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Perfect matchings in inhomogeneous random bipartite graphs in random environment

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ABSTRACT

In this note we study inhomogeneous random bipartite graphs in random environment. These graphs can be thought of as an extension of the classical Erdős-Rényi random bipartite graphs in a random environment. We show that the expected number of perfect matchings obeys a precise asymptotic.

RESUMEN

En esta nota estudiamos grafos aleatorios bipartitos inhomogéneos en un ambiente aleatorio. Estos grafos pueden ser pensados como una extensión de los grafos bipartitos aleatorios clásicos de Erdős-Rényi en un ambiente aleatorio. Mostramos que el número esperado de pareos obedece un comportamiento asintótico preciso.

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1 Introduction

In their seminal paper [7], Erdős and Rényi studied a certain type of *random graphs*, which in the case of bipartite graphs correspond to the following. Consider a bipartite graph with set of vertices given by $W = \{w_1, \dots, w_n\}$ and $M = \{m_1, \dots, m_n\}$. Let $p \in [0,1]$, Σ be a probability space and consider the independent random variables $X_{(ij)}$ defined on Σ with law

$$X_{(ij)}(x) = \begin{cases} 1 & \text{with probability } p; \\ 0 & \text{with probability } 1-p, \end{cases}$$

for $x \in \Sigma$. Denote by $G_n(x)$ the bipartite graph with vertex set $W \cup M$ and edges $E(x)$, where the edge (w_i, m_j) belongs to $E(x)$ if and only if $X_{(ij)}(x) = 1$. Let $\mathrm{pm}(G_n(x))$ be the number of *perfect matchings* of the graph $G_n(x)$ (see Sec. 3 for precise definitions). Erdős and Rényi [8, p. 460] observed that the mean of the number of perfect matchings was given by

$$\mathbb{E}(\mathrm{pm}(G_n(x))) = n!p^n. \tag{1.1}$$

This number has been also studied by Bollobás and McKay [5, Theorem 1] in the context of $k-$regular random graphs and by O'Neil [11, Theorem 1] for random graphs having a fixed (large enough) proportion of edges. We refer to the text by Bollobás [4] for further details on the subject of random graphs.

This paper is devoted to study certain sequences of inhomogeneous random bipartite graphs $G_{n,\omega}$ in a random environment $\omega \in \Omega$ (definitions are given in Sec. 2). Inhomogeneous random graphs have been intensively studied over the last years (see [6], where non-bipartite graphs are also considered). Our main result (see Theorem 3.2 for precise statement) is that there exists a constant $c \in (0,1)$ such that for almost every environment $\omega \in \Omega$ and for large $n \in \mathbb{N}$

$$\mathbb{E}_{n,\omega}(\mathrm{pm}(G_{n,\omega}(x))) \asymp n!c^n, \tag{1.2}$$

where the meaning of the asymptotic $\asymp$ will be explained later. Moreover, we have an explicit formula for the number c.

The result in equation (1.2) should be understood in the sense that the mean number of perfect matchings for inhomogeneous random bipartite graphs in a random environment is asymptotically the same as the one of Erdős-Rényi bipartite graphs in which $p = c$. Note that p is a constant that does not depend on n. The number c is the so-called *scaling mean* of a function related to the random graphs. Scaling means were introduced, in more a general setting, in [2] and are described in Sec. 3.

2 Inhomogeneous random bipartite graphs in random environment

Consider the following generalization of the Erdős-Rényi bipartite graphs. Let $W = \{w_1, \dots, w_n\}$ and $M = \{m_1, \dots, m_n\}$ be two disjoint sets of vertices. For every pair $1 \le i, j \le n$, let $a_{ij} \in [0, 1]$ and consider the independent random variables $X_{(ij)}$, with law

$$X_{(ij)}(x) = \begin{cases} 1 & \text{with probability } a_{(ij)}; \\ 0 & \text{with probability } 1 - a_{(ij)}. \end{cases}$$

Denote by $G_n(x)$ the bipartite graph with vertices W, M and edges $E(x)$, where the edge (w_i, m_j) belongs to $E(x)$ if and only if $X_{(ij)}(x) = 1$. As it is clear from the definition all vertex of the graph do not play the same role. This contrasts with the (homogenous) Erdős-Renyi graphs (see [6] for details). We remark that in relation to the graphs we are considering it is possible to include the stochastic block model (see [10]) that is used, for example, in problems of community detection, in the context of machine learning. In this note we consider inhomogeneous random bipartite graphs in random environments, that is, the laws of $X_{(ij)}$ (and hence the numbers $a_{(ij)}$) are randomly chosen following an exterior environment law. This approach to stochastic processes has developed since the groundbreaking work by Solomon [12] on Random Walks in Random Environment and subsequent work of a large community (see [3] for a survey on the subject).

The model we propose is to consider the vertex sets W, M as the environment and to consider that the number $a_{(ij)}$, which is the probability that the edge connecting w_i with m_j occurs in the graph, is a random variable depending on w_i and m_j. We now describe precisely this model.

The space of environments is as follows. Fix $\alpha \in \mathbb{N}$ and a stochastic vector $(p_1, p_2, \dots, p_\alpha)$. Endow the set $\{1, \dots, \alpha\}$ with the probability measure P_W defined by $P_W(\{i\}) = p_i$. Denote by Ω_W the product space $\prod_{i=1}^{\infty}\{1, 2, \dots, \alpha\}$ and by μ_W the corresponding product measure. Let (Ω_M, μ_M) be the analogous probability measure space for the set $\{1, 2, \dots, \beta\}$ and the stochastic vector $(q_1, q_2, \dots, q_\beta)$. The *space of environments* is $\Omega = \Omega_W \times \Omega_M$ with the measure $\mu_\Omega = \mu_W \times \mu_M$ and an *environment* is an element $\omega \in \Omega$. Note that every environment defines two sequences

$$W(\omega) = (w_1, w_2, \dots) \in \Omega_W \quad \text{and} \quad M(\omega) = (m_1, m_2, \dots) \in \Omega_M.$$

For each environment $\omega \in \Omega$ we now define the edge distribution $X_{\omega,(ij)}$. Let $F = [f_{sr}]$ be a $\alpha \times \beta$ matrix with entries f_{sr} satisfying $0 \le f_{sr} \le 1$ and let $f : \{1, 2, \dots, \alpha\} \times \{1, 2, \dots, \beta\} \to [0, 1]$ be the function defined by $f(w, m) = f_{wm}$. For each $\omega \in \Omega$ let

$$a_{(ij)}(\omega) := f\left(w_i(\omega), m_j(\omega)\right) = f_{w_i(\omega), m_j(\omega)}. \tag{2.1}$$

Given an environment $\omega \in \Omega$ the corresponding *edge distributions* are the random variables $X_{\omega,(ij)}$

with laws

$$X_{\omega,(ij)}(x) = \begin{cases} 1 & \text{with probability } a_{(ij)}(\omega); \\ 0 & \text{with probability } 1 - a_{(ij)}(\omega). \end{cases}$$

Given an environment $\omega \in \Omega$, we construct a sequence of random bipartite graphs $G_{n,\omega}$ considering the sets of vertices

$$W_{n,\omega} = (w_1(\omega), \ldots, w_n(\omega)) \quad \text{and} \quad M_{n,\omega} = (m_1(\omega), \ldots, m_n(\omega)),$$

and edge distributions $X_{\omega,(ij)}$ given by the values of $a_{(ij)}(\omega)$ as in (2.1). We denote by $\mathbb{P}_{n,\omega}$ the law of the random graph $G_{n,\omega}$.

Example 2.1. *Given a choice of an environment $\omega \in \Omega$, the probability that the bipartite graph $G_{n,\omega}(x)$ equals the complete bipartite graph $K_{n,n}$, using independence of the edge variables, is*

$$\mathbb{P}_{n,\omega}\left(G_{n,\omega}(x) = K_{n,n}\right) = \prod_{1\le i,j\le n} \mathbb{P}_{n,\omega}(X_{\omega,(ij)} = 1) = \prod_{1\le i,j\le n} a_{(ij)}(\omega).$$

3 Counting Perfect Matchings

Recall that a perfect matching of a graph G is a subset of edges containing every vertex exactly once. We denote by $\mathrm{pm}(G)$ the number of perfect matchings of G. When the graph G is bipartite, and the corresponding bipartition of the vertices has the form $W = \{w_1, w_2, \ldots, w_n\}$ and $M = \{m_1, m_2, \ldots, m_n\}$, a perfect matching can be identified with a bijection between W and M, and hence with a permutation $\sigma \in S_n$. From this, the total number of perfect matchings can be computed as

$$\mathrm{pm}(G) = \sum_{\sigma\in S_n} x_{1\sigma(1)} x_{2\sigma(2)} \cdots x_{n\sigma(n)}, \tag{3.1}$$

where x_{ij} are the entries of the incidence matrix X_G of G, that is $x_{ij} = 1$ if (w_i, m_j) is an edge of G and $x_{ij} = 0$ otherwise. Of course, the right hand side of (3.1) is the *permanent*, $\mathrm{per}(X_G)$, of the matrix X_G.

In the framework of Section 2, we estimate the number of perfect matchings for the sequence of inhomogeneous random bipartite graphs $G_{n,\omega}$, for a given environment $\omega \in \Omega$. More precisely, we obtain estimates for the growth of the mean of

$$\mathrm{pm}(G_{n,\omega}(x)) = \mathrm{per}(X_{G_{n,\omega}(x)}) = \sum_{\sigma\in S_n} X_{\omega,(1\sigma(1))} \cdots X_{\omega,(n\sigma(n))}. \tag{3.2}$$

Denote by $\mathbb{E}_{n,\omega}$ the expected value with respect to the probability $\mathbb{P}_{n,\omega}$. Since the edges are independent and $\mathbb{E}_{n,\omega}(X_{\omega,(ij)}) = a_{ij}(\omega)$ we have

$$\begin{aligned}\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right) &= \mathbb{E}_{n,\omega}\left(\sum_{\sigma\in S_n} X_{\omega,(1\sigma(1))}\cdots X_{\omega,(n\sigma(n))}\right)\\ &= \sum_{\sigma\in S_n} a_{(1\sigma(1))}(\omega)\cdots a_{(n\sigma(n))}(\omega)\\ &= \mathrm{per}(A_n(\omega)),\end{aligned}$$

where the entries of the matrix are $(A_n(\omega))_{ij} = a_{(ij)}(\omega)$. The main result of this note describes the growth of this expected number for perfect matchings.

The following number is a particular case of a quantity introduced by the authors in a more general setting in [2].

Definition 3.1. *Let F be an $\alpha\times\beta$ matrix with non-negative entries (f_{rs}). Let $\vec{p} = (p_1,\dots,p_\alpha)$ and $\vec{q} = (q_1,\dots,q_\beta)$ be two stochastic vectors. The* scaling mean *of F with respect to $\vec{p}$ and $\vec{q}$ is defined by*

$$\mathrm{sm}_{\vec{p},\vec{q}}(F) := \inf_{(x_r)\in\mathbb{R}_+^\alpha,(y_s)\in\mathbb{R}_+^\beta}\left(\prod_{r=1}^{\alpha} x_r^{-p_r}\right)\left(\prod_{s=1}^{\beta} y_s^{-q_s}\right)\left(\sum_{r=1}^{\alpha}\sum_{s=1}^{\beta} x_r f_{rs} y_s p_r q_s\right).$$

The scaling mean is increasing with respect to the entries of the matrix and lies between the minimum and the maximum of the entries (see [2] for details and more properties). We stress that the scaling mean can be exponentially approximated using a simple iterative process (see Section 5).

The main result in this note is the following,

Theorem 3.2 (Main Theorem)**.** *Let $(G_{n,\omega})_{n\geq 1}$ be a sequence of random bipartite graphs on a random environment $\omega\in\Omega$. If for every pair (r,s) we have $f_{rs}>0$ then the following pointwise convergence holds*

$$\lim_{n\to\infty}\left(\frac{\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right)}{n!}\right)^{1/n} = \mathrm{sm}_{\vec{p},\vec{q}}(F), \tag{3.3}$$

for $\mu_W\times\mu_M$-almost every environment $\omega\in\Omega$.

Remark 3.3. *As discussed in the introduction Theorem 3.2 shows that there exists a constant $c\in(0,1)$, such that for almost every environment $\omega\in\Omega$ and for $n\in\mathbb{N}$ sufficiently large*

$$\mathbb{E}_{n,\omega}(\mathrm{pm}(G_{n,\omega}(x))) \asymp n!c^n.$$

Namely $c = \mathrm{sm}_{\vec{p},\vec{q}}(F)$. This result should be compared with the corresponding one obtained by Erdős and Rényi for their class of random graphs, that is

$$\mathbb{E}(\mathrm{pm}(G_n(x))) = n!p^n.$$

Thus, we have shown that for large values of n the growth of the number of perfect matchings for random graphs in a random environment behaves like the simpler model studied by Erdős and Rényi with $p = \mathrm{sm}_{\vec{p},\vec{q}}(F)$.

Remark 3.4. *Theorem 3.2 shows that the expected number of perfect matchings is a quenched variable, in the sense of that it does not depend on the environment* ω *(see for instance [P]).*

Remark 3.5. *Using the Stirling formula, the limit in (3.3) can be stated as*

$$\lim_{n\to\infty}\left(\frac{1}{n}\log\left(\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right)\right)-\log n\right)=\log \mathrm{sm}_{\vec{p},\vec{q}}(F)-1,$$

which gives a quenched result for the growth of the perfect matching entropy for the sequence of graphs $G_{\omega,n}$ *(see [1]).*

Remark 3.6. *Note that we assume a uniform ellipticity condition on the values of the probabilities* $a_{(ij)}$ *as in (2.1). A similar assumption appears in the setting of Random Walks in Random Environment (see [3, p. 355]).*

We now present some concrete examples.

Example 3.7. *Let* $\alpha = \beta = 2$ *and* $p_1 = p_2 = q_1 = q_2 = 1/2$. *Therefore, the space of environments is the direct product of two copies of the full shift on two symbols endowed with the* $(1/2, 1/2)-$*Bernoulli measure. The edge distribution matrix* F *is a* 2×2 *matrix with entries belonging to* $(0,1)$. *In [2, Example 2.11], it was shown that*

$$\mathrm{sm}_{\vec{p},\vec{q}}\begin{pmatrix} f_{11} & f_{12}\\ f_{21} & f_{22}\end{pmatrix}=\frac{\sqrt{f_{11}f_{22}}+\sqrt{f_{12}f_{21}}}{2}.$$

Therefore, Theorem 3.2 implies that

$$\lim_{n\to\infty}\left(\frac{\mathbb{E}_{n,\omega}\left(\mathrm{pm}(G_{n,\omega})\right)}{n!}\right)^{1/n}=\frac{\sqrt{f_{11}f_{22}}+\sqrt{f_{12}f_{21}}}{2},$$

for almost every environment $\omega \in \Omega$.

Example 3.8. *More generally let* $\alpha \in \mathbb{N}$ *with* $\alpha \geq 2$ *and* $\beta = 2$. *Consider the two stochastic vectors* $\vec{p} = (p_1, p_2, \ldots, p_\alpha)$ *and* $\vec{q} = (q_1, q_2)$. *The space of environments is the direct product of a full shift on* α *symbols endowed with the* $\vec{p}$*-Bernoulli measure with a full shift on two symbols endowed with the* $\vec{q}$*-Bernoulli measure. The edge distribution matrix* F *is a* $\alpha \times 2$ *matrix with entries* $f_{r1}, f_{r2} \in (0,\infty)$, *where* $r \in \{1,\ldots,\alpha\}$. *Denote by* $\chi \in \mathbb{R}^+$ *the unique positive solution of the equation*

$$\sum_{r=1}^{\alpha}\frac{p_r f_{r1}}{f_{r1}+f_{r2}\chi}=q_1.$$

Then

$$\mathrm{sm}_{\vec{p},\vec{q}}(F)=\mathrm{sm}_{\vec{p},\vec{q}}\begin{pmatrix} f_{11} & f_{12}\\ \vdots & \vdots\\ f_{\alpha 1} & f_{\alpha 2}\end{pmatrix}=q_1^{q_1}\left(\frac{q_2}{\chi}\right)^{q_2}\prod_{r=1}^{\alpha}\left(f_{r1}+f_{r2}\chi\right)^{p_r}.$$

Therefore, Theorem 3.2 implies that

$$\lim_{n\to\infty}\left(\frac{\mathbb{E}_{n,\omega}\left(\mathrm{pm}(\mathrm{G}_{\mathrm{n},\omega})\right)}{n!}\right)^{1/n} = q_1^{q_1}\left(\frac{q_2}{\chi}\right)^{q_2}\prod_{r=1}^{\alpha}\left(f_{r1}+f_{r2}\chi\right)^{p_r},$$

for almost every environment $\omega \in \Omega$. The quantity in the right hand side first appeared in the work by Halász and Székely in 1976 *[9], in their study of symmetric means. In [2, Theorem 5.1] using a completely different approach we recover their result.*

4 Proof of the Theorem

The *shift map* $\sigma_W : \Omega_W \to \Omega_W$ is defined by

$$\sigma_W(w_1, w_2, w_3, \dots) = (w_2, w_3, \dots).$$

The shift map σ_W is a μ_W-preserving, that is, $\mu_W(\Lambda) = \mu_W(\sigma_W^{-1}(\Lambda))$ for every measurable set $\Lambda \subset \Omega_W$, and it is ergodic, that is, if $\Lambda = \sigma_W^{-1}(\Lambda)$ then $\mu_W(\Lambda)$ equals 1 or 0. Analogously for σ_M and μ_M. We define a function $\Phi : \Omega_W \times \Omega_M \to \mathbb{R}$ by

$$\Phi(\vec{w}, \vec{m}) = f_{w_1 m_1}.$$

Thus

$$\Phi(\sigma_W^{i-1}(\vec{w}), \sigma_M^{j-1}(\vec{m})) = f_{w_i m_j} = a_{(ij)}(\omega).$$

That is, the matrix $A_n(\omega)$ has entries $a_{(ij)}(\omega) = \Phi(\sigma_W^{i-1}(\vec{w}), \sigma_M^{j-1}(\vec{m}))$. We are in the exact setting in order to apply the Law of Large Permanents see [2, Theorem 4.1].

Theorem (Law of Large Permanents). Let (X, μ), (Y, ν) be Lebesgue probability spaces, let $T\colon X \to X$ and $S\colon Y \to Y$ be ergodic measure preserving transformations, and let $g\colon X \times Y \to \mathbb{R}$ be a positive measurable function essentially bounded away from zero and infinity. Then for $\mu \times \nu$-almost every $(x, y) \in X \times Y$, the $n \times n$ matrix

$$M_n(x,y) = \begin{pmatrix} g(x,y) & g(Tx,y) & \cdots & g(T^{n-1}x,y) \\ g(x,Sy) & g(Tx,Sy) & \cdots & g(T^{n-1}x,Sy) \\ \vdots & \vdots & & \vdots \\ g(x,S^{n-1}y) & g(Tx,S^{n-1}y) & \cdots & g(T^{n-1}x,S^{n-1}y) \end{pmatrix}$$

verifies

$$\lim_{n\to\infty}\left(\frac{\mathrm{per}\left(M_n(x,y)\right)}{n!}\right)^{1/n} = \mathrm{sm}_{\mu,\nu}(g)$$

pointwise, where $\mathrm{sm}_{\mu,\nu}(g)$ is the scaling mean of g defined as

$$\mathrm{sm}_{\mu,\nu}(g) = \inf_{\phi,\psi}\frac{\iint_{X\times Y}\phi(x)g(x,y)\psi(y)d\mu d\nu}{\exp\left(\int_X \log\phi(x)d\mu\right)\exp\left(\int_Y \log\psi(y)d\nu\right)},$$

where the functions ϕ and ψ are assumed to be measurable, positive and such that their logarithms are integrable.

We apply this Law of Large Permanents setting $X = \Omega_W, Y = \Omega_M$, $T = \sigma_W, S = \sigma_M$, $g = \Phi$ and recalling that $f_{rs} > 0$. We have

$$\mathrm{sm}_{\mu_W,\mu_M}(\Phi) = \mathrm{sm}_{\vec{p},\vec{q}}(F)$$

as a consequence of an alternative characterization of the scaling mean given in (see [2, Proposition 3.5]). This concludes the proof of the Main Theorem. ■

Remark 4.1. *We have chosen to present our result in the simplest possible setting. That is, the environment space being products of full-shifts endowed with Bernoulli measures. Using the general form of the Law of Large Permanent above our results can be extended for inhomogeneous random graphs in more general random environments.*

5 An algorithm to compute the scaling mean

The purpose of this section is to show that the scaling mean is the unique fixed point of a contraction. Therefore it can be computed, or approximated, using a suitable iterative process. It should be stressed that, on the other hand, it has been shown that no such algorithm exists to compute the permanent.

Denote by $\mathcal{B}^\alpha \subset \mathbb{R}^\alpha$ and by $\mathcal{B}^\beta \subset \mathbb{R}^\beta$ the positive cones. Define the following maps forming a (non-commutative) diagram:

$$\begin{array}{ccc} \mathcal{B}^\alpha & \xrightarrow{\mathtt{I}_1} & \mathcal{B}^\alpha \\ {\scriptstyle \mathtt{K}_1}\big\uparrow & & \big\downarrow{\scriptstyle \mathtt{K}_2} \\ \mathcal{B}^\beta & \xleftarrow[\mathtt{I}_2]{} & \mathcal{B}^\beta \end{array}$$

by the formulas:

$$(\mathtt{I}_1(\vec{x}))_i := \frac{1}{x_i}\,, \quad (\mathtt{I}_2(\vec{y}))_i := \frac{1}{y_i}\,,$$

$$(\mathtt{K}_1(\vec{x}))_j := \sum_{i=1}^{\beta} f_{ij} x_i p_i\,, \quad (\mathtt{K}_2(\vec{y}))_j := \sum_{j=1}^{\alpha} f_{ij} y_j q_j.$$

Let $\mathtt{T} : \mathcal{B}^\alpha \mapsto \mathcal{B}^\alpha$ be the map defined by $\mathtt{T} := \mathtt{K}_1 \circ \mathtt{I}_2 \circ \mathtt{K}_2 \circ \mathtt{I}_1$. The map $\mathtt{T}$ is a contraction for a suitable Hilbert metric. Indeed, for $\vec{x}, \vec{z} \in \mathcal{B}^\alpha$ define the following (pseudo)-metric

$$d(\vec{x}, \vec{z}) := \log\left(\frac{\max_i x_i/z_i}{\min_i x_i/z_i}\right).$$

It was proven in [2, Lemma 3.4]

Lemma 5.1. *We have that*

$$d(T(\vec{x}), T(\vec{z})) \le \left(\tanh\frac{\delta}{4}\right)^2 d(\vec{x}, \vec{z}),$$

where

$$\delta \leq 2\log\left(\frac{\max_{i,j} f_{ij}}{\min_{ij} f_{ij}}\right) < \infty.$$

The following results was proved in [2, Lemma 3.3]

Lemma 5.2. *The map* $\mathtt{T}$ *has a unique (up to positive scaling) fixed point* $\vec{x}_{\mathtt{T}} \in \mathcal{B}^{\alpha}$*. Moreover, defining* $\vec{y}_{\mathtt{T}} := \mathtt{K}_2 \circ \mathtt{I}_1(\vec{x}_{\mathtt{T}})$ *one has that*

$$\mathrm{sm}(f) = \prod_{i=1}^{\alpha} x_i^{p_i} \prod_{j=1}^{\beta} y_j^{q_j}.$$

Therefore, it possible to find good approximations of the scaling mean using an iterative process.

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