^M(t)|dx \\
\leq & \ C_\Omega \left(\int_{\{|u^N-u^M|\leq 1\}} |u^N(t)-u^M(t)|^2 dx \right)^{\frac{1}{2}} + \int_{\{|u^N-u^M|\geq 1\}} |u^N(t)-u^M(t)|dx \\
\leq & \ C_2(\Omega) \left(\int_\Omega J_1(u^N(t)-u^M(t))dx \right)^{\frac{1}{2}} + \int_\Omega J_1(u^N(t)-u^M(t))dx.
\end{aligned}$$

By (5.26), we deduce that $(u^N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $C(0,T;L^1(\Omega))$. Hence $(u^N)_{N\in\mathbb{N}}$ converges to u in $C(0,T;L^1(\Omega))$. □

Step 2: Existence of entropy solution. Now, we prove that the limit function u is an entropy solution of the problem (P). Since $u^N(0) = U^0 = u_0$ for all $N \in \mathbb{N}$, we have $u(0,.) = u_0$, and inequality (5.21) implies

$$\begin{aligned}
& \int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_k(\overline{u}^N-\varphi) - T_k(u^N-\varphi) \right\rangle ds + \int_0^t \int_\Omega a(x,\nabla \overline{u}^N).\nabla T_k(\overline{u}^N-\varphi)dxds \\
& + \int_0^t \int_\Omega b(\overline{u}^N) T_k(\overline{u}^N-\varphi)dxds \qquad (5.29) \\
& \leq \int_0^t \left\langle \frac{\varphi}{\partial s}, T_k(u^N-\varphi) - T_k(u^N-\varphi) \right\rangle ds + \int_\Omega J_k(u^N(0)-\varphi(0))dx - \int_\Omega J_k(u^N(t)-\varphi(t))dx \\
& + \int_0^t \int_\Omega f_N T_k(\overline{u}^N-\varphi)dxds.
\end{aligned}$$

Let $\overline{k} = k + \|\varphi\|_\infty$. Then

$$\begin{aligned}
\int_0^t \int_\Omega a(x,\nabla \overline{u}^N).\nabla T_k(\overline{u}^N-\varphi)dxds &= \int_0^t \int_\Omega a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla T_k(T_{\overline{k}}(\overline{u}^N)-\varphi)dxds \\
&= \int_0^t \int_\Omega [a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla T_{\overline{k}}(\overline{u}^N) \\
&\quad - a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla \varphi] \mathbf{1}_{Q(N,k)} dxds,
\end{aligned}$$

where $Q(N,k) = \left\{ |T_{\overline{k}}(\overline{u}^N) - \varphi| \leq k \right\}$. Thus, the inequality (5.29) becomes

$$\begin{aligned}
& \int_0^t \left\langle \frac{\partial u^N}{\partial s}, T_k(\overline{u}^N-\varphi) - T_k(u^N-\varphi) \right\rangle ds - \int_0^t \int_\Omega a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla \varphi \mathbf{1}_{Q(N,k)} \\
& + \int_0^t \int_\Omega [a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla T_{\overline{k}}(\overline{u}^N)] \mathbf{1}_{Q(N,k)} + \int_0^t \int_\Omega b(\overline{u}^N) T_k(\overline{u}^N-\varphi)dxds \qquad (5.30) \\
\leq & -\int_0^t \left\langle \frac{\partial \varphi}{\partial s}, T_k(u^N-\varphi) \right\rangle ds + \int_\Omega J_k(u^N(0)-\varphi(0))dx - \int_\Omega J_k(u^N(t)-\varphi(t))dx \\
& + \int_0^t \int_\Omega f_N T_k(\overline{u}^N-\varphi)dxds.
\end{aligned}$$

On the one hand, thanks to Lemma 5.5 $a(x,\nabla T_{\overline{k}}(\overline{u}^N))$ converges weakly to $a(x,\nabla T_{\overline{k}}(u))$ in $\left(L^{p'(\cdot)}(\Omega)\right)^d$. Therefore,

$$\lim_{N\to\infty}\int_0^t\int_\Omega a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla\varphi\mathbf{1}_{Q(N,k)}=\int_0^t\int_\Omega a(x,\nabla T_{\overline{k}}(u)).\nabla\varphi\mathbf{1}_{Q(k)},\tag{5.31}$$

where $Q(k)=\left\{|T_{\overline{k}}(u)-\varphi|\le k\right\}$. Moreover, $a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla T_{\overline{k}}(\overline{u}^N)$ is nonnegative and converges a.e. in Q_T to $a(x,\nabla T_{\overline{k}}(u)).\nabla T_{\overline{k}}(u)$ (see Lemma 5.5). Therefore by Fatou's lemma, we obtain

$$\liminf_{N\to\infty}\int_0^t\int_0^t\int_\Omega[a(x,\nabla T_{\overline{k}}(\overline{u}^N)).\nabla T_{\overline{k}}(\overline{u}^N)]\mathbf{1}_{Q(N,k)}dxds\ge\int_0^t\int_0^t\int_\Omega[a(x,\nabla T_{\overline{k}}(u)).\nabla T_{\overline{k}}(u)]\mathbf{1}_{Q(k)}dxds.$$

For the fourth term of (5.30), we have

$$\int_0^t\int_\Omega b(\overline{u}^N)T_k(\overline{u}^N-\varphi)dxds=\int_0^t\int_\Omega(b(\overline{u}^N)-b(\varphi))T_k(\overline{u}^N-\varphi)dxds+\int_0^t\int_\Omega b(\varphi)T_k(\overline{u}^N-\varphi)dxds.$$

The quantity $(b(\overline{u}^N)-b(\varphi))T_k(\overline{u}^N-\varphi)$ is is nonnegative and since for all $s\in\mathbb{R}$, $s\mapsto b(s)$ is continuous, we obtain

$$(b(\overline{u}^N)-b(\varphi))T_k(\overline{u}^N-\varphi)\to(b(u)-b(\varphi))T_k(\overline{u}^N-\varphi)\quad\text{a.e. in }\ \Omega.$$

Then, it follows by Fatou's lemma that

$$\liminf_{N\to\infty}\int_0^t\int_\Omega(b(\overline{u}^N)-b(\varphi))T_k(\overline{u}^N-\varphi)dxds\ge\int_0^t\int_\Omega(b(u)-b(\varphi))T_k(u-\varphi)dxds.$$

We have $b(\varphi)\in L^1(Q_T)$. Since $T_k(\overline{u}^N-\varphi)$ converges weakly$-*$ to $T_k(u-\varphi)$ and $b(\varphi)\in L^1(Q_T)$, it follows that

$$\liminf_{N\to\infty}\int_0^t\int_\Omega b(\varphi)T_k(\overline{u}^N-\varphi)dxds\ge\int_0^t\int_\Omega b(\varphi)T_k(u-\varphi)dxds.$$

By Lemma 5.7 , we deduce that $u^N(t)\to u(t)$ in $L^1(\Omega)$ for all $t\in[0,T]$, which implies that

$$\int_\Omega J_k(u^N(t)-\varphi(t))dx\to\int_\Omega J_k(u(t)-\varphi(t))dx\quad\forall t\in[0,T].\tag{5.32}$$

We follow the method used in the proof of equality (5.24) to show that

$$\lim_{N\to\infty}\int_0^t\left\langle\frac{\partial u^N}{\partial s},T_k(\overline{u}^N-\varphi)-T_k(u^N-\varphi)\right\rangle ds=0.\tag{5.33}$$

Finally, letting $N\to\infty$ and using the above results, the continuity of b and the facts that

$$f_N\to f\quad\text{in }L^1(Q_T),$$
$$T_{\overline{k}}(\overline{u}^N-\varphi)\to T_{\overline{k}}(u-\varphi)\quad\text{in }L^\infty(Q_T),$$

we deduce that u is an entropy solution of the nonlinear parabolic problem (P). □

6 Conclusion

In this paper we prove the existence and uniqueness of an entropy solution for a non- linear parabolic equation with homogeneous Neumann boundary conditions and initial data in L^1 by a time discretization technique.

This method turns out to be better suited for the study of parabolic problems under Neumann-type boundary conditions. However, this technique assumes that the associated elliptic problem is well posed. This study opens up new perspectives, we could always in the context of the Sobolev space with variable exponents look at the problem with measure data or consider the function b as maximal monotone graph.

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