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On Severi varieties as intersections of a minimum number of quadrics

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ABSTRACT

Let $\mathscr{V}$ be a variety related to the second row of the Freudenthal-Tits Magic square in N-dimensional projective space over an arbitrary field. We show that there exist $M \leq N$ quadrics intersecting precisely in $\mathscr{V}$ if and only if there exists a subspace of projective dimension $N-M$ in the secant variety disjoint from the Severi variety. We present some examples of such subspaces of relatively large dimension. In particular, over the real numbers we show that the Cartan variety (related to the exceptional group $\mathsf{E}_6(\mathbb{R})$) is the set-theoretic intersection of 15 quadrics.

RESUMEN

Sea $\mathscr{V}$ una variedad relacionada a la segunda fila del cuadrado Mágico de Freudenthal-Tits en el espacio proyectivo N-dimensional sobre un cuerpo arbitrario. Mostramos que existen $M \leq N$ cuádricas intersectandose precisamente en $\mathscr{V}$ si y solo si existe un subespacio de dimensión proyectiva $N-M$ en la variedad secante disjunta de la variedad de Severi. Presentamos algunos ejemplos de tales subespacios de dimensión relativamente grande. En particular, sobre los números reales, mostramos que la variedad de Cartan (relacionada al grupo excepcional $\mathsf{E}_6(\mathbb{R})$) es la intersección conjuntista de 15 cuádricas.

Keywords and Phrases: Cartan variety, quadrics, exceptional geometry, Severi variety, quaternion veronesian.

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1 Introduction

It is well known that the Grassmannians of the (split) spherical buildings related to semi-simple algebraic groups over algebraically closed fields can be described as the intersection of a number of quadrics, see [7] for the complex case, and [3] and [10] for the more general case. In this paper, we consider the Grassmannians (or "varieties") related to the second row of the Freudenthal-Tits Magic square. Over the complex numbers, these are the so-called "Severi varieties". However, these can be considered over any field $\mathbb{K}$ (not necessarily algebraically closed anymore), and these geometries will be also called Severi varieties. A Severi variety lives in a projective space of dimension $N = 5$, 8, 14 or 26 and is the set-theoretic and scheme-theoretic intersection of $N + 1$ quadrics, the equations of which carry a particularly elegant combinatorics, see [11]. The question we'd like to put forward in this paper is whether we can describe the Severi varieties set-theoretically with fewer quadrics, and ultimately try to find the minimum number of quadrics the intersection of which is precisely the given Severi variety. Our motivation is entirely curiosity and beauty; the latter under the form of a rather unexpected connection we found.

We will show that the $N + 1$ quadrics referred to above are linearly independent from each other. Also, every quadric containing the given Severi variety is a linear combination of these $N + 1$ quadrics. These two facts point, in our opinion, to the conjecture that no set of N quadrics can intersect precisely in the Severi variety. However, the quadric Veronese surface (the case $N = 5$ Severi variety) over fields of characteristic 2 is the set-theoretic intersection of three quadrics, see Lemma 4.20 in [6]. Moreover, it was stated in [2], however without proof, that in the case $N = 8$, the Severi variety is the set-theoretic intersection of only 6 quadrics. Hence the above conjecture is false. In general, we will show the following equivalence:

Main Result. *There exist $M \leq N$ quadrics intersecting precisely in the given Severi variety $\Longleftrightarrow$ there exists a subspace of projective dimension $N - M$ in the secant variety disjoint from the Severi variety.*

A more detailed and precise statement will be provided in Section 3. In fact, that statement and its proof allow one, in principle, to describe all equivalence classes of systems of $M \leq N$ quadratic equations exactly describing a given Severi variety. As an application, we will do this explicitly in the simplest case, $N = 5$. For the other cases we content ourselves with giving examples for relatively small M. In particular we will exhibit the real Cartan variety (the Grassmanian of type $\mathsf{E}_{6,1}$ in 26-dimensional real projective space) as the intersection of only 15 quadrics (whereas initially, we had 27 of them). It would require additional methods and ideas to pin down the minimal M for each case and each field, so we consider that to be out of the scope of this paper.

About the method of our proof: Usually, the equations of the $N + 1$ initial quadrics are partial derivatives of a cubic form (which has to be taken for granted). In the present paper, we start with the combinatorics of the equations of the quadrics and derive the cubic form from that. This enables us to make a few geometric observations and interpretations which lead to a proof of the Main Result.

Since the secant variety of a Severi variety always contains at least one point outside the variety, we recover in our special case of Severi variety already the general result of Kronecker saying that any projective variety in $\mathbb{P}^N_{\mathbb{K}}$ is a set theoretic intersection of (at most) N hypersurfaces (in our case quadrics), see Corollary 2 in [5]. One could also ask the equivalent question for the scheme-theoretic intersection of quadrics, but we did not consider that. It seems to us that the answer we give in the present paper for the Segre variety is also valid in the scheme-theoretic sense, but the minimal examples for the line Grassmannian and the Cartan variety are not.

2 Preliminaries

2.1 The varieties

The main objects in this paper are the *quadric Veronese surface* $\mathscr{V}_2(\mathbb{K})$ over any field $\mathbb{K}$, the *Segre variety* $\mathscr{S}_{2,2}(\mathbb{K})$ corresponding to the product of two projective planes over $\mathbb{K}$, the *line Grassmannian* $\mathscr{G}_{2,6}(\mathbb{K})$ of projective 5-space over $\mathbb{K}$, and the *Cartan variety* $\mathscr{E}_6(\mathbb{K})$ associated to the 27-dimensional module of the (split) exceptional group of Lie type E_6 over the field $\mathbb{K}$. These varieties can be defined as intersections of quadrics (and we will do so in Subsection 4.1 below), but it might be insightful to also have the classical definition, which we now present. In what follows, $\mathbb{K}$ is an arbitrary field and $\mathbb{P}^N_{\mathbb{K}}$ or $\mathbb{P}^N$ denotes the N-dimensional projective space over $\mathbb{K}$, which we suppose to be coordinatized with homogeneous coordinates from $\mathbb{K}$ after an arbitrary choice of a basis.

The quadric Veronese surface $\mathscr{V}_2(\mathbb{K})$—This is the image of the *Veronese map* $\nu : \mathbb{P}^2 \to \mathbb{P}^5 : (x,y,z) \mapsto (x^2,y^2,z^2,yz,zx,xy)$.

The Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$—This is the image of the *Segre map* $\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8 : (x,y,z;u,v,w) \mapsto (xu,yu,zu,xv,yv,zv,xw,yw,zw)$.

We may view the set of 3×3 matrices over $\mathbb{K}$ as a 9-dimensional vector space, and the set of symmetric 3×3 matrices as a 6-dimensional subspace. Then we may consider the corresponding projective spaces of (projective) dimension 8 and 5, respectively, in the classical way by considering the 1-spaces as the points. In this way, the Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$ corresponds exactly with the rank 1 matrices; explicitly

$$\mathbb{K}(xu,yu,zu,xv,yv,zv,xw,yw,zw) \leftrightarrow \mathbb{K}\begin{pmatrix} xu & yu & zu \\ xv & yv & zv \\ xw & yw & zw \end{pmatrix}.$$

Similarly, the quadric Veronese surface $\mathscr{V}_2(\mathbb{K})$ corresponds exactly with the rank 1 symmetric matrices; explicitly

$$\mathbb{K}(x^2,y^2,z^2,yz,zx,xy) \leftrightarrow \mathbb{K}\begin{pmatrix} x^2 & yx & zx \\ xy & y^2 & zy \\ xz & yz & z^2 \end{pmatrix}.$$

In particular, $\mathscr{V}_2(\mathbb{K})$ is a subvariety of $\mathscr{S}_{2,2}(\mathbb{K})$ obtained by intersecting with a 5-dimensional subspace.

There exist other Segre varieties; in general $\mathscr{S}_{n,m}(\mathbb{K})$ is defined as the image in $\mathbb{P}^{nm-1}$ of the map $(x_i,y_j)_{1\le i\le n,1\le j\le m} \mapsto (x_iy_j)_{1\le i\le n,1\le j\le m}$. The images of the marginal maps defined by either fixing the x_i, $1 \le i \le n$, or the y_j, $1 \le j \le m$, are called the *generators* of the variety (in case of $\mathscr{S}_{2,2}(\mathbb{K})$ the generators are 2-dimensional projective subspaces).

The line Grassmannian $\mathscr{G}_{2,6}(\mathbb{K})$—Denote the set of lines of $\mathbb{P}^5$, or equivalently, the set of 2-spaces of $\mathbb{K}^6$ by $\binom{\mathbb{K}^6}{\mathbb{K}^2}$. Then $\mathscr{G}_{2,6}(\mathbb{K})$ is the image of the Plücker map

$$\binom{\mathbb{K}^6}{\mathbb{K}^2} \to \mathbb{P}^{14} : \langle (x_1,x_2,\ldots,x_6).(y_1,y_2,\ldots,y_6)\rangle \mapsto (x_iy_j - x_jy_i)_{1\le i<j\le 6}.$$

Denote the coordinate of $\mathbb{P}^{14}$ corresponding to the entry $x_iy_j - x_jy_i$ by p_{ij}, $1 \le i < j \le 6$. By restricting to $y_1 = y_2 = y_3 = x_4 = x_5 = x_6 = 0$, we see that $\mathscr{S}_{2,2}(\mathbb{K})$ is a subvariety of $\mathscr{G}_{2,6}(\mathbb{K})$ obtained by intersecting with an 8-dimensional projective subspace with equation $p_{12} = p_{13} = p_{23} = p_{45} = p_{46} = p_{56} = 0$.

The Cartan variety $\mathscr{E}_6(\mathbb{K})$—This variety is traditionally defined using a trilinear or cubic form, and we postpone this to Subsection 4.1. It is an exceptional variety in the sense that it cannot be defined, using classical notions like Plücker or Grassmann coordinates, from a projective space.

The above varieties share the following properties, see [9]. Set $N = 2+3M$, with $M = 1,2,4,8$. Let $\mathscr{V}$ be one of the varieties $\mathscr{V}_2(\mathbb{K})$, $\mathscr{S}_{2,2}(\mathbb{K})$, $\mathscr{G}_{2,6}(\mathbb{K})$ or $\mathscr{E}_6(\mathbb{K})$, in $\mathbb{P}^N$, with $M = 1,2,4,8$, respectively. Then there exists a unique set $\mathscr{H}$ of $(M+1)$-dimensional subspaces, called *host spaces*, satisfying

(1) every pair of points of $\mathscr{V}$ is contained in at least one host space;

(2) the intersection of $\mathscr{V}$ with any host space is a non-degenerate quadric of maximal Witt index in the host space.

Borrowing some terminology from the theory of parapolar spaces, we shall refer to the quadrics in (2) as *symps*. Also, we shall call two points of the variety *collinear* when all points of the joining projective line belong to the variety.

If we specialize $\mathbb{K} = \mathbb{C}$, then $\mathscr{V}$ is sometimes called a *Severi variety*; these are the only complex varieties with the property that their secant varieties are not the whole projective space, but the secant variety of every variety of the same dimension in a lower dimensional projective space coincides with the ambient space. So we will also refer to these varieties over an arbitrary field as the *Severi varieties*.

2.2 A generalized quadrangle

The introduction of an appropriate cubic form and explicit descriptions using coordinates will be greatly facilitated by using the language of finite generalized quadrangles. A finite *generalized quadrangle (of order* (s,t)*)* is an incidence system $\Gamma = (\mathscr{P}, \mathscr{L})$ of finitely many points ($\mathscr{P}$) and lines ($\mathscr{L}$), where each line is a subset of $\mathscr{P}$, such that each line contains $s+1$ points, through each point pass $t+1$ lines, and for each point p and each line L with $p \notin L$, there exists a unique point-line pair (q, M) such that $p \in M$ and $q \in L \cap M$. We are only interested in generalized quadrangles of order $(2,t)$, and then, by 1.2.2 and 1.2.3 of [8], necessarily $t \in \{1,2,4\}$. Moreover, by 5.2.3 and 5.3.2 of [8], for each $t \in \{1,2,4\}$, there is a unique generalized quadrangle $\mathsf{GQ}(2,t)$ of order $(2,t)$ and $\mathsf{GQ}(2,1)$ is contained in $\mathsf{GQ}(2,2)$ as a subgeometry, and $\mathsf{GQ}(2,2)$ is contained in $\mathsf{GQ}(2,4)$ as a subgeometry.

In the rest of this paper, we denote by $\Gamma = (\mathscr{P}, \mathscr{L})$ the generalized quadrangle $\mathsf{GQ}(2,4)$. An explicit construction of Γ runs as follows, see Section 6.1 of [8]. Let $\mathscr{P}'$ be the set of all 2-subsets of the 6-set $\{1,2,3,4,5,6,\}$, and define

$$\mathscr{P} = \mathscr{P}' \cup \{1,2,3,4,5,6\} \cup \{1',2',3',4',5',6'\}.$$

Denote briefly the 2-subset $\{i,j\}$ by ij, for all appropriate i,j. Let $\mathscr{L}'$ be the set of partitions of $\{1,2,3,4,5,6\}$ into 2-subsets and define

$$\mathscr{L} = \mathscr{L}' \cup \{\{i, j', ij\} \mid i,j \in \{1,2,3,4,5,6\}, i \neq j\}.$$

Then $\Gamma = (\mathscr{P}, \mathscr{L})$ is a model of $\mathsf{GQ}(2,4)$. The subgeometry $\Gamma' = (\mathscr{P}', \mathscr{L}')$ is a model of $\mathsf{GQ}(2,2)$. Further restriction to

$$\mathscr{P}'' = \{ij \mid i \in \{1,2,3\}, j \in \{4,5,6\}\},$$

with induced line set

$$\mathscr{L}'' = \{\{14,25,36\},\{15,26,34\},\{16,24,35\},\{14,26,35\},\{15,24,36\},\{16,25,34\}\},$$

produces a model $\Gamma'' = (\mathscr{P}'', \mathscr{L}'')$ of $\mathsf{GQ}(2,1)$, which we sometimes refer to as a 3×3 grid.

The sets $\{1,2,3,4,5,6\}$ and $\{1',2',3',4',5',6'\}$ have the property that they both do not contain any pair of collinear points, and that non-collinearity is a paring between the two sets. Such a pair of 6-sets is usually called a *double six*.

Finally, we need the notion of a *partial spread*, which is just a set of disjoint lines. A *spread* is a partial spread that partitions the point set. Every generalized quadrangle of order $(2,t)$, $t = 1,2,4$, satisfies the following property (again, see Section 6.1 of [8]):

($*$) *Every pair of disjoint lines is contained in a unique generalized subquadrangle of order* $(2,1)$

Three mutually disjoint lines of a subquadrangle of order $(2,1)$ will be called a *regulus*. Property ($*$) can be reformulated as "every pair of disjoint lines is contained in a unique regulus". A partial spread which is closed under taking reguli of pairs of its members is called *regular*. The $\mathsf{GQ}(2,4)$ contains regular spreads; a maximum regular partial spread of $\mathsf{GQ}(2,2)$ has size 3, and obviously the $\mathsf{GQ}(2,1)$ contains exactly two regular spreads. In this paper, we will fix the following regular spread $\mathscr{S}$ of Γ, which induces maximum regular partial spreads in Γ' and Γ'':

$$\begin{aligned}\mathscr{S} &= \{\{14,25,36\},\{15,26,34\},\{16,24,35\},\{12,2,1'\},\{23,3,2'\},\{13,1,3'\},\\ &\quad \{45,4,5'\},\{56,5,6'\},\{46,6,4'\}\}\,.\end{aligned}$$

The lines $\{14,25,36\},\{15,26,34\},\{16,24,35\}$ form a maximum regular partial spread in both Γ' and Γ''.

3 Main result

In this paper, we prove the following connection between the minimum number of quadrics needed to describe a Severi variety and the largest dimension of a projective subspace in the secant variety disjoint from the variety itself.

Theorem 3.1. *Let $\mathscr{V}$ be either the quadratic Veronese surface $\mathscr{V}_2(\mathbb{K})$, the Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$, the line Grassmannian $\mathscr{G}_{2,6}(\mathbb{K})$, or the Cartan variety $\mathscr{E}_6(\mathbb{K})$, in N-dimensional projective space $\mathbb{P}^N$ over $\mathbb{K}$, with $N=5,8,14,26$, respectively. Then $\mathscr{V}$ is the intersection of $N-d$ quadrics and no less, where d is the dimension of a maximum dimensional projective subspace of $\mathbb{P}^N$ entirely consisting of points lying on a secant of $\mathscr{V}$, or in the nucleus plane if $\mathscr{V}=\mathscr{V}_2(\mathbb{K})$ with $\operatorname{char}\mathbb{K}=2$, but not on $\mathscr{V}$. More precisely, the equivalence classes of the systems of $N-d$ linearly independent quadrics intersecting precisely in $\mathscr{V}$ are in natural bijective correspondence with the d-dimensional projective subspaces of $\mathbb{P}^N$ entirely consisting of points lying on a secant of $\mathscr{V}$, or in the nucleus plane if $\mathscr{V}=\mathscr{V}_2(\mathbb{K})$ with $\operatorname{char}\mathbb{K}=2$, but not on $\mathscr{V}$.*

To fix the ideas, we provide a full proof for the variety $\mathscr{E}_6(\mathbb{K})$. The other cases are completely similar. We comment on them along the way, if differences arise.

4 Proof of Theorem 3.1

4.1 A cubic form

The Cartan variety $\mathscr{E}_6(\mathbb{K})$ is the intersection of 27 well chosen degenerate quadrics. The equations of these quadrics can be described as follows. Let $\mathbb{K}^{27}$ be the vector space underlying $\mathbb{P}^{26}$ and denote by $\langle v \rangle$ the point of $\mathbb{P}^{26}$ corresponding to the nonzero vector $v \in \mathbb{K}^{27}$. Recall that $\Gamma = (\mathscr{P}, \mathscr{L})$ is the generalized quadrangle of order $(2,4)$ and $\mathscr{S}$ is a regular spread of Γ. Label the standard basis vectors of $\mathbb{K}^{27}$ with the points of Γ; so the standard basis is $\{e_p : p \in \mathscr{P}\}$. Each point $p \in \mathscr{P}$ defines a unique quadratic form Q_p given in coordinates by

$$Q_p(v) = X_{q_1} X_{q_2} - \sum_{\{p,r_1,r_2\} \in \mathscr{L} \setminus \mathscr{S}} X_{r_1} X_{r_2},$$

where $\{p, q_1, q_2\} \in \mathscr{S}$. Now define the map $\phi : \mathbb{K}^{27} \to \mathbb{K}^{27} : v \mapsto (Q_p(v))_{p \in \mathscr{P}}$. Our basic observation is the following identity.

Observation 4.1. *For all $v \in \mathbb{K}^{27}$ we have $\phi(\phi(v)) = C(v)v$, where*

$$C(v) = \sum_{\{p,q,r\} \in \mathscr{S}} X_p X_q X_r - \sum_{\{p,q,r\} \in \mathscr{L} \setminus \mathscr{S}} X_p X_q X_r.$$

Also, $\phi(v) = \nabla C(v)$ (the gradient in the classical sense).

Proof. The last assertion is obvious. We show the first one. We have to prove the following identity for each point $p \in \mathscr{P}$:

$$Q_{q_1}(v) Q_{q_2}(v) - \sum_{\{p,r_1,r_2\} \in \mathscr{L} \setminus \mathscr{S}} Q_{r_1}(v) Q_{r_2}(v) \quad = \quad C(v) X_p, \tag{4.1}$$

where $\{p, q_1, q_2\} \in \mathscr{S}$ and $v = (X_q)_{q \in \mathscr{P}}$. Since each $Q_q(v)$, $q \in \mathscr{P}$, has five terms of degree 2 in the coordinates of v, the above sum has 125 terms of degree 4. Since each $Q_q(v)$ has a unique term containing X_p, there are five terms containing X_p^2 and another 40 containing X_p but not X_p^2. The terms with X_p^2 are easily seen to be

$$X_p^2 X_{q_1} X_{q_2} \quad - \sum_{\{p,r_1,r_2\} \in \mathscr{L} \setminus \mathscr{S}} X_p^2 X_{r_1} X_{r_2}. \tag{4.2}$$

For each line $\{q_1, s, s'\} \in \mathscr{L}$, we have the combined terms $X_p X_{q_1}$ of $Q_{q_2}(v)$ and $-X_s X_{s'}$ of $Q_{q_1}(v)$, resulting in a term $-X_p X_{q_1} X_s X_{s'}$ in the left hand side of Equation (4.1). Note that $\{q_1, s, s'\} \notin \mathscr{S}$. Similarly for the lines through q_2. We conclude that the terms of $Q_{q_1} Q_{q_2}$ containing X_p but not X_p^2 are given by

$$- \sum_{i=1}^{2} \sum_{\{q_i,s,s'\} \in \mathscr{L} \setminus \mathscr{S}} X_p X_{q_i} X_s X_{s'}. \tag{4.3}$$

Now let $r \in \mathscr{P}$ be collinear to p but distinct from q_1 and q_2. Let $\{r, s, s'\} \in \mathscr{L}$, with $p \notin \{s, s'\}$. First suppose that $\{r, s, s'\} \in \mathscr{S}$. Let $r' \in \mathscr{P}$ be such that $\{p, r, r'\} \in \mathscr{L} \setminus \mathscr{S}$. Then have the combined terms $-X_p X_r$ of $Q_{r'}(v)$ and $X_s X_{s'}$ of $Q_{q_1}(v)$, resulting in a term $-X_p X_r X_s X_{s'}$ in the left hand side of Equation (4.1). If $\{r, s, s'\} \in \mathscr{L} \setminus \mathscr{S}$, then we obtain the same term, but with the opposite sign. These terms, together with those of Expressions (4.2) and (4.3) already provide the full right hand side of Equality (4.1). The remaining $125 - 5 - 40 = 80$ terms in the left hand side of Equality (4.1) should now cancel pairwise. Disregarding the signs, they are all of the form $X_{s_1}X_{s'_1}X_{s_2}X_{s'_2}$, where $\{r_i, s_i, s'_i\} \in \mathscr{L}$, $i = 1, 2$, with $\{p, r_1, r_2\} \in \mathscr{L}$. These three lines are contained in a unique grid

$$\begin{pmatrix} p & r_1 & r_2 \\ q & s_1 & s_2 \\ q' & s'_1 & s'_2 \end{pmatrix}, \quad \text{for some } q, q' \in \mathscr{P},$$

where the rows and columns correspond to lines of Γ. Hence in the term $Q_q(v)Q_{q'}(v)$ also appears a term $X_{s_1}X_{s_2}X_{s'_1}X_{s'_2}$, up to sign. We now have to see that the signs are opposite. If $\{p, r_1, r_2\} \in \mathscr{S}$, then both signs are $+$, but the terms nevertheless cancel since $Q_q(v)Q_{q'}(v)$ appears with a minus sign in Equality (4.1). Note that it does not make any difference whether $\{q, s_1, s_2\} \in \mathscr{S}$ or not, since, by the regularity property of $\mathscr{S}$ we have $\{q, s_1, s_2\} \in \mathscr{S}$ if and only if $\{q', s'_1, s'_2\} \in \mathscr{S}$.

Now suppose $\{p, r_1, r_2\} \in \mathscr{L} \setminus \mathscr{S}$. We may also assume that $\{p, q, q'\} \in \mathscr{L} \setminus \mathscr{S}$, as otherwise we are back in the previous case by interchanging the roles of $\{r_1, r_2\}$ and $\{q, q'\}$. If exactly one of the other lines of the grid belongs to the spread $\mathscr{S}$, then the signs are opposite. The regularity of $\mathscr{S}$ implies that at most one other line belongs to $\mathscr{S}$; we now claim that every 3×3 grid of Γ contains at least one spread line. Indeed, we count 12 grids with three spread lines and $9 \cdot 12 = 108$ grids with a unique spread line. In total there are 45 lines, each in 16 grids, but each also counted 6 times. Hence there are 120 3×3 grids in total, which shows our claim and the observation. □

Comments on the other cases.

(i) The Grassmannian variety $\mathscr{G}_{2,6}(\mathbb{K})$ arises from the Cartan variety above by setting $X_p = 0$ for all points p in a double six. Indeed, the analogue of the construction above considers Γ' in place of Γ and a maximal regular partial spread $\mathscr{S}'$ in place of $\mathscr{S}$ ($\mathscr{S}'$ consists just of three disjoint lines of a grid). That this works can be seen through the model of Γ, Γ' and $\mathscr{S}$ given in Subsection 2.2. Since $\mathscr{G}_{2,6}(\mathbb{K})$ is the intersection of all quadrics with equation $p_{ij}p_{k\ell} + p_{ik}p_{\ell j} + p_{i\ell}p_{jk} = 0$ (as follows from Theorem 3.8 in [6]), it suffices to make a choice between each p_{ij} and p_{ji} in order to get the signs lined up with the above rule and the choice of $\mathscr{S}'$. But this can simply be done by retaining p_{ij} for $i \in \{1, 2, 3\}$ and $j \in \{4, 5, 6\}$, and $(ij) \in \{(12), (23), (31), (45), (56), (64)\}$, as an elementary calculation shows.

(ii) The Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$ arises from the Cartan variety by setting $X_p = 0$ for all points outside a regulus of spread lines. This can easily be seen through the construction in Subsection 2.1, denoting the point of the grid associated to the entry (i, j) in the 3×3 matrix by q_{ij} and the corresponding coordinate by x_{ij}, we let the grid be defined by the lines $\{q_{ij}, q_{k\ell}, q_{mn}\}$ with $\{i, k, m\} = \{j, \ell, n\} = \{1, 2, 3\}$. If we choose the spread lines as

$\{q_{11}, q_{22}, q_{33}\}$, $\{q_{12}, q_{23}, q_{31}\}$ and $\{q_{13}, q_{32}, q_{21}\}$, then we see that Q_p is exactly the co-factor of the entry corresponding to p in the matrix $(x_{ij})_{1\le i,j\le 3}$. This indeed defines $\mathscr{S}_{2,2}(\mathbb{K})$ as can be deduced from Theorem 4.94 in [6], or from [2].

(*iii*) The quadric Veronese variety $\mathscr{V}_2(\mathbb{K})$ arises from the Cartan variety by setting $X_p = 0$ for all points outside a regulus $\{L_1, L_2, L_3\}$ of spread lines and $X_{p_1} = X_{p_2}$ for collinear points $p_i \in L_i$, $i = 1, 2$. Indeed, in the previous paragraph, choosing $L_3 = \{q_{11}, q_{22}, q_{33}\}$, collinear points outside this line correspond to symmetric entries of the matrix. Here, the gradient is not identical to ϕ; the last three coordinates of the gradient are twice the last three coordinates of ϕ, hence there is special behaviour in characteristic 2.

Denoting by $v.w$ the ordinary dot product of v and w in $\mathbb{K}^{27}$, we observe the following.

Observation 4.2. *For arbitrary $v, w \in \mathbb{K}^{27}$ and $t \in \mathbb{K}$, we have*

$$C(v + tw) = C(v) + t\phi(v).w + t^2 v.\phi(w) + t^3 C(w). \tag{4.4}$$

Proof. It is clear that the coefficient of t^0 and t^3 are $C(v)$ and $C(w)$, respectively. It remains to explain the coefficient of t, as the one of t^2 is obtained by switching the roles of v and w. Now, obviously, the coefficient of t is linear in w, hence if suffices to set $w = e_p$ for $p \in \mathscr{P}$. Then we see that the coefficient of t in $C(v+te_p)$ is equal to $\frac{\partial C(v)}{\partial X_p} e_p = Q_p(v)e_p$. Now Identity (4.4) follows. □

Hence we deduce that the *adjoint square* $v^\sharp$ in the sense of Aschbacher [1], is, up to reordering the coordinates, exactly equal to $\phi(v)$. Hence $C(v)$ is the cubic form related to $\mathscr{E}_6(\mathbb{K})$ and the Chevalley group $\mathsf{E}_6(\mathbb{K})$ acts on $\mathbb{P}^{26}$ with three orbits, which are easily seen to be defined as

(*i*) the points of the variety $\mathscr{E}_6(\mathbb{K})$, namely those corresponding to the vectors v with $\phi(v) = \vec{o}$. These points are the *white points*;

(*ii*) the points off the variety $\mathscr{E}_6(\mathbb{K})$ corresponding to the vectors v with $C(v) = 0$. These points are the *grey points*;

(*iii*) the points corresponding to vectors v with $C(v) \neq 0$. These points are the *black points*.

We have taken the notions of white, grey and black from Aschbacher [1]. See also Cohen [4] for a very comprehensive introduction.

Comments on the other cases. For the quadric Veronese variety $\mathscr{V}_2(\mathbb{K})$ the group has more than three orbits; in this case, and if $\operatorname{char}\mathbb{K} = 2$, the grey points also comprise all points of the *nucleus plane*.

It now follows from (*i*), (*ii*) and Identity (4.4) that the projective null set of the cubic form C is exactly the secant variety of $\mathscr{E}_6(\mathbb{K})$.

Observation 4.3. *Let v be a nonzero vector of $\mathbb{K}^{27}$.*

(*i*) *The point $\langle v\rangle$ is a white point if and only if $\phi(v)=\vec{o}$;*

(*ii*) *the point $\langle v\rangle$ is grey if and only if $\phi(v)\neq\vec{o}$ and the point $\langle\phi(v)\rangle$ is white;*

(*iii*) *the point $\langle v\rangle$ is black if and only if $\langle\phi(v)\rangle$ is a black point.*

Proof. (*i*) This follows immediately from the definition of white points above.

(*ii*) By definition, the point $\langle v\rangle$ is grey if and only if $\phi(v)\neq\vec{o}$ and $C(v)=0$. The latter is equivalent to $\phi(\phi(v))=\vec{o}$, which is equivalent to $\phi(v)$ being white by (*i*).

(*iii*) Suppose $\langle v\rangle$ is black. If $\langle\phi(v)\rangle$ is white or grey, then $C(\phi(v))=0$, implying $\phi(\phi(\phi(v)))=\vec{o}$. But the left hand side is equal to $\phi(C(v)v)=C(v)^2\phi(v)\neq\vec{o}$, a contradiction. Now suppose $\phi(v)$ is black. Then $\phi(\phi(\phi(v)))$ is a non-zero multiple of $\phi(v)$, and so $\phi(\phi(v))$ cannot be equal to $\vec{o}$, implying $C(v)\neq 0$ and $\langle v\rangle$ is black. □

It follows from the previous observation that $\langle\phi(v)\rangle$ is never a grey point. We record this for further reference.

Corollary 4.4. *For each $v\in\mathbb{K}^{27}$, $\langle\phi(v)\rangle$ is never a grey point.*

We also observe that transitivity of the automorphism group of $\mathscr{E}_6(\mathbb{K})$ implies the following.

Observation 4.5. *Let v be a nonzero vector of $\mathbb{K}^{27}$. Then $\langle v\rangle$ is a white point if and only if there exists a grey point $\langle w\rangle$ with $\langle\phi(w)\rangle=\langle v\rangle$.*

Proof. If $\langle w\rangle$ is grey, then by Observation 4.3 (*ii*), $\langle\phi(w)\rangle$ is white. Now let $\langle v\rangle$ be a white point. Let $\langle w_0\rangle$ be a grey point (for instance the point $\langle e_p+e_q\rangle$ with p and q collinear points of Γ). Then by Observation 4.3 (*ii*), $\langle\phi(w_0)\rangle$ is white. Let g be an automorphism of $\mathscr{E}_6(\mathbb{K})$ mapping $\langle\phi(w_0)\rangle$ to $\langle v\rangle$. Then $\langle w_0^g\rangle$ is grey and $\langle\phi(w_0^g)\rangle=\langle\phi(w_0)^g=\langle v\rangle$. □

Observation 4.6. *For every white point $\langle v\rangle$, the set $\{\langle w\rangle\mid\vec{o}\neq\phi(w)\in\langle v\rangle\}$ is the set of grey points of a (9-dimensional) host space of $\mathbb{P}^{26}$ (hence generated by the points of some fixed symp).*

Proof. Let $p\in\mathscr{P}$ be arbitrary. Let $\langle w\rangle$ be a grey point belonging to the host space $U_p:=\langle e_q\mid p\perp q\in\mathscr{P}\rangle$. Then clearly $\phi(w)$ is a nonzero multiple of e_p. By transitivity of the automorphism group, we thus see that for every white point $\langle v\rangle$, the set $\{\langle w\rangle\mid\vec{o}\neq\phi(w)\in\langle v\rangle\}$ is the set of grey points of a union of host spaces of $\mathbb{P}^{26}$. Suppose that we have the union of at least two host spaces. By transitivity, we may assume that two of these host spaces are U_p and U_q, with $p,q\in\mathscr{P}$. But we already know that these map to $\langle e_p\rangle$ and $\langle e_q\rangle$, respectively, which are distinct. The assertion now follows. □

4.2 A lemma

Lemma 4.7. *Let Q be a quadratic form whose null set contains $\mathscr{E}_6(\mathbb{K})$. Then Q is a linear combination (with constant coefficients in $\mathbb{K}$) of the Q_p, $p \in \mathscr{P}$.*

Proof. Let Q be given by the polynomial

$$Q(v) = \sum_{\{p,q\} \subseteq \mathscr{P}} a_{\{p,q\}} X_p X_q,$$

with $a_{\{p,q\}} \in \mathbb{K}$. Since all points corresponding to the standard basis vectors belong to $\mathscr{E}_6(\mathbb{K})$, we have $a_{\{p\}} = 0$, for all $p \in \mathscr{P}$. Now let $p, q \in \mathscr{P}$ be distinct but non-collinear in Γ. Then one easily checks that $\langle e_p + e_q \rangle \in \mathscr{E}_6(\mathbb{K})$. Hence the coefficient $a_{\{p,q\}}$ of $X_p X_q$ in $Q(v)$ is also 0.

Now consider a line $L \in \mathscr{S}$ and a line $M \in \mathscr{L} \setminus \mathscr{S}$ with $L \cap M = \{p\}$, $p \in \mathscr{P}$. Let $L = \{p, q_1, q_2\}$ and $M = \{p, r_1, r_2\}$. Then clearly the point $\langle e_{q_1} + e_{q_2} + e_{r_1} + e_{r_2} \rangle$ belongs to $\mathscr{E}_6(\mathbb{K})$. This implies that $a_{\{q_1,q_2\}} = -a_{\{r_1,r_2\}} =: a_p$. Now it is clear that $Q(v) = \sum_{p \in \mathscr{P}} a_p Q_p(v)$, proving the lemma. □

Noting that, for collinear points $q_1, q_2 \in \mathscr{P}$, the vector $e_{q_1} + e_{q_2}$ belongs to the null set of each quadratic form Q_p, $p \in \mathscr{P}$, except for the unique point p with $\{p, q_1, q_2\} \in \mathscr{L}$, we see that

Observation 4.8. *The set $\{Q_p : p \in \mathscr{P}\}$ is a linearly independent set of quadratic forms and no proper subset of it intersects precisely in $\mathscr{E}_6(\mathbb{K})$.*

Comments on the other cases. Care has to be taken for the case $\mathscr{V}_2(\mathbb{K})$, not only since the automorphism group can have more than three orbits on the points (and on the hyperplanes) of the surrounding projective space, but also since this case behaves in an exceptional way for small fields. Let us provide some quick details. With respect to the representation given as definition in Subsection 2.1, we have

$$\phi(x_1, x_2, x_3, x_{23}, x_{31}, x_{12}) = \\ (x_2x_3 - x_{23}^2, x_3x_1 - x_{31}^2, x_1x_2 - x_{12}^2, x_{31}x_{12} - x_1x_{23}, x_{12}x_{23} - x_2x_{31}, x_{23}x_{31} - x_3x_{12}),$$

and

$$C(x_1, x_2, x_3, x_{23}, x_{31}, x_{12}) = x_1x_2x_3 + 2x_{12}x_{23}x_{31} - x_1x_{23}^2 - x_2x_{31}^2 - x_3x_{12}^2.$$

Observations 4.5 and 4.6 need an alternative proof, since the group does not act transitively on the grey points. However, one calculates easily that

$$\phi(0, 0, 0, k, -\ell, 0) = (-k^2, -\ell^2, 0, 0, 0, -k\ell) = -\nu(k, \ell, 0)$$

and

$$\phi(1, 1, a^2 + b^2, -b, -a, 0) = (a^2, b^2, 1, b, a, ab) = \nu(a, b, 1),$$

which covers all points of $\mathscr{V}_2(\mathbb{K})$. This shows the nontrivial direction of Observation 4.5. Observation 4.6 follows from a similar calculation and is left to the reader. Lemma 4.7 only holds for fields with at least four elements. To prove this, one just expresses that a generic point $\nu(x, y, z)$ satisfies a quadratic equation, and one argues that, if the field has at least 4 elements, then the corresponding quadratic form is a linear combination of the $X_iX_j - X_{ij}^2$, $ij \in \{12, 23, 31\}$, and the $X_iX_{jk} - X_{ki}X_{ij}$, $ijk \in \{123, 231, 312\}$. For $|\mathbb{K}| \leq 3$, all points of $\mathscr{V}_2(\mathbb{K})$ satisfy $X_1X_{23} - X_2X_{23} = 0$ since $x = x^3$ for all $x \in \mathbb{K}$, and this is not a linear combination of the basic quadratic equations. Finally, Observation 4.8 is false, see Subsection 5.2 below.

4.3 Reducing the number of quadrics—End of the proof

Lemma 4.7 and Observation 4.8 indicate that we need all 27 quadratic forms to describe $\mathscr{E}_6(\mathbb{K})$ as the intersection of quadrics. However, making suitable linear combinations, we can actually reduce the number of quadrics. To do this, let U be a subspace of $\mathbb{K}^{27}$ such that all its non-zero vectors correspond to grey points, and we use the same notation U for the corresponding subspace of $\mathbb{P}^{26}$. Let $\{H_i : i \in I\}$ be a minimal set of hyperplanes of $\mathbb{K}^{27}$ whose intersection is exactly U (then $|I| + \dim U = 27$, where $\dim U$ is the vector dimension of U). For each

$$H_i \leftrightarrow \sum_{p \in \mathscr{P}} a_p^{(i)} X_p = 0, \quad i \in I,$$

define the quadratic form Q_i given by

$$Q_i(v) = \sum_{p \in \mathscr{P}} a_p^{(i)} Q_p(v).$$

Note that for a vector $v \in \mathbb{K}^{27}$ we have $Q_i(v) = 0$ if and only if $\phi(v) \in H_i$.

Clearly, the null set of each Q_i contains the vectors corresponding to $\mathscr{E}_6(\mathbb{K})$. Conversely, suppose some nonzero vector v belongs to the null set of each Q_i, $i \in I$. Then, by construction, $\langle \phi(v) \rangle \in U$. If $\phi(v) \neq \vec{o}$, this would mean that $\langle \phi(v) \rangle$ is a grey point, contradicting Corollary 4.4.

Conversely, suppose $\mathscr{E}_6(\mathbb{K})$ is the intersection of the null sets of a number of quadratic forms Q_i, $i \in I$. By Lemma 4.7, each quadratic form Q_i is a linear combination of the Q_p, $p \in \mathscr{P}$, say

$$Q_i(v) = \sum_{p \in \mathscr{P}} a_p^{(i)} Q_p(v), \quad a_p^{(i)} \in \mathbb{K}.$$

For $i \in I$, let the hyperplane H_i be given by the equation

$$H_i \leftrightarrow \sum_{p \in \mathscr{P}} a_p^{(i)} X_p = 0.$$

Suppose there is a white or black point $\langle v \rangle$ contained in each hyperplane H_i, $i \in I$. Then Lemma 4.5 and the definition of $C(v)$ implies that there exists $w \in \mathbb{K}^{27}$ with $\phi(w) = v$ and with $\langle w \rangle$ grey or black. It follows that w is in the null set of each Q_i, $i \in I$, contradicting the fact that $\langle w \rangle$ is not white. This actually shows the last claim of Theorem 3.1, and the first claim also follows.

Hence the minimum number of quadrics completely describing $\mathscr{E}_6(\mathbb{K})$ as their intersection is equal to $27 - d'$, where $d' = d + 1$ is the dimension of a maximum dimensional subspace containing no vectors corresponding to white or black points.

5 Examples and applications

In this section, we determine the exact value of d for some specific cases. Our results will show that d strongly depends on the field $\mathbb{K}$ and therefore the determination of d for every field $\mathbb{K}$ is beyond the scope of this paper.

We begin with some general observations.

5.1 General observations

To ease notation, we will identify the projective version of ϕ with ϕ, *i.e.*, we will write $\langle \phi(v) \rangle$ as $\phi(\langle v \rangle)$. This projective version is then not defined on the points of $\mathscr{E}_6(\mathbb{K})$, and it induces an involutive bijection from the set of black points onto itself.

In this section, let U be a subspace of $\mathbb{P}^{26}$ entirely consisting of grey points; we will briefly call this a *grey subspace*. Then $\phi(U)$ corresponds to a set of points of $\mathscr{E}_6(\mathbb{K})$. We prove some properties of $\phi(U)$.

Lemma 5.1. *Let $p, q \in U$, $p \neq q$. Denote the line joining p and q by L, and note that $L \subseteq U$. Then*

- *(i) If $\phi(p) = \phi(q)$, then $\phi(L) = \phi(p)$;*
- *(ii) if $\phi(p)$ and $\phi(q)$ are collinear on $\mathscr{E}_6(\mathbb{K})$, then ϕ is bijective on L and $\phi(L)$ is a conic on $\mathscr{E}_6(\mathbb{K})$ which is contained in a singular plane of $\mathscr{E}_6(\mathbb{K})$;*
- *(iii) if $\phi(p)$ and $\phi(q)$ are not collinear on $\mathscr{E}_6(\mathbb{K})$, then ϕ is bijective on L and $\phi(L)$ is a conic on $\mathscr{E}_6(\mathbb{K})$ which is not contained in a singular plane of $\mathscr{E}_6(\mathbb{K})$.*

Proof. Define the cross product $v \times w$ as the linearization of ϕ, *i.e.*, $v \times w = \phi(v+w) - \phi(v) - \phi(w)$. Denote the projective version also by $\times$, *i.e.*, $\langle v \rangle \times \langle w \rangle = \langle v \times w \rangle$. Then one calculates that, for all $\lambda, \mu \in \mathbb{K}$,

$$\phi(\lambda v + \mu w) = \lambda^2 \phi(v) + \lambda\mu(v \times w) + \mu^2 \phi(w). \tag{5.1}$$

If $v \times w$ is linearly dependent on $\phi(v)$ and $\phi(w)$, then also $\phi(v+w)$ is a linear combination of $\phi(v)$ and $\phi(w)$. Suppose first that $\phi(v)$ and $\phi(w)$ are not collinear on $\mathscr{E}_6(\mathbb{K})$. Then, since the only points of $\mathscr{E}_6(\mathbb{K})$ on the line $\langle v, w\rangle$ are $\langle v\rangle$ and $\langle w\rangle$, and since $\langle \phi(v+w)\rangle$ is by assumption a point of $\mathscr{E}_6(\mathbb{K})$, we deduce without loss of generality $\phi(\langle v+w\rangle) = \phi(\langle v\rangle)$. Hence $\langle v\rangle$ and $\langle v+w\rangle$ are contained in the same host space, implying $\langle w\rangle$ is also, and we are in Situation (i), a contradiction. Hence Equation (5.1) defines a conic.

Now suppose that $\phi(v)$ and $\phi(w)$ are collinear on $\mathscr{E}_6(\mathbb{K})$. Let ξ and ζ be the symplecta the host spaces of which contain $\langle v\rangle$ and $\langle w\rangle$, respectively. Let $U = \xi \cap \zeta$. Select maximal singular subspaces $V \subseteq \xi$ and $W \subseteq \zeta$ disjoint from U. Then simple dimension arguments show that every point of V is collinear to a unique point of W. Moreover $\langle V, W\rangle \cap \mathscr{E}_6(\mathbb{K})$ is a Segre variety $\mathscr{S}$ isomorphic to $\mathscr{S}_{1,4}(\mathbb{K})$, and every 4-dimensional generator of that Segre variety is contained in a unique symp also containing U. This follows from the similar but easy to check fact for $\mathscr{S}_{2,2}(\mathbb{K})$ and the fact that $\mathscr{S}_{2,2}(\mathbb{K})$ is amply contained in $\mathscr{E}_6(\mathbb{K})$ (by [11]). Now $v = v_1 + v_2$, with $\langle v_1\rangle \in U$ and $v_2 \in V$, and $w = w_1 + w_2$, with $\langle w_1\rangle \in U$ and $\langle w_2\rangle \in W$. Notice that $p \times q = \phi(p+q)$ for points $p, q \in \mathscr{E}_6(\mathbb{K})$. If $\langle v_2\rangle$ and $\langle w_2\rangle$ are not contained in the same 1-dimensional generator of $\mathscr{S}$, then $\langle v_2 + w_2\rangle$ is not contained in $\mathscr{S}$ and hence $\phi(v_2 + w_2)$ is not contained in a symplecton through U (as each host space through U intersects $\langle \mathscr{S}\rangle$ in a 4-dimensional generator of $\mathscr{S}$). Consequently in that case,

$$v \times w = v_1 \times w_2 + v_2 \times w_1 + v_2 \times w_2 = \phi(v_1 + w_2) + \phi(v_2 + w_1) + \phi(v_2 + w_2)$$

is linearly independent from $\phi(v)$ and $\phi(w)$ (since $\phi(v_2 + w_1)$ is a (possibly trivial) multiple of $\phi(\langle v\rangle)$ and $\phi(v_1 + w_2)$ a multiple of $\phi(w)$). So in this case, (ii) holds.

So we may assume that $\langle v_2\rangle$ and $\langle w_2\rangle$ are collinear on $\mathscr{E}_6(\mathbb{K})$, *i.e.*, $v_2 \times w_2 = \vec{o}$. It then follows that there exists a unique point p on the line through $\langle v_1\rangle$ and $\langle w_1\rangle$ collinear with both $\langle v_2\rangle$ and $\langle w_2\rangle$ (if some point q on that line were collinear to $\langle v_2\rangle$ but not to $\langle w_2\rangle$, then ζ would be determined by $\langle w_2\rangle$ and q and would contain $\langle v_2\rangle$). Hence there exists $\ell \in \mathbb{K}^\times$ with $v_2 \times (v_1 + \ell w_1) = \vec{o} = w_2 \times (v_1 + \ell w_1)$. Then, using the bilinearity of the cross-product, we calculate

$$v \times w = v_1 \times w_2 + v_2 \times w_1 = -\ell w_1 \times w_2 - \ell^{-1} v_1 \times v_2 = -\ell\phi(w) - \ell^{-1}\phi(v).$$

Hence, substituting this in Equation (5.1), we obtain

$$\phi(\lambda v + \mu w) = (\lambda^2 - \ell^{-1}\lambda\mu)\phi(v) + (\mu^2 - \ell\lambda\mu)\phi(w). \tag{5.2}$$

which becomes $\vec{o}$ for $\mu = \ell\lambda$, a contradiction.

The lemma now follows. □

We call lines of type (i) *short*, lines of type (ii) *flat* and lines of type (iii) *conical*. We now have the following result.

Proposition 5.2. *Set $d = \dim(U)$, where we use projective dimensions.*

(1) *If all lines of U are short, then $\phi(U)$ is a point.*

(2) *If only all lines in a hyperplane of U are short, then either all other lines are flat, or all other lines are conical. In both cases $\phi(U)$ is a quadric of Witt index 1 spanning a $(d+1)$-dimensional space in $\mathbb{P}^{26}$, which is singular in the flat case, and in the conical case the quadric is contained in a symp as a subquadric.*

(3) *In all cases $\phi(U)$ is the quotient (or projection) of a Veronese variety $\mathscr{V}_d(\mathbb{K})$, where the image of a conic is either a conic, or a single point. If U does not contain short lines, then $\dim\langle\phi(U)\rangle \geq d$.*

(4) *If U contains two disjoint planes containing only short lines, then every line intersecting both planes is conical.*

For $\mathscr{G}_{2,6}(\mathbb{K})$, the last statement becomes:

(4′) *If U contains two disjoint short lines, then every line intersecting both lines is conical.*

Proof. We start with noting that (1) is obvious: all points of U are contained in the same host space.

Let $e_0, \ldots, e_d$ be a (vector) basis of U. Then, using the definition of the cross product and the bilinearity of it, we calculate that $\phi(U)$ is the image of the map

$$(\lambda_0, \ldots, \lambda_d) \mapsto \sum_{i=0}^{d} \lambda_i^2 \phi(e_i) + \sum_{i=0}^{d-1} \sum_{j=i+1}^{d} \lambda_i \lambda_j (e_i \times e_j), \tag{5.3}$$

which is a Veronese variety $\mathscr{V}_d(\mathbb{K})$ if all $\phi(e_i)$ and $e_i \times e_j$ are linearly independent. But if not, then this is just an obvious quotient of $\mathscr{V}_d(\mathbb{K})$. If $\phi(U)$ does not contain short lines, then no point of the subspace from which one projects lies on a tangent, and since tangents at one point fill the whole tangent space, the latter are isomorphically projected. Hence (3).

To show (2), we may assume that all lines of the subspace $H := \langle e_1, \ldots, e_d \rangle$ are short. Hence there exist constants $k_1, \ldots, k_{d-1}$ such that $\phi(e_i) = k_i \phi(e_d)$, $k_i \in \mathbb{K}$, $i = 1, \ldots, d-1$. Then $\phi(e_i + e_j)$ is a multiple of $\phi(e_d)$, $i, j \in \{1, \ldots, d\}$, $i \neq j$, and so we may write $e_i \times e_j = \ell_{ij} \phi(e_d)$, $i, j \in \{1, \ldots, d\}$, $i < j$, for some $\ell \in \mathbb{K}$. The mapping (5.3) becomes

$$(\lambda_0, \ldots, \lambda_d) \mapsto \lambda_0^2 \phi(e_0) + \left(\sum_{i=1}^{d} k_i \lambda_i^2 + \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \ell_{ij} \lambda_i \lambda_j \right) \phi(e_d) + \sum_{i=1}^{d} \lambda_0 \lambda_i (e_0 \times e_i). \tag{5.4}$$

If $\phi(e_0), \phi(e_d)$ and all $e_0 \times e_i$, $i = 1, \ldots, d$, are linearly independent from each other, then, with respect to that basis, and denoting the coordinate corresponding to $e_0 \times e_i$ by X_i, the one corresponding to $\phi(e_0)$ by X_0 and the one corresponding to $\phi(e_d)$ by X_{d+1}, it is an elementary exercise to calculate that a point is in the image of the map (5.4) if and only if its coordinates $(X_0, \ldots, X_{d+1})$ satisfy

$$X_0 X_{d+1} = \sum_{i=1}^{d} k_i X_i^2 + \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \ell_{ij} X_i X_j. \tag{5.5}$$

Note that the right hand side of Equation (5.5) is an anisotropic quadratic form; indeed, suppose there exist $x_i \in \mathbb{K}$, $i = 1, \ldots, d$, such that $\sum_{i=1}^{d} k_i x_i^2 + \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \ell_{ij} x_i x_j = 0$. Then setting $\lambda_0 = 0$ and $\lambda_i = x_i$, we see that the right hand side of the map in (5.4) becomes $\vec{o}$, a contradiction, as this would yield a white point in U.

Hence Equation (5.5) defines a quadric Q of Witt index 1. If $\phi(e_0), \phi(e_d)$ and all $e_0 \times e_i$, $i = 1, \ldots, d$, are not linearly independent from each other, then $\phi(S)$ is a projection of Q. However, considering a point p in $\langle Q \rangle$ in the subspace from which we project, we can select a plane α through p containing two points of $\phi(U)$, and then α contains a conic, which is either not projected bijectively, or projected into a line, both of which are contradictions to Lemma 5.1. Hence $\phi(U)$ spans a space of dimension $d + 1$.

If some line L of U is flat, then, for each point $p \in L \setminus H$, $\phi(p)$ and $\phi(L \cap H)$ are collinear on $\mathscr{E}_6(\mathbb{K})$. But $\phi(L \cap H) = \phi(H) = \phi(q)$, for each $q \in H$. Hence all lines of U intersecting L in some point not in H are flat. Replacing L with each such a line, we obtain that all lines of U not contained in H are flat.

This completes the proof of (2). We now address (4). Suppose that α and β are two disjoint planes all of whose lines are short, and suppose for a contradiction that there is a flat line L intersecting α and β in some point $\langle v \rangle$ and $\langle w \rangle$, respectively. Then $\phi(\alpha) = \phi(\langle v \rangle)$ and $\phi(\beta) = \phi(\langle w \rangle)$ are collinear on $\mathscr{E}_6(\mathbb{K})$. We now use the same notation as in the proof of Lemma 5.1 (*ii*). So ξ and ζ are the symplecta with $\alpha \subseteq \langle \xi \rangle$ and $\beta \subseteq \langle \zeta \rangle$, and $U = \xi \cap \zeta$. Also, V and W are maximal singular subspaces of ξ and ζ, respectively, disjoint from U. Let α_2 and β_2 be the projection of α and β, respectively, from U onto V and W, respectively. Since $\langle V, W \rangle \cap \mathscr{E}_6(\mathbb{K})$ is a Segre variety, a dimension argument implies that some point $\langle v_2 \rangle$ of α_2 is collinear on $\mathscr{E}_6(\mathbb{K})$ with some point $\langle w_2 \rangle$ of β_2. But, as one can read in the last part of the proof of Lemma 5.1, this leads to a contradiction. □

Corollary 5.3. *With the above notation, if U intersects the space spanned by a symp ξ in a subspace of dimension 1, 2 or 4 in the cases $\mathscr{V} = \mathscr{S}_{2,2}(\mathbb{K})$, $\mathscr{G}_{2,6}(\mathbb{K})$ or $\mathscr{E}_6(\mathbb{K})$, respectively, then either U is contained in $\langle \xi \rangle$, or $\mathscr{V} = \mathscr{G}_{2,6}(\mathbb{K})$ and all lines of U that intersect $\langle \xi \rangle$ are flat.*

Proof. Set $d = \dim U$. Without loss of generality, we may assume that $U \cap \langle \xi \rangle$ is a hyperplane of U. Then Proposition 5.2 implies that $\phi(U)$ is contained in a subspace W of $\mathbb{P}^{6d-4}$ of dimension $d + 1 = 3, 4, 6$ for the respective cases. So W can only be a singular subspace of $\mathscr{V}$ if $\mathscr{V} = \mathscr{G}_{2,6}(\mathbb{K})$. If W is not singular, then $\phi(U)$ is quadric of Witt index 1 arising as the intersection of a symp with a subspace of dimension 3, 4 and 6, respectively. But such subspaces always have lines in common with the symp, since they intersect each maximal singular subspace of the symp in a line, by an obvious dimension argument, a contradiction. □

Now we consider the separate varieties in turn. Notice first that, since $\mathscr{V}_2(\mathbb{K}) \subseteq \mathscr{S}_{2,2}(\mathbb{K}) \subseteq \mathscr{G}_{2,6}(\mathbb{K}) \subseteq \mathscr{E}_6(\mathbb{K})$, each example for a certain variety carries over to the next variety, as ordered in the inclusions just given.

5.2 The quadric Veronesean $\mathscr{V}_2(\mathbb{K})$

Recall that $\mathscr{V}_2(\mathbb{K})$ is given by the image of the Veronese map

$$\mathbb{P}^2 \to \mathbb{P}^5 : (x, y, z) \mapsto (x^2, y^2, z^2, yz, zx, xy).$$

The line given by the points $(0, 0, 0, k, \ell, 0)$ entirely consists of grey points, hence in general, $6 - 2 = 4$ quadrics suffice to describe $\mathscr{V}_2(\mathbb{K})$. After a little calculation, ordering the coordinates like $(X_1, X_2, X_3, X_{23}, X_{31}, X_{12})$, these turn out to be $X_1X_2 = X_{12}^2$, $X_3X_1 = X_{31}^2$, $X_2X_3 = X_{23}^2$, and any one of $X_1X_{23} = X_{31}X_{12}$, $X_2X_{31} = X_{12}X_{23}$ or $X_3X_{12} = X_{23}X_{31}$. In characteristic 2, the whole nucleus plane consists of grey points and hence the first three equations suffice (see also Lemma 4.20 in Hirschfeld & Thas [6]). This somehow reflects the property of the gradient being identically zero in the last three coordinates.

We now determine all grey planes, showing in particular that in characteristic not equal to 2 there do not exist such planes, and in characteristic 2 only the nucleus plane is a grey plane, except if the underling field is $\mathbb{F}_2$. We are grateful to J. Thas for hinting the use of conic bundles in the below argument (our original proof consisted merely of boring calculations).

So suppose π is a grey plane containing at least one point p contained in a secant L. Obviously there are no flat lines. Then Corollary 5.3 implies that π only contains conical lines. Proposition 5.2(3) now implies that ϕ is bijective from π onto $\mathscr{V}_2(\mathbb{K})$. Hence the map ρ mapping each point $p \in \pi$ to the unique conic C on $\mathscr{V}_2(\mathbb{K})$ with $p \in \langle C \rangle$ is a bijection. Let $L \cap \mathscr{V}_2(\mathbb{K}) = \{x, y\}$. Consider the bundle $\mathscr{B}$ of conics of $\mathbb{P}^2$ defined by intersecting $\mathscr{V}_2(\mathbb{K})$ with all hyperplanes containing the solid $\langle \pi, L \rangle$. By the bijectivity of ρ, each conic D on $\mathscr{V}_2(\mathbb{K})$ containing x generates, together with π and L, a hyperplane H_D. Hence $H_D \cap \mathscr{V}_2(\mathbb{K})$ is a degenerate conic in $\mathbb{P}^2$, which also contains y. So if $y \notin D$, then $H_D \cap \mathscr{V}_2(\mathbb{K})$ contains a conic of $\mathscr{V}_2(\mathbb{K})$ through y. It follows that $\mathscr{B}$ consists solely of degenerate conics. But an arbitrary pair of members of $\mathscr{B}$ not containing the line of $\mathbb{P}^2$ corresponding to the conic of $\mathscr{V}_2(\mathbb{K})$ containing x and y generates a bundle containing exactly three degenerate members. Hence $|\mathbb{K}| = 2$. In this case one can easily check that π is the unique plane in a solid spanned by the complement in $\mathscr{V}_2(\mathbb{F}_2)$ of a conic (a conic corresponding to a line of $\mathbb{P}^2_{\mathbb{F}_2}$). Hence there are seven such planes. Each such plane intersects the nucleus plane in a unique point, namely the unique point of the solid not lying in a plane spanned by any three of the four points of $\mathscr{V}_2(\mathbb{F}_2)$ it contains.

5.3 The Segre variety $\mathscr{S}_{2,2}(\mathbb{K})$

This is the only case with a uniform answer for arbitrary fields. Indeed, we will show that there always exists a grey plane π, and never a grey solid.

First, if we represent $\mathscr{S}_{2,2}(\mathbb{K})$ as the 3×3 rank 1 matrices, up to a scalar, then we can define π as the plane containing all skew-symmetric matrices (with 0 on every diagonal entry). It is easy to see that a skew-symmetric matrix, which always has determinant 0, has rank 1 if and only if it is the 0-matrix. Here every line of π is conical and $\phi(\pi)$ is a Veronesean isomorphic to $\mathscr{V}_2(\mathbb{K})$ embedded in $\mathscr{S}_{2,2}(\mathbb{K})$. It follows that the following system of equations in the unknowns $X_{00},\ldots,X_{22}$ defines $\mathscr{S}_{2,2}(\mathbb{K})$:

$$\begin{cases} X_{11}X_{22} = X_{12}X_{21} \\ X_{00}X_{22} = X_{02}X_{20} \\ X_{00}X_{11} = X_{01}X_{10} \\ X_{10}X_{02} + X_{01}X_{20} = (X_{12}+X_{21})X_{00} \\ X_{01}X_{12} + X_{10}X_{21} = (X_{02}+X_{20})X_{11} \\ X_{02}X_{21} + X_{20}X_{12} = (X_{01}+X_{10})X_{22} \end{cases}$$

In characteristic 2, the plane π is the nucleus plane of the Veronese surface contained in $\mathscr{S}_{2,2}(\mathbb{K})$ obtained by restricting $\mathscr{S}_{2,2}(\mathbb{K})$ to the symmetric (rank 1) 3×3 matrices.

Now suppose there exists a grey solid S. If S contains a short line, then considering any plane in S containing that short line, Corollary 5.3 leads to a contradiction. If S contains only conical lines, then let L_1 and L_2 be two non-intersecting lines of S. Let ξ_i be the symp containing $\phi(L_i)$, $i=1,2$. Clearly $\xi_1\neq\xi_2$ as otherwise every point of $\phi(L_1)$ is collinear to two points of $\phi(L_2)$, yielding flat lines. Hence ξ_1 and ξ_2 intersect nontrivially and since $\phi(L_1)$ is an ovoid of ξ_1, some point $x_1\in\phi(L_1)$ is collinear to a point of the intersection $\xi_1\cap\xi_2$. Then x_1 is collinear to a line of ξ_2, and since $\phi(L_2)$ is an ovoid of ξ_2, x_1 is collinear to some point $x_2\in\phi(L_2)$, a contradiction (as $\langle x_1,x_2\rangle$ is then the image under ϕ of a flat line of S). Hence there is at least one flat line $L\subseteq S$. Let π be the plane spanned by $\phi(L)$. If $\phi(S)\subseteq\pi$, then S only contains flat lines. By Lemma 5.2 (3) the dimension of π is at least 3, a contradiction. Hence there is some point $p\in S$ with $\phi(p)\notin\pi$. Since there is a unique point in π collinear to $\phi(p)$, we can pick two points $x_1,x_2\in L$ such that $\phi(x_i)$ is not collinear to $\phi(p)$, $i=1,2$. Let ξ_i be the symp determined by $\phi(x_i)$ and $\phi(p)$, $i=1,2$. Then $\xi_1\cap\xi_2$ is obviously equal to the line through $\phi(p)$ intersecting π. The argument above shows that for each point q_1 on the line $\langle x_1,p\rangle$, the point $\phi(q_1)$ is collinear in $\mathscr{S}_{2,2}(\mathbb{K})$ to a unique point $\phi(q_2)$, with $q_2\in\langle x_2,p\rangle$. But clearly $\phi(\langle q_1,q_2\rangle)$ is contained in a plane disjoint from π, contradicting the fact that $\langle p,L\rangle$ is a projective plane in S. So we ruled out all possibilities for S to exist.

Nevertheless one can sometimes find other grey planes. For instance, if $\mathbb{P}^2$ admits a linear collineation without fixed points, then one can find such a plane in the span of two disjoint singular planes of $\mathscr{S}_{2,2}(\mathbb{K})$. Such a plane only has flat lines. As an example suppose $\mathbb{K}$ is a field admitting a cubic extension $\mathbb{L}$; let the corresponding cubic polynomial be given by $x^3 - Tx^2 + Qx - N$, with $T, Q, N \in \mathbb{K}$. The plane containing the points

$$\begin{pmatrix} m & k - Qm & -\ell \\ \ell - Tm & Nm & k \\ 0 & 0 & 0 \end{pmatrix}, k, \ell, m \in \mathbb{K},$$

is grey, as one can calculate (in the calculations one might need the fact that also the polynomial $x^3 + Qx^2 + TNx + N^2$ is irreducible; its roots in the cubic extension $\mathbb{L}$ are the opposites of the pairwise products of the roots of the original polynomial). Applying ϕ we obtain that the mapping

$$(k, l, m) \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k^2 + N\ell m - Qkm & \ell^2 - T\ell m + km & (N - QT)m^2 - l\ell + Tkm + Q\ell m \end{pmatrix},$$

$k, \ell, m \in \mathbb{K}$, induces a bijection from $\mathbb{P}^2$ onto a singular plane of $\mathscr{S}_{2,2}(\mathbb{K})$, where each line is mapped to a conic. In fact, these conics form the net of all conics passing through three given conjugate points in the plane over the cubic extension $\mathbb{L}$.

Remark 5.4. *One might wonder how the net of conics in $\mathbb{P}^2$ of the last example can be a projection of the quadric Veronese surface, as required by Proposition 5.2 (3). To see this directly, one considers the above net of conics in $\mathbb{P}^2$, take its image under the Veronese map, and project the Veronese surface from the intersection of the hyperplanes spanned by the image of three linearly independent members of the net. This intersection is a plane consisting merely of black points.*

5.4 The line Grassmannian $\mathscr{G}_{2,6}(\mathbb{K})$

By the previous subsection, there always exists a grey plane. But we can do better for certain fields, in particular, if the field $\mathbb{K}$ admits a quadratic extension (separable or not). We will see that in this case we can find a grey 5-dimensional subspace of $\mathbb{P}^{14}$. But we start with a curious example in the case that $\mathbb{P}^3$ admits a linear collineation without fixed elements.

Example 5.5 (Dimension 3)**.** *Therefore, we consider a point $p \in \mathscr{G}_{2,6}(\mathbb{K})$ and the subspace $U_p \subseteq \mathbb{P}^{14}$ generated by all singular lines on p. Then $\dim U_p = 8$ and $\mathscr{G}_{2,6}(\mathbb{K}) \cap U_p$ is a cone with vertex p and base $\mathscr{S}_{1,3}(\mathbb{K})$ (the latter is indeed the residue at p). Consider any base space W; that is, a 7-dimensional subspace of U_p not containing p. Then $\mathscr{S} := W \cap \mathscr{G}_{2,6}(\mathbb{K}) \cong \mathscr{S}_{1,3}(\mathbb{K})$. Take two singular solids S_1, S_2 of $\mathscr{S}$. The mapping $\theta : S_1 \to S_2 : x_1 \mapsto x_2$ defined by $\langle x_1, x_2 \rangle \subseteq \mathscr{S}$ is a (linear) collineation from S_1 to S_2. Now let φ be a linear collineation of S_2 without fixed elements.*

Then let $\mathscr{S}' \cong \mathscr{S}_{1,3}(\mathbb{K})$ be the Segre variety with as set of maximal singular 1-dimensional subspaces the set of lines $\{\langle p, p^{\theta\varphi}\rangle\}$, and select a 3-dimensional singular subspace S' of $\mathscr{S}'$ distinct from S_1 and S_2. We claim that $S' \cap \mathscr{S} = \emptyset$. Indeed, each point in S' is on a unique line intersecting both S_1 and S_2, and if that line would belong to $\mathscr{S}$ and intersect S_i in x_i, $i = 1,2$, then $x_1^\theta = x_2 = x_1^{\theta\varphi}$, implying x_2 is a fixed point of φ, a contradiction.

Suppose now that two points $p', q' \in S'$ are contained in a common host space of some symplecton ξ of $\mathscr{G}_{2,6}(\mathbb{K})$. Then it is easy to see that $\xi \cap S_i = L_i$ is a line, $i = 1,2$. In the solid $\langle L_1, L_2\rangle$ there is a unique line L' containing p' and intersecting L_i in some point p_i; then $p_1^{\theta\varphi} = p_2$. Likewise, there is a unique line M' containing q' and intersecting L_i in some point q_i; then $q_1^{\theta\varphi} = q_2$. Hence $L_1^{\theta\varphi} = \langle p_1, q_1\rangle^{\theta\varphi} = \langle p_2, q_2\rangle = L_2$. But the latter also coincides with L_1^θ (as L_1, L_2 is contained in a hyperbolic quadric completely contained in $\mathscr{S}$). Hence φ fixes L_2, a contradiction.

We conclude that $\phi(S')$ is the bijective projection of a Veronese variety $\mathscr{V}_3(\mathbb{K})$ into a hyperbolic quadric in some 5-dimensional projective space (that quadric corresponds to the point p; it consists of the images under ϕ of the symplecta passing through p). This is a rather remarkable situation. But that inclusion can abstractly be seen directly by sending a point x of S_2 to the image of the line $\langle x, x^\varphi\rangle$ under the Klein correspondence. We deduce that every plane of the Klein quadric contains a unique conic of that image.

If, in the above, φ has no fixed points, but does admit fixed lines, then we can still find S' and it is still a grey solid. But $\phi(S')$ is the union of elliptic quadratic surfaces (in 3-dimensional subspaces). An extreme situation is that the fixed lines of φ form a spread of S', in which case $\phi(S')$ coincides with one such elliptic quadric. It is clear that this situation arises if and only if $\mathbb{K}$ admits a quadratic extension. But in this case we can extend S' to a 5-dimensional grey subspace, as evidenced by the next example.

Example 5.6 (Dimension 5)**.** *Let $x^2 - Tx + N$ be an irreducible quadratic polynomial over $\mathbb{K}$ (with coefficients in $\mathbb{K}$), defining the quadratic extension $\mathbb{L}$ of $\mathbb{K}$. Let p_1, p_2, p_3 be three points on a line of the quadrangle Γ' of order $(2,2)$, and suppose $\{p_1, p_2, p_3\}$ is a spread line. Let $\{p_i, q_i, r_i\}$ and $\{p_i, s_i, t_i\}$ be the other two lines passing through p_i, $i = 1,2,3$. We may choose this notation such that $\{q_1, q_2, q_3\}$ and $\{r_1, r_2, r_3\}$ are the other two spread lines in Γ', and the other six lines of Γ' are $\{s_1, q_2, t_3\}$, $\{s_1, t_2, r_3\}$ and cyclic permutations of the indices. (For an explicit realization inside the model given in Subsection 2.2, see Example 5.8.) Define the following subspace:*

$$\begin{cases} 0 = X_{p_i}, & i = 1,2,3, \\ 0 = X_{r_i} + X_{q_i}, & i = 1,2,3, \\ 0 = X_{t_i} + NX_{s_i} + TX_{q_i}, & i = 1,2,3. \end{cases}$$

Since we have nine linearly independent equations, this defines a 5-dimensional projective subspace U. In order to apply ϕ we write a generic point of U with coordinates $X_{p_i} = 0$, $i = 1,2,3$, the coordinates X_{q_i} and X_{s_i} are considered as running parameters, $i = 1,2,3$, and the coordinates

X_{r_i} and X_{t_i} linearly depend on these parameters as given above, namely $X_{r_i} = -X_{q_i}$ and $X_{t_i} = -NX_{s_i} - TX_{q_i}$. We denote the coordinate vector of such a generic point by $v_{X_{q_1},X_{q_2},X_{q_3};X_{s_1},X_{s_2},X_{s_3}}$, or simply v in the sequel. Calculating $\phi(v)$ we obtain a vector with p_i-coordinate equal to

$$X_{q_i}^2 + TX_{q_1}X_{s_i} + NX_{s_i}^2, \quad i = 1,2,3.$$

Clearly, such coordinate is 0 if and only if $X_{q_i} = X_{s_i} = 0$, showing that no point of U is white. Calculating $C(v)$, we simply obtain 0, showing that U is a grey space.

One now sees that the short lines in U form a regular spread; they are the point set of a projective plane $\mathbb{P}^2_{\mathbb{L}}$ the lines of which are the 3-dimensional subspaces of U generated by two distinct short lines. This is the spread representation of $\mathbb{P}^2_{\mathbb{L}}$. We now claim that ϕ transforms this representation into the corresponding Hermitian Veronesean of $\mathbb{P}^2_{\mathbb{L}}$. Indeed, let δ be one of the roots in $\mathbb{L}$ of the polynomial $x^2 - Tx + N$, and let $x \mapsto \overline{x}$ be the corresponding Galois involution of $\mathbb{L}$. Note that $\overline{a + b\delta} = a + Tb - b\delta$, $a, b \in \mathbb{K}$. Denoting the p-coordinate of $\phi(v)$ by Y_p, $p \in \mathscr{P}$, a straightforward calculation reveals:

$$\begin{aligned}
(\overline{X_{q_2} + X_{s_2}\delta})(X_{q_3} + X_{s_3}\delta) &= Y_{r_1} + Y_{t_1}\delta, \\
(\overline{X_{q_1} + X_{s_1}\delta})(X_{q_1} + X_{s_1}\delta) &= Y_{p_1}, \\
Y_{s_1} &= NY_{t_1}, \\
Y_{q_1} &= Y_{r_1} - TY_{t_1},
\end{aligned}$$

and the same equation for cyclic permutations of the indices, which shows that $\phi(U)$ is projectively equivalent to the point set

$$\{(\overline{X}_1X_1, \overline{X}_2X_2, \overline{X}_3X_3, \overline{X}_2X_3, \overline{X}_3X_1, \overline{X}_1X_2) \mid X_1, X_2, X_3 \in \mathbb{L}\},$$

where the first three coordinates are considered to belong to $\mathbb{K}$, and the last three to $\mathbb{K} \times \mathbb{K}$ via the obvious identification $a + b\delta \to (a, b)$. This shows our claim.

5.5 The Cartan variety $\mathscr{E}_6(\mathbb{K})$

By the previous subsections, there always exists a grey plane, and if $\mathbb{K}$ admits a quadratic extension, there is always a grey 5-space. We can slightly generalise the latter, and we can also give an example of a grey 11-dimensional space if $\mathbb{K}$ is the centre of a quaternion division algebra, or $\mathrm{char}\,\mathbb{K} = 2$ and $\mathbb{K}$ admits a degree 4 inseparable field extension. Also, we will show that there always exists a grey 4-space, whatever the field.

Example 5.7 (Dimensions 4 and 5). *Let $\Gamma' = (\mathscr{P}', \mathscr{L}')$ be a subquadrangle of $\Gamma = (\mathscr{P}, \mathscr{L})$ of order $(2,2)$. Let W be the 12-dimensional vector subspace of $\mathbb{K}^{27}$ generated by the e_p not belonging to Γ'. The points outside $\mathscr{P}'$ form a double six $\{p_1, \ldots, p_6, q_1, \ldots, q_6\}$, where $\{p_1, \ldots, p_6\}$ and $\{q_1, \ldots, q_6\}$*

are cocliques and p_i *is collinear to* q_j *if and only if* $i \neq j$, *for all* $i,j \in \{1,\ldots,6\}$. *Let* $w \in W$ *have coordinates* $(x_p)_{p\in\mathscr{P}}$ *(with* $x_p = 0$ *if* $p \in \mathscr{P}'$*). Then* $\phi(w) = 0$ *if and only if* $x_{p_i}x_{q_j} = \pm x_{p_j}x_{q_i}$, *for all* $i,j \in \{1,\ldots,6\}$ *with* $i \neq j$, *and where each sign depends on the position of the spread* $\mathscr{S}$. *However, changing the sign of the coordinates related to the points of one single six collinear to the points of one single three with respect to the grid in* Γ' *defined by intersecting* $\mathscr{L}'$ *with* $\mathscr{S}$, *we see that all signs become positive. This means that, denoting the projective subspace defined by* W *also by* W, *the intersection* $W \cap \mathscr{E}_6(\mathbb{K})$ *is a Segre variety* $\mathscr{S}_{1,5}(\mathbb{K})$. *As before, given a fixed point free linear collineation of* $\mathbb{P}^5$, *one can select a* 5*-dimensional subspace* U *of* W *disjoint from* $\mathscr{E}_6(\mathbb{K})$, *which is automatically a grey subspace. If* $\mathbb{K}$ *admits a separable quadratic extension, then we may choose* W *such that it contains a regular spread of short lines, and* $\phi(U)$ *is a Hermitian Veronesean variety on* $\mathscr{E}_6(\mathbb{K})$, *as in Example 5.6. However, note that in the inseparable case, the corresponding spread is elementwise fixed only by the identity. We hence conjecture that also in the separable case, the current* 5*-space is not projectively equivalent to the one of Example 5.6 (meaning the current subspace* U *is not contained in the space spanned by any subvariety of* $\mathscr{E}_6(\mathbb{K})$ *isomorphic to* $\mathscr{G}_{2,6}(\mathbb{K})$*).*

Now let $\mathbb{K}$ *be arbitrary and let* M *be a* 6×6 *upper triangular matrix with entries in* $\mathbb{K}$, *with* 1*s on the diagonal and such that* $M - I$ *(with* I *the identity matrix) has rank 5. Then the corresponding linear collineation* θ *of* $\mathbb{P}^5$ *has a unique fixed point. Let* U' *be a* 5*-space in* W *constructed as above from* θ*; then* $U' \cap \mathscr{E}_6(\mathbb{K})$ *is a point* p *corresponding to the unique fixed point of* θ. *Hence any hyperplane of* U' *not containing* p *is a grey* 4*-space.*

Example 5.8 (Dimension 11)**.** *Let* $x_1^2 - Tx_1x_2 + Nx_2^2 - \ell x_3^2 + \ell T x_3x_4 - \ell N x_4^2$ *be the norm form of a quaternion division algebra* $\mathbb{H}$ *over* $\mathbb{K}$, *with* $\ell, T, N \in \mathbb{K}$, *or with* $T = 0$ *and* $\operatorname{char}\mathbb{K} = 2$, *and then we assume it is just an inseparable field extension of degree 4.*

It is convenient to work with the explicit description of Γ *and* $\mathscr{S}$ *given in Subsection 2.2. The current example will extend Example 5.6 with*

$$\begin{array}{rcl} (p_1,p_2,p_3) & = & (25,14,36), \\ (q_1,q_2,q_3) & = & (34,26,15), \\ (r_1,r_2,r_3) & = & (16,35,24), \\ (s_1,s_2,s_3) & = & (46,56,45), \\ (t_1,t_2,t_3) & = & (13,23,12). \end{array}$$

The subspace U *we want to define can be described by a system of fifteen equations, nine of which are given in Example 5.6 (using the above identification). The other six read (denoting the coordinate corresponding to the point* i *by* X_i *and the one corresponding to* i' *by* X_i', $i = 1,\ldots,6$*):*

$$\left\{ \begin{array}{rcll} 0 & = & X_i' - \ell X_j, & (i,j) = (2,5),(1,4),(3,6), \\ 0 & = & X_i' - \ell N X_j - \ell T X_i, & (i,j) = (5,2),(4,1),(6,3). \end{array} \right.$$

In completely the same way as in Example 5.6, one checks that U *is a grey* 11*-dimensional subspace, and that its image under* ϕ *is the corresponding quaternion Veronesean of* $\mathbb{P}^2_{\mathbb{H}}$.

As a corollary of the last example, we obtain that every quaternion Veronese variety of the plane $\mathbb{P}^2_{\mathbb{H}}$, with $\mathbb{H}$ a quaternion division algebra over the field $\mathbb{K}$, or a degree 4 inseparable field extension in characteristic 2, is a projection of the Veronese variety $\mathscr{V}_{11}(\mathbb{K})$.

Over the real numbers, we can choose $-\ell = N = 1$ and $T = 0$. It follows that in this case $\mathscr{E}_6(\mathbb{R})$ has a particularly nice description as the intersection of fifteen quadrics, whose forms can be given as follows. Choose a fixed spread line L of $\mathscr{S}$. Three of the forms are Q_p, with $p \in L$. The other twelve forms are all of shape $Q_a + Q_b$, where $\{a, b, p\} \in \mathscr{L}$ is a line of Γ with $p \in L$.

5.6 Conclusion

We conclude by noting that we gave a full answer for the minimality of the number of quadrics describing a Severi variety in the cases of $\mathscr{V}_2(\mathbb{K})$ and $\mathscr{S}_{2,2}(\mathbb{K})$. For the two other case, we were only able to give some examples (yielding bounds) over fields with certain properties. Since we think that some of the dimensions we obtained are pretty high, we conjecture that

(C1) If $\mathbb{K}$ admits a quadratic extension, then the maximum projective dimension of a grey subspace for $\mathscr{G}_{2,6}(\mathbb{K})$ is 5.

(C2) If $\mathbb{K}$ admits a quaternion division algebra, or a degree 4 inseparable field extension in characteristic 2, then the maximum projective dimension of a grey subspace for $\mathscr{E}_6(\mathbb{K})$ is 11.

Remark 5.9. *We note that the minimum number of quadrics found in the present paper for a certain variety, is exactly equal to the dimension of the vector space related to the variety of the previous case, ranking the cases in increasing dimension, and adding a trivial variety in the beginning consisting of three spanning points in a projective plane (three* 1*-spaces generating a* 3*-dimensional vector space; this is the line-residue of the long root geometry of type* D_4 *which is sometimes added as zeroth column in the fourth row of the Freudenthal-Tits magic square; the Severi varieties are the line-residues of the other varieties of the fourth column). We do not think this is a coincidence; further research should give evidence for this.*

Finally, one could wonder which quadrics one can obtain by linearly combining the 27 basic quadrics in the case $\mathscr{E}_6(\mathbb{K})$*, or 9 and 15 basic quadrics in the cases* $\mathscr{S}_{2,2}(\mathbb{K})$ *and* $\mathscr{G}_{2,6}(\mathbb{K})$*, respectively. It is proved in a yet unpublished manuscript of A. De Schepper and M. Victoor that there are exactly three possibilities (corresponding to the "duals" of the white, grey and black points): For* $\mathscr{E}_6(\mathbb{K})$*, these are non-degenerate parabolic quadrics (hence of maximal Witt index) and degenerate quadrics with an* 8*- or* 16*-dimensional radical (projective dimension) and hyperbolic base. Similarly,* *for* $\mathscr{G}_{2,6}(\mathbb{K})$*, we have non-degenerate parabolic quadrics and degenerate quadrics with a* 4*- or* 8*-dimensional radical and hyperbolic base; for* $\mathscr{S}_{2,2}(\mathbb{K})$*, we have non-degenerate parabolic quadrics and degenerate quadrics with a* 2*- or* 4*-dimensional radical and hyperbolic base.*

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