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Positive solutions of nabla fractional boundary value problem

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ABSTRACT

In this article, we consider the following two-point discrete fractional boundary value problem with constant coefficient associated with Dirichlet boundary conditions.

$$\begin{cases} -\left(\nabla^{\nu}_{\rho(a)}u\right)(t) + \lambda u(t) = f(t, u(t)), & t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = u(b) = 0, \end{cases}$$

where $1 < \nu < 2$, $a, b \in \mathbb{R}$ with $b-a \in \mathbb{N}_3$, $\mathbb{N}^{b}_{a+2} = \{a+2, a+3, \dots, b\}$, $|\lambda| < 1$, $\nabla^{\nu}_{\rho(a)}u$ denotes the ν^{th}-order Riemann–Liouville nabla difference of u based at $\rho(a) = a-1$, and $f : \mathbb{N}^{b}_{a+2} \times \mathbb{R} \to \mathbb{R}^{+}$.
We make use of Guo–Krasnosels'kiĭ and Leggett–Williams fixed-point theorems on suitable cones and under appropriate conditions on the non-linear part of the difference equation. We establish sufficient requirements for at least one, at least two, and at least three positive solutions of the considered boundary value problem. We also provide an example to demonstrate the applicability of established results.

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RESUMEN

En este artículo consideramos el siguiente problema de valor en la frontera de dos puntos discreto fraccional con coeficientes constantes asociado a condiciones de frontera de tipo Dirichlet

$$\begin{cases} -\big(\nabla^{\nu}_{\rho(a)}u\big)(t) + \lambda u(t) = f(t, u(t)), \quad t \in \mathbb{N}^b_{a+2}, \\ u(a) = u(b) = 0, \end{cases}$$

donde $1 < \nu < 2$, $a, b \in \mathbb{R}$ con $b - a \in \mathbb{N}_3$, $\mathbb{N}^b_{a+2} = \{a + 2, a + 3, \dots, b\}$, $|\lambda| < 1$, $\nabla^{\nu}_{\rho(a)}u$ denota la nabla diferencia de Riemann–Liouville de u de orden ν basada en $\rho(a) = a - 1$, y $f : \mathbb{N}^b_{a+2} \times \mathbb{R} \to \mathbb{R}^+$.
Usamos los teoremas de punto fijo de Guo–Krasnosels'kiĭ y Leggett–Williams en conos adecuados y bajo condiciones apropiadas en la parte nolineal de la ecuación en diferencias. Establecemos requerimientos suficientes para al menos una, al menos dos, y al menos tres soluciones positivas del problema de valor en la frontera considerado. También entregamos un ejemplo para mostrar la aplicabilidad de los resultados.

Keywords and Phrases: Nabla fractional difference, boundary value problem, Dirichlet boundary conditions, positive solution, existence, fixed-point.

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1 Introduction

Nabla fractional calculus is a branch of mathematics that deals with arbitrary order differences and sums in the backward sense. The theory of nabla fractional calculus is still in its early stages, with the most important contributions appearing in the last two decades. Gray & Zhang [15] and Miller & Ross in [34] first introduced the concept of nabla fractional difference and sum. Atici & Eloe [2] developed the Riemann–Liouville type nabla fractional difference operator. They also studied the nabla fractional initial value problem, and established the exponential law, product rule, and nabla Laplace transform in this line. Several mathematicians [2, 3, 4, 5, 6, 7, 8, 16, 17, 21, 22] have contributed to the development of the theory of discrete fractional calculus in line with the theory of continuous fractional calculus. For historical references on continuous fractional calculus, see [28, 31, 32]. As a result of their works, today discrete fractional calculus has turned into a fruitful field of research in science and engineering. We refer here to recent monographs [9, 12, 29] and the references therein, which are important resources pertaining to this field of work.

The study of boundary value problems (BVPs) has a long past and can be followed back to the work of Euler and Taylor on vibrating strings. On the discrete fractional side, there is a sudden growth in interest for the development of nabla fractional BVPs. Many authors have studied nabla fractional BVPs recently. To name a few, Goar [11] and Ikram [18] worked with self-adjoint Caputo nabla BVPs. Gholami *et al.* [10] obtained the Green's function for a non-homogeneous Riemann–Liouville nabla BVP with Dirichlet boundary conditions. Jonnalagadda [19, 20, 23] analysed some qualitative properties of two-point non-linear Riemann–Liouville nabla fractional BVPs associated with a variety of boundary conditions.

As pointed out earlier, many authors have studied the discrete fractional two-point boundary value problem like in [4, 19] and recently authors in [23] have worked with general nabla fractional difference equation with constant coefficients coupled with Dirichlet conditions, which resulted in for the first time Green's function in terms of discrete Mittag–Leffler function along with a few properties of the same. Compared to discrete Taylor monomial, discrete Mittag–Leffler function is an infinite series because of which it poses a challenge while proving positivity of Green's function. In the article, [23] the authors have overcome this challenge of proving positivity of Green's function. In the present article, we use the positivity of Green's function and prove an important lemma which helps us deal with conical mappings by proving that a ratio of infinite series is increasing or decreasing with respect to the ratio of its coefficient. To the best of our knowledge, no work has been done with Leggett–Williams fixed-point theorem in the nabla setting.

We consider the following boundary value problem

$$\begin{cases} -\big(\nabla^{\nu}_{\rho(a)}u\big)(t) + \lambda u(t) = f(t, u(t)), & t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = u(b) = 0, \end{cases} \tag{1.1}$$

where $1 < \nu < 2$, $a, b \in \mathbb{R}$ with $b - a \in \mathbb{N}_3$, $\mathbb{N}_{a+2}^{b} = \{a+2, a+3, \dots, b\}$, $|\lambda| < 1$, $\nabla_{\rho(a)}^{\nu} u$ denotes the ν^{th}-order Riemann–Liouville nabla difference of u based at $\rho(a) = a - 1$, and $f : \mathbb{N}_{a+2}^{b} \times \mathbb{R} \to \mathbb{R}^{+}$.

The present paper is organized as follows: Section 2 contains preliminaries on nabla fractional calculus. In Section 3, we establish some properties of the Green's function associated with the nabla fractional boundary value problem (1.1) and construct the existence of at least one, at least two and at least three positive solutions with the help of Guo–Krasnosel'skiĭ and Leggett–Williams fixed-point theorems on suitable cones and under appropriate conditions on the non-linear part of the difference equation. Finally, we conclude this article with an example to demonstrate the applicability of our results.

2 Preliminaries

Denote the set of all real numbers and positive integers by $\mathbb{R}$ and $\mathbb{Z}^{+}$, respectively. We use the following notations, definitions and known results of nabla fractional calculus [12]. Assume empty sums and products are 0 and 1, respectively.

Definition 2.1. *For $a \in \mathbb{R}$, the sets $\mathbb{N}_a$ and $\mathbb{N}_a^b$, where $b - a \in \mathbb{Z}^{+}$, are defined by*

$$\mathbb{N}_a = \{a, a+1, a+2, \dots\}, \quad \mathbb{N}_a^b = \{a, a+1, a+2, \dots, b\}.$$

Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by $\big(\nabla u\big)(t) = u(t) - u(t-1)$, for $t \in \mathbb{N}_{a+1}$, and the N^{th}-order nabla difference of u is defined recursively by $\big(\nabla^N u\big)(t) = \Big(\nabla\big(\nabla^{N-1} u\big)\Big)(t)$, for $t \in \mathbb{N}_{a+N}$.

Definition 2.2 ([12])**.** *For $t \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, the generalized rising function (many authors employ the Pochhammer symbol [33] to denote the same) is defined by*

$$t^{\overline{r}} = \frac{\Gamma(t+r)}{\Gamma(t)}.$$

Here $\Gamma(\cdot)$ denotes the Euler gamma function. Also, if $t \in \{\dots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\dots, -2, -1, 0\}$, then we use the convention that $t^{\overline{r}} = 0$.

Definition 2.3 ([12])**.** *Let t, $a \in \mathbb{R}$ and $\mu \in \mathbb{R} \setminus \{\dots, -2, -1\}$. The μ^{th}-order nabla fractional Taylor monomial is given by*

$$H_{\mu}(t, a) = \frac{(t-a)^{\overline{\mu}}}{\Gamma(\mu + 1)},$$

provided the right-hand side exists.

We observe the following properties of the nabla fractional Taylor monomials.

Lemma 2.4 ([18, 19]). *Let $\mu > -1$ and $s \in \mathbb{N}_a$. Then the following hold:*

(1) *If $t \in \mathbb{N}_{\rho(s)}$, then $H_\mu(t, \rho(s)) \geq 0$ and if $t \in \mathbb{N}_s$, then $H_\mu(t, \rho(s)) > 0$.*

(2) *If $t \in \mathbb{N}_s$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is an increasing function of s.*

(3) *If $t \in \mathbb{N}_{s+1}$ and $-1 < \mu < 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of t.*

(4) *If $t \in \mathbb{N}_{\rho(s)}$ and $\mu > 0$, then $H_\mu(t, \rho(s))$ is a decreasing function of s.*

(5) *If $t \in \mathbb{N}_{\rho(s)}$ and $\mu \geq 0$, then $H_\mu(t, \rho(s))$ is a non-decreasing function of t.*

(6) *If $t \in \mathbb{N}_s$ and $\mu > 0$, then $H_\mu(t, \rho(s))$ is an increasing function of t.*

(7) *If $0 < v \leq \mu$, then $H_v(t, a) \leq H_\mu(t, a)$, for each fixed $t \in \mathbb{N}_a$.*

Definition 2.5 ([12]). *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u is given by*

$$\left(\nabla_a^{-\nu} u\right)(t) = \sum_{s=a+1}^{t} H_{\nu-1}(t, \rho(s)) u(s), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.6 ([12]). *Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N - 1 < \nu \leq N$. The ν^{th}-order Riemann–Liouville nabla difference of u is given by*

$$\left(\nabla_a^{\nu} u\right)(t) = \Big(\nabla^N \left(\nabla_a^{-(N-\nu)} u\right)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Lemma 2.7 ([13]). *Let a, b be two real numbers such that $0 < a \leq b$ and $1 < \alpha < 2$. Then $\frac{(a-s)^{\overline{\alpha-1}}}{(b-s)^{\overline{\alpha-1}}}$ is a decreasing function of s for $s \in \mathbb{N}_0^{a-1}$.*

Lemma 2.8 ([12]). *Assume the successive fractional nabla Taylor monomials are well defined.*

(1) *Let $\nu > 0$ and $\alpha \in \mathbb{R}$. Then, $\nabla_a^{-\nu} H_\alpha(t, a) = H_{\alpha+\nu}(t, a)$, for $t \in \mathbb{N}_a$.*

(2) *Let ν, $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}_1$ such that $n - 1 < \nu \leq n$. Then, $\nabla_a^{\nu} H_\alpha(t, a) = H_{\alpha-\nu}(t, a)$, for $t \in \mathbb{N}_{a+n}$.*

Finally, we present the definition of the nabla Mittag–Leffler function which is the nabla analogue of classical Mittag-Leffler function [14, 30].

Definition 2.9 ([12]). *Let α, β, $\lambda \in \mathbb{R}$ such that $\alpha > 0$ and $|\lambda| < 1$. The nabla Mittag–Leffler function is defined by*

$$E_{\lambda,\alpha,\beta}(t, a) = \sum_{n=0}^{\infty} \lambda^n H_{\alpha n+\beta}(t, a), \quad \text{for} \quad t \in \mathbb{N}_a.$$

Theorem 2.10 ([23])**.** *Assume $1 < \nu < 2$, $-1 < \lambda < 1$ and $h : \mathbb{N}_{a+2} \to \mathbb{R}$. The unique solution of the nabla fractional boundary value problem*

$$\begin{cases} -\big(\nabla^{\nu}_{\rho(a)}u\big)(t) + \lambda u(t) = h(t), \quad t \in \mathbb{N}^{b}_{a+2}, \\ u(a) = u(b) = 0, \end{cases} \tag{2.1}$$

is given by

$$u(t) = \sum_{s=a+2}^{b} G(t,s)h(s), \quad t \in \mathbb{N}^{b}_{a}, \tag{2.2}$$

where

$$G(t,s) = \begin{cases} G_1(t,s) = \dfrac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)), & s \in \mathbb{N}^{b}_{t+1}, \\ G_2(t,s) = \dfrac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) - E_{\lambda,\nu,\nu-1}(t,\rho(s)), & s \in \mathbb{N}^{t}_{a+2}. \end{cases} \tag{2.3}$$

Now, we state some positive properties of the Green's function (2.3).

Lemma 2.11 ([23])**.** *Assume $1 < \nu < 2$ and $t \in \mathbb{N}_{a+2}$. For each $0 \leq \lambda < 1$, denote by*

$$g(\lambda) = \sum_{n=0}^{\infty} \lambda^n H_{\nu n + \nu - 3}(t, \rho(a)) \tag{2.4}$$

$$= \sum_{n=0}^{\infty} \lambda^n \frac{\Gamma(t - a + \nu n + \nu - 2)}{\Gamma(t - a + 1)\Gamma(\nu n + \nu - 2)}. \tag{2.5}$$

Then there exists a unique $\bar{\lambda} = \bar{\lambda}(t) \in (0,1)$ such that

$$g(\bar{\lambda}) = 0. \tag{2.6}$$

Take $\lambda^* = \min_{t \in \mathbb{N}^{b}_{a+2}} \bar{\lambda}(t)$. Then, $0 < \lambda^* < 1$.

We observe the following properties of the nabla Mittag-Leffler function

Lemma 2.12 ([23])**.** *Assume $1 < \nu < 2$ and $0 \leq \lambda < 1$. Then,*

(1) $0 < H_{\nu-1}(t,\rho(a)) \leq E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *for* $t \in \mathbb{N}_a$;

(2) $E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *is an increasing function with respect to t for* $t \in \mathbb{N}_a$;

(3) $0 < H_{\nu-2}(t,\rho(a)) \leq \nabla E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *for* $t \in \mathbb{N}_{a+1}$;

(4) $\nabla E_{\lambda,\nu,\nu-1}(t,\rho(a))$ *is a decreasing function with respect to t for $t \in \mathbb{N}_{a+1}$ and* $\lambda \in (0,\lambda^*]$;

(5) $E_{\lambda,\nu,\nu-1}(t,\rho(s)) \leq E_{\lambda,\nu,\nu-1}(t,a)$ *for $t \in \mathbb{N}_s$ and* $s \in \mathbb{N}_{a+1}$;

(6) $\nabla E_{\lambda,\nu,\nu-1}(t,\rho(s)) \geq \nabla E_{\lambda,\nu,\nu-1}(t,a)$ *for $t \in \mathbb{N}_s$, $s \in \mathbb{N}_{a+1}$ and* $\lambda \in (0,\lambda^*]$.

Lemma 2.13 ([27]). *Let* (a_n) *and* (b_n) $(n = 0, 1, 2, \dots)$ *be real numbers and let the power series* $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *be convergent for* $|x| < r$. *If* $b_n > 0,\ n = 0, 1, 2, \dots$ *and the sequence* $\left(\frac{a_n}{b_n}\right)_{n\geq 0}$ *is (strictly) increasing (decreasing), then the function* $\frac{A(x)}{B(x)}$ *is also (strictly) increasing (decreasing) on* $[0, r)$.

Theorem 2.14 ([23]). *Assume* $1 < \nu < 2$ *and* $0 \leq \lambda < 1$ *such that* $\lambda \in (0, \lambda^*]$. *The Green's function* $G(t, s)$ *defined in* (2.3) *satisfies* $G(t, s) \geq 0$ *for each* $(t, s) \in \mathbb{N}_a^b \times \mathbb{N}_{a+2}^b$. *In particular,* $G(a, s) = G(b, s) = 0$ *and* $G(t, s) > 0$ *for each* $(t, s) \in \mathbb{N}_{a+1}^{b-1} \times \mathbb{N}_{a+2}^b$.

3 Multiple Positive Solutions

In this section, we establish sufficient conditions on existence of at least one, at least two and at least three positive solutions of (1.1) using Guo–Krasnosel'skiĭ and Leggett–Williams fixed-point theorems on conical shells.

Definition 3.1. *Let* $\mathcal{B}$ *be a Banach space over* $\mathbb{R}$. *A closed nonempty convex set* $K \subset \mathcal{B}$ *is called a cone provided,*

(i) $\lambda_1 u \in K$, *for all* $u \in K$ *and* $\lambda_1 \geq 0$.

(ii) $u \in K$ *and* $-u \in K$ *implies* $u = 0$.

Definition 3.2. *A functional* α_2 *is said to be a non-negative continuous concave functional on a cone* K *of a real Banach space* β, *if* $\alpha_2 : K \to [0, \infty)$ *is continuous and*

$$\alpha_2(tx + (1-t)y) \geq t\alpha_2(x) + (1-t)\alpha_2(y),$$

for all $x, y \in K$ *and* $t \in [0, 1]$.

Definition 3.3. *An operator is called completely continuous, if it is continuous and maps bounded sets into precompact sets.*

Theorem 3.4 (Guo–Krasnosel'skiĭ fixed-point theorem, [24]). *Let* $\mathcal{B}$ *be a Banach space and* $\mathcal{K} \subseteq \mathcal{B}$ *be a cone. Assume that* Ω_1 *and* Ω_2 *are open sets contained in* $\mathcal{B}$ *such that* $0 \in \Omega_1$ *and* $\overline{\Omega}_1 \subseteq \Omega_2$. *Assume further that* $T : \mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow \mathcal{K}$ *is a completely continuous operator. If, either*

(1) $\|Tu\| \leq \|u\|$ *for* $u \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$ *for* $u \in \mathcal{K} \cap \partial\Omega_2$; *or*

(2) $\|Tu\| \geq \|u\|$ *for* $u \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in \mathcal{K} \cap \partial\Omega_2$;

holds, then T *has at least one fixed-point in* $\mathcal{K} \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

The following results are useful for the main results of this section.

Lemma 3.5. *Let a, b be two real numbers such that $0 < a \leq b$ and $1 < \nu < 2$. Then $\dfrac{E_{\lambda,\nu,\nu-1}(a,\rho(s))}{E_{\lambda,\nu,\nu-1}(b,\rho(s))}$ is a decreasing function of s for $s \in \mathbb{N}_0^{a-1}$.*

Proof. For each $s \in \mathbb{N}_0^{a-1}$, denote by

$$a_n = H_{\nu n+\nu-1}(a,\rho(s)) \quad \text{and} \quad b_n = H_{\nu n+\nu-1}(b,\rho(s)), \quad n \in \mathbb{N}_0.$$

Clearly, a_n and b_n for $n \in \mathbb{N}_0$ are real numbers. Further, denote by

$$A(\lambda) = E_{\lambda,\nu,\nu-1}(a,\rho(s)) \quad \text{and} \quad B(\lambda) = E_{\lambda,\nu,\nu-1}(b,\rho(s)).$$

We know that the power series $A(\lambda)$ and $B(\lambda)$ are convergent for $|\lambda| < 1$. Also, $b_n > 0,\ n \in \mathbb{N}_0$ and the sequence

$$\left(\frac{a_n}{b_n}\right)_{n\geq 0} = \left(\frac{H_{\nu n+\nu-1}(a,\rho(s))}{H_{\nu n+\nu-1}(b,\rho(s))}\right)_{n\geq 0}$$

is strictly decreasing, by Lemma 2.7. Then, by Lemma 2.13, the function

$$\frac{A(\lambda)}{B(\lambda)} = \frac{E_{\lambda,\nu,\nu-1}(a,\rho(s))}{E_{\lambda,\nu,\nu-1}(b,\rho(s))}$$

is also strictly decreasing on $[0,1)$ for each $s \in \mathbb{N}_0^{a-1}$. The proof is complete. □

Theorem 3.6. *There exists a number $\gamma \in (0,1)$, such that*

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \geq \gamma \max_{t\in\mathbb{N}_a^b} G(t,s) = \gamma G(s-1,s), \tag{3.1}$$

for $\lambda \in (0,\lambda^*]$ and $c, d \in \mathbb{N}_{a+1}^{b-1}$ such that $c = a + \left\lceil \dfrac{b-a+1}{4} \right\rceil$ and $d = a + 3\left\lfloor \dfrac{b-a+1}{4} \right\rfloor$.

Proof. It follows from the proof of Theorem 2.14 in [23] that for each $\lambda \in (0,\lambda^*]$, $G(t,s)$ is an increasing function of t for $\in \mathbb{N}_a^{s-1}$ and is a decreasing function of t for $\in \mathbb{N}_s^b$. Thus, we have

$$\max_{t\in\mathbb{N}_a^b} G(t,s) = G(s-1,s) \text{ for } s \in \mathbb{N}_{a+2}^b.$$

Consider

$$\frac{G(t,s)}{G(s-1,s)} = \begin{cases} \dfrac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)}, & s \in \mathbb{N}_{t+1}^b, \\ \dfrac{E_{\lambda,\nu,\nu-1}(t,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)} - \dfrac{E_{\lambda,\nu,\nu-1}(t,\rho(s))E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,\rho(s))E_{\lambda,\nu,\nu-1}(s-1,a)}, & s \in \mathbb{N}_{a+2}^t. \end{cases}$$

Now, for $s > t$ and $c \leq t \leq d$, $G_1(t,s)$ is an increasing function with respect to t. Then, we have

$$\min_{t\in\mathbb{N}_c^d} G_1(t,s) = G_1(c,s) = \frac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)), \quad s \in \mathbb{N}_{t+1}^b.$$

For $t > s$ and $c \leq t \leq d$, $G_2(t,s)$ is a decreasing function with respect to t. Then, we have

$$\min_{t\in\mathbb{N}_c^d} G_2(t,s) = G_2(d,s) = \frac{E_{\lambda,\nu,\nu-1}(d,a)}{E_{\lambda,\nu,\nu-1}(b,a)} E_{\lambda,\nu,\nu-1}(b,\rho(s)) - E_{\lambda,\nu,\nu-1}(d,\rho(s)), \quad s \in \mathbb{N}_{a+2}^t.$$

Thus,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) = \begin{cases} G_1(c,s), & \text{for } s\in\mathbb{N}_d^b, \\ \min\{G_2(d,s), G_1(c,s)\}, & \text{for } s\in\mathbb{N}_{c+1}^{d-1}, \\ G_2(d,s), & \text{for } s\in\mathbb{N}_{a+2}^{c}, \end{cases}$$
$$= \begin{cases} G_2(d,s), & \text{for } s\in\mathbb{N}_{a+2}^{r}, \\ G_1(c,s), & \text{for } s\in\mathbb{N}_r^b, \end{cases}$$

where $c<r<d$. Consider

$$\frac{\min_{t\in\mathbb{N}_c^d} G(t,s)}{G(s-1,s)} = \begin{cases} \dfrac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)}, & s\in\mathbb{N}_r^b, \\ \dfrac{E_{\lambda,\nu,\nu-1}(d,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)} - \dfrac{E_{\lambda,\nu,\nu-1}(d,\rho(s))E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,\rho(s))E_{\lambda,\nu,\nu-1}(s-1,a)}, & s\in\mathbb{N}_{a+2}^r. \end{cases}$$

Hence,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \geq \gamma(s)\max_{t\in\mathbb{N}_a^b} G(t,s), \tag{3.2}$$

where

$$\gamma(s) = \min\left[\frac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)}, \frac{E_{\lambda,\nu,\nu-1}(d,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)} - \frac{E_{\lambda,\nu,\nu-1}(d,\rho(s))E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,\rho(s))E_{\lambda,\nu,\nu-1}(s-1,a)}\right].$$

For $s\in\mathbb{N}_r^b$, denote by

$$\gamma_1(s) = \frac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)} \geq \frac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(b-1,a)}.$$

Similarly, for $s\in\mathbb{N}_{a+2}^r$, we take

$$\gamma_2(s) = \frac{E_{\lambda,\nu,\nu-1}(d,a)}{E_{\lambda,\nu,\nu-1}(s-1,a)} - \frac{E_{\lambda,\nu,\nu-1}(d,\rho(s))E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,\rho(s))E_{\lambda,\nu,\nu-1}(s-1,a)}.$$

By Lemma 3.5, we see that $\dfrac{E_{\lambda,\nu,\nu-1}(d,\rho(s))}{E_{\lambda,\nu,\nu-1}(b,\rho(s))}$ is a decreasing function for $s\in\mathbb{N}_{a+2}^r$. Then,

$$\begin{aligned} \gamma_2(s) &\geq \frac{1}{E_{\lambda,\nu,\nu-1}(s-1,a)}\left[E_{\lambda,\nu,\nu-1}(d,a) - \frac{E_{\lambda,\nu,\nu-1}(d,a+1)E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,a+1)}\right] \\ &> \frac{1}{E_{\lambda,\nu,\nu-1}(d,a)}\left[E_{\lambda,\nu,\nu-1}(d,a) - \frac{E_{\lambda,\nu,\nu-1}(d,a+1)E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,a+1)}\right]. \end{aligned}$$

Thus,

$$\min_{t\in\mathbb{N}_c^d} G(t,s) \geq \gamma\max_{t\in\mathbb{N}_a^b} G(t,s), \tag{3.3}$$

where

$$\gamma = \min\left[\frac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(b-1,a)}, 1 - \frac{E_{\lambda,\nu,\nu-1}(d,a+1)E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,a+1)E_{\lambda,\nu,\nu-1}(d,a)}\right].$$

Since $G_1(c,s) > 0$ and $G_2(d,s) > 0$, we have $\gamma(s) > 0$ for all $s \in \mathbb{N}_{a+2}^b$, implying that $\gamma > 0$. It would be suffice to prove that one of the terms $\dfrac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(b-1,a)}, 1-\dfrac{E_{\lambda,\nu,\nu-1}(d,a+1)E_{\lambda,\nu,\nu-1}(b,a)}{E_{\lambda,\nu,\nu-1}(b,a+1)E_{\lambda,\nu,\nu-1}(d,a)}$ is less than 1. It follows from Lemma 2.12 that

$$\frac{E_{\lambda,\nu,\nu-1}(c,a)}{E_{\lambda,\nu,\nu-1}(b-1,a)} < 1.$$

Therefore, we conclude that $\gamma \in (0,1)$. The proof is complete. □

By Theorem 2.10, we observe that u is a solution of (1.1) if and only if u is a solution of the summation equation

$$u(t) = \sum_{s=a+2}^{b} G(t,s) f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{3.4}$$

Note that any solution $u : \mathbb{N}_a^b \to \mathbb{R}$ of (1.1) can be viewed as a real $(b-a+1)$-tuple vector. Consequently, $u \in \mathbb{R}^{b-a+1}$. Define the operator $T : \mathbb{R}^{b-a+1} \to \mathbb{R}^{b-a+1}$ by

$$\big(Tu\big)(t) = \sum_{s=a+2}^{b} G(t,s) f(s,u(s)), \quad t \in \mathbb{N}_a^b. \tag{3.5}$$

Clearly, u is a fixed-point of T if and only if u is a solution of (1.1). We use the fact that $\mathbb{R}^{b-a+1}$ is a Banach space equipped with the maximum norm $\|u\| = \max_{t\in\mathbb{N}_a^b} |u(t)|$, for any $u \in \mathbb{R}^{b-a+1}$. Denote by

$$\mathcal{B} = \{u : \mathbb{N}_a^b \to \mathbb{R} \mid u(a) = u(b) = 0\} \subseteq \mathbb{R}^{b-a+1}. \tag{3.6}$$

Clearly $\mathcal{B}$ is a Banach space equipped with the maximum norm *i.e.*

$$\|u\| = \max_{t\in\mathbb{N}_a^b} |u(t)|.$$

Since T is defined on a discrete finite domain, it is trivially completely continuous. Define the cone

$$K = \{u \in \mathcal{B} : u(t) \geq 0 \text{ for } t \in \mathbb{N}_a^b, \text{ and } \min_{t\in\mathbb{N}_c^d} u(t) \geq \gamma\|u\|\}. \tag{3.7}$$

Lemma 3.7. *For $\lambda \in (0,\lambda^*]$ the operator T maps K into itself.*

Proof. Let $u \in K$. Clearly, $(Tu)(t) \geq 0$, whenever $u \in K$. Consider

$$\begin{aligned}
\min_{t\in\mathbb{N}_c^d} (Tu)(t) &= \min_{t\in\mathbb{N}_c^d} \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \geq \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_c^d} [G(t,s)]\, f(s,u(s)) \\
&\geq \sum_{s=a+2}^{b} \gamma \max_{t\in\mathbb{N}_a^b} [G(t,s)]\, f(s,u(s)) \geq \gamma \max_{t\in\mathbb{N}_a^b} \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \\
&= \gamma \max_{t\in\mathbb{N}_a^b} \left| \sum_{s=a+2}^{b} G(t,s) f(s,u(s)) \right| \\
&= \gamma \|Tu\|.
\end{aligned}$$

Thus, we have $T : K \to K$ and it is completely continuous. The proof is complete. □

Take

$$\eta = \frac{1}{\sum_{s=a+2}^{b} G(s-1,s)}.$$

Theorem 3.8. *Assume $f(t,u(t))$ satisfies the following conditions for $0 < r_1 < r_2$*

(i) There exists a number $r_1 > 0$ such that $f(t,u(t)) \leq \eta r_1$, whenever $0 \leq u \leq r_1$.

(ii) There exists a number $r_2 > 0$ such that $f(t,u(t)) \geq \frac{\eta r_2}{\gamma}$, whenever $\gamma r_2 \leq u \leq r_2$.

Then, for $\lambda \in (0, \lambda^*]$ the BVP (1.1) has at least one positive solution.

Proof. We know that $T : K \to K$ is completely continuous. Define the set

$$\Omega_1 = \{u \in K : \|u\| < r_1\}.$$

Clearly, $\Omega_1 \subseteq \beta$ is an open set with $0 \in \Omega_1$. Since $\|u\| = r_1$ for $u \in \partial\Omega_1$, condition (i) holds for all $u \in \partial\Omega_1$. So, it follows that

$$\begin{aligned}\|Tu\| &= \max_{t\in\mathbb{N}_a^b} \sum_{s=a+2}^{b} G(t,s)f(s,u(s)) \leq \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_a^b} G(t,s)f(s,u(s)) \leq \eta r_1 \sum_{s=a+2}^{b} G(s-1,s)\\ &= r_1 = \|u\|.\end{aligned}$$

implying that $\|Tu\| \leq \|u\|$ whenever $u \in K \cap \partial\Omega_1$. On the other hand, define the set

$$\Omega_2 = \{u \in K : \|u\| < r_2\}.$$

Clearly, $\Omega_2 \subseteq \beta$ is an open set and $\overline{\Omega}_1 \subseteq \Omega_2$. Since $\|u\| = r_2$ for $u \in \partial\Omega_2$, condition (ii) holds for all $u \in \partial\Omega_2$.

Thus, we have

$$\begin{aligned}\|Tu\| &\geq \min_{t\in\mathbb{N}_c^d} \sum_{s=a+2}^{b} G(t,s)f(s,u(s)) \geq \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_c^d} G(t,s)f(s,u(s))\\ &\geq \gamma \sum_{s=a+2}^{b} G(s-1,s)f(s,u(s)) \geq \eta r_2 \sum_{s=a+2}^{b} G(s-1,s)\\ &= r_2 = \|u\|\end{aligned}$$

implying that $\|Tu\| \geq \|u\|$ whenever $u \in K \cap \partial\Omega_2$. Hence by part 1 of Theorem 3.4, T has at least one fixed-point in $K \cap (\overline{\Omega}_1 \backslash \Omega_1)$, say u_0 satisfying $r_1 < \|u_0\| < r_2$ $\square$

Theorem 3.9. *Assume $f(t,u(t))$ satisfies the following conditions*

(i) There exists a number $r_2 > 0$ such that $f(t,u(t)) \leq \eta r_2$, whenever $0 \leq u \leq r_2$.

(ii) $\displaystyle\lim_{u\to 0^+} \min_{t\in\mathbb{N}_a^b} \frac{f(t,u(t))}{u} = \infty, \quad \lim_{u\to\infty} \min_{t\in\mathbb{N}_a^b} \frac{f(t,u(t))}{u} = \infty.$

Then, for $\lambda \in (0,\lambda^*]$ the BVP (1.1) has at least two positive solution.

Proof. Let us choose a number $N > 0$ such that

$$\frac{N\gamma}{\eta} > 1,$$

by condition (ii) there exists a number $r^* > 0$ such that $r^* < r_1 < r_2$ and $f(t,u(t)) \geq Nu$ for $u \in [0,r^*]$ and $t \in \mathbb{N}_a^b$. Define the set $\Omega_{r^*} = \{u \in \mathcal{K} : \|u\| < r^*\}$. It can easily be shown that $\|Tu\| > \|u\|$, for $u \in \partial\Omega_{r^*} \cap \mathcal{K}$.

Next for the same N, we can find a number $R_1 > 0$ such that $f(t,u) \geq Nu$ for $u \geq R_1$ and $t \in \mathbb{N}_a^b$. Choose R such that $R = \max\left\{r_2, \frac{R_1}{\gamma}\right\}$. Define the set $\Omega_R = \{u \in \mathcal{K} : \|u\| < R\}$. We can show that $\|Tu\| > \|u\|$, for $u \in \partial\Omega_R \cap \mathcal{K}$.
Finally define the set

$$\Omega_2 = \{u \in \mathcal{K} : \|u\| < r_2\}.$$

Since $\|u\| = r_2$ condition (i) holds for all $u \in \partial\Omega_2$. Then, we have

$$\begin{aligned}\|Tu\| &= \max_{t\in\mathbb{N}_b^a} \sum_{s=a+2}^{b} G(t,s)f(s,u(s)) \leq \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_b^a} [G(t,s)]\, f(s,u(s) \\ &\leq r_2\eta \sum_{s=a+2}^{b} G(s-1,s) = r_2.\end{aligned}$$

Implying $\|Tu\| \leq \|u\|$, for $u \in \partial\Omega_{r_2} \cap \mathcal{K}$. Hence, we conclude that T has at least two fixed-points say $u_1 \in \Omega_2 \backslash \hat{\Omega}_{r^*}$ and $u_2 \in \Omega_R \backslash \hat{\Omega}_2$, where $\hat{\Omega}$ denoted the interior of the set Ω. In particular (1.1) has at least two positive solutions, say u_1 and u_2 satisfying $0 < \|u_1\| < r_2 < \|u_2\|$. The proof is complete. ☐

We state here the Leggett–Williams fixed-point theorem as follows. The proof can be found in [26] and also, we would like to refer here a paper by Kwong [25] on the same.

Denote

$$\begin{aligned}K_c =&\{u \in K : \|u\| < c\},\\ K_{\alpha_2}(a,b) =&\{u \in K : a \leq \alpha_2(u), \|u\| \leq b\},\end{aligned}$$

where α_2 is defined as in Definition 3.2.

Theorem 3.10 ([1]). *Let $T : \bar{K}_c \to \bar{K}_c$ be completely continuous and α_2 be a non-negative continuous concave functional on K, such that $\alpha_2(u) \leq \|u\|$, for all $u \in \bar{K}_c$. Suppose there exists $0 < d < a < b \leq c$, such that*

(1) $\{u \in K_{\alpha_2}(a, b) : \alpha_2(u) > a\} \neq \emptyset$ *and* $\alpha_2(Tu) > a$, *for* $u \in K_{\alpha_2}(a, b)$;

(2) $\|Tu\| < d$, *for* $\|u\| \leq d$;

(3) $\alpha_2(Tu) > a$, *for* $u \in K_{\alpha_2}(a, c)$ *with* $\|Tu\| > b$.

Then, T has at least three fixed-points u_1, u_2, u_3 satisfying

$$\|u_1\| < d, \quad a < \alpha_2(u_2),$$

and

$$\|u_3\| > d \text{ and } \alpha_2(u_3) < a.$$

We introduce here growth conditions on the non-linear function f in line with [1].

Theorem 3.11. *Suppose there exists numbers $a', b', d' \in \mathbb{R}^+$, where $0 < d' < a' < \gamma b' < b'$, such that f satisfies the following*

(1) $f(t, u(t)) > \dfrac{a'\eta}{\gamma}$, *if* $u \in [a', b']$;

(2) $f(t, u(t)) < d'\eta$, *if* $u \in [0, d']$;

(3) *There exists c' such that $c' > b'$ and if $u \in [0, c']$ then $f(t, u(t)) < c'\eta$;*

Then, the boundary value problem (1.1) *for $\lambda \in (0, \lambda^*]$ has at least three positive solutions.*

Proof. Define a non-negative continuous concave functional $\alpha_2 : K \to [0, \infty)$ with $\alpha_2(u) \leq \|u\|$, for all $u \in K$, by

$$\alpha_2(u) = \min_{t \in \mathbb{N}_c^d} u(t).$$

Claim 1: If there exists a positive number r such that $u \in [0, r]$ implies $f(u) < r\eta$, then $T : \bar{K}_r \to K_r$.

Suppose that $u \in \bar{K}_r$. Then,

$$\begin{aligned}
\|Tu\| = \max_{t \in \mathbb{N}_a^b} \left[\sum_{s=a+2}^{b} G(t, s) f(s, u(s)) \right] &\leq \sum_{s=a+2}^{b} \max_{t \in \mathbb{N}_a^b} [G(t, s)] f(s, u(s)) \\
&= \sum_{s=a+2}^{b} G(s-1, s) f(s, u(s)) \\
&< r\eta \sum_{s=a+2}^{b} G(s-1, s) = r.
\end{aligned}$$

Thus, $T : \bar{K}_r \to K_r$. Hence, we have that if condition (3) holds, then there exists a number c' such that $c' > b'$ and $T : \bar{K}_{c'} \to K_{c'}$. Note that with $r = d'$ and using condition (2), we get that $T : \bar{K}_{d'} \to K_{d'}$.

Claim 2: $\{u \in K_{\alpha_2}(a', b') : \alpha_2(u) > a'\} \neq \emptyset$ and $\alpha_2(Tu) > a'$ for $u \in K_{\alpha_2}(a', b')$.

Since $u = \frac{a'+b'}{2} \in \{u \in K_{\alpha_2}(a', b') : \alpha_2(u) > a'\} \neq \emptyset$. Let $u \in K_{\alpha_2}(a', b')$. By using condition (1), we have

$$\begin{aligned}\alpha_2(Tu) &= \min_{t\in\mathbb{N}_c^d}\left[\sum_{s=a+2}^{b} G(t,s)f(s,u(s))\right] \geq \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_c^d}\left[G(t,s)\right] f(s,u(s)) \\ &\geq \gamma \sum_{s=a+2}^{b} G(s-1,s)f(s,u(s)) > a'\end{aligned}$$

Thus, if $u \in K_{\alpha_2}(a', b')$, then $\alpha_2(Tu) > a'$.

Claim 3: If $u \in K_{\alpha_2}(a', c')$ and $\|Tu\| > b'$ then $\alpha_2(Tu) > a'$.

Suppose $u \in K_{\alpha_2}(a', c')$ and $\|Tu\| > b'$. Then,

$$\begin{aligned}\alpha_2(Tu) &= \min_{t\in\mathbb{N}_c^d}\left[\sum_{s=a+2}^{b} G(t,s)f(s,u(s))\right] \geq \sum_{s=a+2}^{b} \min_{t\in\mathbb{N}_c^d}\left[G(t,s)\right] f(s,u(s)) \\ &\geq \gamma \sum_{s=a+2}^{b} \max_{t\in\mathbb{N}_a^b}\left[G(t,s)\right] f(s,u(s)) \geq \gamma \max_{t\in\mathbb{N}_c^d}\left[\sum_{s=a+2}^{b} G(t,s)f(s,u(s))\right] \\ &= \gamma\|Tu\| > \gamma b' > a'.\end{aligned}$$

Thus, $\alpha_2(Tx) > a'$.

Hence all the hypothesis of the Theorem 3.10 are satisfied. Therefore, the boundary value problem (1.1) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$\|u_1\| < d', \quad a' < \alpha_2(u_2),$$

and

$$\|u_3\| > d' \text{ and } \alpha_2(u_3) < a'.$$

The proof is complete. $\square$

Example

In this section, we have constructed a suitable example to illustrate the applicability of the established results.

Example 3.12. *Take* $\nu = 1.5$, $a = 0, b = 5$, *and* $f(t, u(t)) = \frac{1}{20}\left(\sqrt{u} + u^2\right)$. *Then,* (1.1) *becomes*

$$\begin{cases} -\left(\nabla^{1.5}_{\rho(0)} u\right)(t) + \lambda u(t) = \dfrac{1}{20}\left(\sqrt{u} + u^2\right), \quad t \in \mathbb{N}_2^5, \\ u(0) = 0 = u(5). \end{cases} \tag{3.8}$$

Choose $\lambda^* = 0.007$. *Then, we get*

$$\eta = \frac{1}{\sum\limits_{s=2}^{5} G(s-1, s)} = \frac{E_{\lambda,1.5,0.5}(5, 0)}{\sum\limits_{s=2}^{5} E_{\lambda,1.5,0.5}(s-1, 0) E_{\lambda,1.5,0.5}(5, s-1)} = 0.2473.$$

By taking $r_2 = 2$, *we have*

$$f(t, u) = \frac{1}{20}\left(\sqrt{u} + u^2\right) \leq \frac{1}{20}\left(\sqrt{r_2} + r_2^2\right) = 0.270 < \eta r_2 = 0.4946,$$

implying that $f(t, u)$ *satisfies conditions (i) and (ii) of Theorem 3.9. Thus, all conditions of Theorem 3.9 are satisfied. Hence,* (3.8) *has at least two positive solutions* u_1 *and* u_2 *such that* $0 < \|u_1\| < 2 < \|u_2\|$.

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