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A STUDY ON PICTURE DOMBI FUZZY GRAPH

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Abstract: *The picture fuzzy graph is a newly introduced fuzzy graph model to handle with uncertain real scenarios, in which simple fuzzy graph and intuitionistic fuzzy graph may fail to model those problems properly. The picture fuzzy graph is used efficiently in real world scenarios which involve several answers to these types: yes, no, abstain and refusal. In this paper, the new idea of dombi picture fuzzy graph is introduced. We also describe some operations on dombi picture graphs, viz. union, join, composition and cartesian product. In addition, we investigated many interesting results regarding the operations. The concept of complement and isomorphism of Picture dombi fuzzy graph are presented in this paper. Some important results on weak and co-weak isomorphism of Picture dombi fuzzy graph are derived.*

Key words*: t-norm, s-norm, Picture Dombi Fuzzy Graph, Union, Composition, Cartesian Product, Join, Complement, Homomorphism, Isomorphism.*

1. Introduction

Menger (1942) presented triangular norms (t-norms) and triangular co-norms (t-conorms) in the framework of probabilistic metric spaces which were later defined and discussed by Schweizer and Berthold (2011). Alsina et al. (1983) proved that t-norms and t-conorms are standard models for intersecting and unifying fuzzy sets, respectively. Since then, many other researchers have presented various types of T-operators for the same purpose (Hamacher, 1978). Zadeh's conventional T-operators, min and max, have been used in almost every application of fuzzy logic particularly in decision-making processes and fuzzy graph theory. It is a well-known fact that from theoretical and experimental aspects other T-operators may work better in some situations, especially in the context of decision-making processes. For example, the product operator may be preferred to the min operator (Dubois et al., 2000). For the selection of appropriate T-

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operators for a given application, one has to consider the properties they possess, their suitability to the model, their simplicity, their software and hardware implementation, etc. As the study on these operators has widened, multiple options are available for selecting T-operators that may be better suited for given research. There are various real-life problems that we cannot explain with the concept of fuzzy set theory. For solving these kinds of problem, K. Atanassov (1986) proposed the idea of an intuitionistic fuzzy set (IFS). In IFS we consider membership function and non-membership function such that their sum is lying in [0; 1]. In IFS theory, the idea of neutrality membership value is not considering. In many real-life situations, the neutral membership degree is needed, like a democratic election station. Human beings generally give opinions having more answers of the type: yes, no, abstain and refusal. For example, in a democratic voting system, 1000 people participated in the election. The election commission issues 1000 ballot paper and one person can take only one ballot for giving his/her vote and A is only one candidate. The results of the election are generally divided into four groups came with the number of ballot papers namely "vote for the candidate (500)", "abstain in the vote (200)", "vote against a candidate (200)" and "refusal of voting (100)". The "abstain in the vote" describes that ballot paper is white which contradicts both "vote for the candidate" and "vote against a candidate" but it considers the vote. However, "refusal of voting" means bypassing the vote. This type of real-life scenarios cannot be handled by intuitionistic fuzzy set. If we use intuitionistic fuzzy sets to describe the above voting system, the information of voting for non-candidates may be ignored. To solve this problem, Cuong and Kreinovich (2014) proposed the concept of picture fuzzy set which is a modified version of the fuzzy set and Intuitionistic fuzzy set. Picture fuzzy set (PFS) allows the idea degree of positive membership, degree of neutral membership and degree of negative membership of an element. Graph theory is an important mathematical tool for handling many real-world problems. Graph theory has various application in different areas like computer science, social sciences, economics, physics, system analysis, chemistry, neural networks, electrical engineering, control theory, transportation, architecture, and communication. Kaufmann (1975) introduces the basic concept of fuzzy graph theory and after that Rosenfeld (1975) describes more idea on the fuzzy graph-theoretic concept. Krassimir T Atanassov introduces the concept of intuitionistic graph theory. Shovan Dogra (2015) describes different types of product of fuzzy graphs. Havare, Özge Çolakoğlu, n.d. discussed on the coronary product of two fuzzy graphs.

In this paper we present the concept of Picture dombi fuzzy graph (PDFG) and discussed the operations like union, join, composition, cartesian product, h-morphism, isomorphism, complement of DPFG's. We also introduce some theorems and examples on PDFG's.

2. Preliminaries

t-norm

A t-norm is a binary mapping $t:[0,1]\times[0,1]\to[0,1]$ which is satisfies the following conditions: $\forall a,b,c,d \in [0,1]$

1. (Boundedness property) $t(0,0)=0, t(a,1)=t(1,a)=a$;
2. (Monotonicity property) $t(a,b)\le t(c,d)$, if $a\le c$ and $b\le d$;
3. (Commutativity property) $t(a,b)=t(b,a)$;
4. (Associativity property) $t(a,t(b,c))=t(t(a,b),c)$.

t-conorm or s-norm

A t-conorm is a binary mapping $s:[0,1]\times[0,1]\to[0,1]$ which is satisfies the following conditions: $\forall a,b,c,d\in[0,1]$

(Boundedness property) $s(1,1)=1$, $s(a,0)=s(0,a)=a$;

(Monotonicity property) $s(a,b)\le t(c,d)$, if $a\le c$ and $b\le d$;

(Commutativity property) $s(a,b)=s(b,a)$;

(Associativity property) $s(a,s(b,c))=s(s(a,b),c)$.

Hamacher norm

Hamacher define t-norm and s-norm as follows: $\forall a,b\in[0,1]$

(t-norm) $t(a,b)=\dfrac{ab}{\gamma+(1-\gamma)(a+b-ab)}$, $\gamma\ge 0$.

(s-norm) $s(a,b)=\dfrac{(\lambda-1)ab+a+b}{1+\lambda ab}$, $\lambda\ge -1$.

Dombi norm

The Dombi norm is given by $\forall a,b\in[0,1]$

(t-norm) $t(a,b)=\dfrac{1}{1+[(\frac{1-a}{a})^{\lambda}+(\frac{1-b}{b})^{\lambda}]^{\frac{1}{\lambda}}}$;

(s-norm) $s(a,b)=\dfrac{1}{1+[(\frac{1-a}{a})^{-\lambda}+(\frac{1-b}{b})^{-\lambda}]^{-\frac{1}{\lambda}}}$.

Remark 1: If we put $\lambda=1$ in Dombi t-norm, we have $t(a,b)=\dfrac{ab}{a+b-ab}$, $\forall a,b\in[0,1]$. If we put $\lambda=1$ in Dombi s-norm, we have $s(a,b)=\dfrac{a+b-2ab}{1-ab}$, $\forall a,b\in[0,1]$.

Fuzzy set

Let X be a universal set. A fuzzy set M of X is the collection of elements α in X s. t., $T(\alpha)\in[0,1]$. Here T is called a membership function of M i.e., $T:X\to[0,1]$.

Fuzzy graph

A f-graph of the graph $G'=(V_G,E_G)$ is a pair $G=(\Upsilon,\Gamma)$, where $\Upsilon:V\to[0,1]$ is a fuzzy set on V_G and $\Gamma:V_G\times V_G\to[0,1]$ is a fuzzy relation on V_G s. t., $\Gamma(x,y)\le\Upsilon(x)\wedge\Upsilon(y)$, $\forall(x,y)\in V_G\times V_G$ (Zadeh, 1965).

Picture Fuzzy set (PFS)

Let $\mathcal{U}$ be an universal set. A PFS $\mathcal{A}$ is defined as follows

$$\mathcal{A}=\{<\xi,\mu_{\mathcal{A}}(\xi),\nu_{\mathcal{A}}(\xi),\eta_{\mathcal{A}}(\xi)>:0\le\mu_{\mathcal{A}}(\xi)+\nu_{\mathcal{A}}(\xi)+\eta_{\mathcal{A}}(\xi)\le 1,\xi\in\mathcal{U}\}.$$

Here $\mu_{\mathcal{A}}:\mathcal{U}\to[0,1]$, $\nu_{\mathcal{A}}:\mathcal{U}\to[0,1]$ and $\eta_{\mathcal{A}}:\mathcal{U}\to[0,1]$ are called positive membership degree, neutral membership degree and negative membership degree respectively. For all $\xi\in\mathcal{U}$, $\pi_{\mathcal{A}}=1-(\mu_{\mathcal{A}}(\xi)+\nu_{\mathcal{A}}(\xi)+\eta_{\mathcal{A}}(\xi))$ is called refusal function of ξ in $\mathcal{A}$.

Picture Fuzzy Relation (PFR)

Let $\mathcal{U}$ and $\mathcal{V}$ be two universal sets. A PFR $\mathcal{R}$ is subset of $\mathcal{U}\times\mathcal{V}$ s. t., $\mathcal{R} = \{< (\alpha,\beta), \mu_{\mathcal{R}}(\alpha,\beta), \nu_{\mathcal{R}}(\alpha,\beta), \eta_{\mathcal{R}}(\alpha,\beta) >: 0 \le \mu_{\mathcal{R}}(\alpha,\beta) + \nu_{\mathcal{R}}(\alpha,\beta) + \eta_{\mathcal{R}}(\alpha,\beta) \le 1, \forall(\alpha,\beta) \in \mathcal{U}\times\mathcal{V}\}$, where $\mu_{\mathcal{R}} : \mathcal{U}\times\mathcal{V} \to [0,1]$, $\nu_{\mathcal{R}} : \mathcal{U}\times\mathcal{V} \to [0,1]$ and $\eta_{\mathcal{R}} : \mathcal{U}\times\mathcal{V} \to [0,1]$ are called positive membership function, neutral membership function and negative membership function respectively.

Dombi Graph

Let $G = (V_G, E_G)$ be a crisp undirected graph contain no self-loop and parallel edges. Also, let $\Upsilon : V \to [0,1]$ membership degree on V and $\Gamma : V\times V \to [0,1]$ be the membership degree on the symmetric fuzzy relation $E \subset V\times V$. Then $\mathcal{G} = (V, \Upsilon, \Gamma)$, is said to be a dombi graph if $\Gamma(a,b) \le \dfrac{\Upsilon(a)\Upsilon(b)}{\Upsilon(a)+\Upsilon(b)-\Upsilon(a)\Upsilon(b)}$, $\forall(ab) \in E$.

Picture Dombi Fuzzy Graph (PDFG)

Let $G = (V_G, E_G)$ be a crisp undirected graph contain no self-loop and parallel edges. Also, let $\Upsilon = (\mu_\Upsilon, \nu_\Upsilon, \eta_\Upsilon)$ s. t., $\mu_\Upsilon : V \to [0,1]$, $\nu_\Upsilon : V \to [0,1]$ and $\eta_\Upsilon : V \to [0,1]$ be the positive membership degree, neutral membership degree and negative membership degree respectively on the PFS V. We consider $\Gamma = (\mu_\Gamma, \nu_\Gamma, \eta_\Gamma)$ s. t., $\mu_\Gamma : V\times V \to [0,1]$, $\nu_\Gamma : V\times V \to [0,1]$ and $\eta_\Gamma : V\times V \to [0,1]$ as the positive membership degree, neutral membership degree and negative membership degree respectively, in the symmetric PFR $E \subset V\times V$. Then $\mathcal{G} = (V, \Upsilon, \Gamma)$, is said to be a PDFG if

1. $\mu_\Gamma(ab) \le \dfrac{\mu_\Upsilon(a)\mu_\Upsilon(b)}{\mu_\Upsilon(a)+\mu_\Upsilon(b)-\mu_\Upsilon(a)\mu_\Upsilon(b)}$, $\forall(ab) \in E_G$;

2. $\nu_\Gamma(ab) \le \dfrac{\nu_\Upsilon(a)\nu_\Upsilon(b)}{\nu_\Upsilon(a)+\nu_\Upsilon(b)-\nu_\Upsilon(a)\nu_\Upsilon(b)}$, $\forall(ab) \in E_G$;

3. $\eta_\Gamma(ab) \ge \dfrac{\eta_\Upsilon(a)+\eta_\Upsilon(b)-2\eta_\Upsilon(a)\eta_\Upsilon(b)}{1-\eta_\Upsilon(a)\eta_\Upsilon(b)}$, $\forall(ab) \in E_G$.

3. Some Operation on PDFG's

Union

The union of two PDFG's $\mathcal{G} = (V_\mathcal{G}, \Upsilon_\mathcal{G}, \Gamma_\mathcal{G})$ and $\mathcal{H} = (V_\mathcal{H}, \Upsilon_\mathcal{H}, \Gamma_\mathcal{H})$ of the graphs $G' = (V_G, E_G)$ and $H' = (V_H, E_H)$ respectively, is denoted by $\mathcal{G}\cup\mathcal{H}$ and is defined as $(V_G \cup V_H, \Upsilon_\mathcal{G} \cup \Upsilon_\mathcal{H}, \Gamma_\mathcal{G} \cup \Gamma_\mathcal{H})$, where $\Upsilon_\mathcal{G} \cup \Upsilon_\mathcal{H} = (\mu_{\Upsilon_\mathcal{G}} \cup \mu_{\Upsilon_\mathcal{H}}, \nu_{\Upsilon_\mathcal{G}} \cup \nu_{\Upsilon_\mathcal{H}}, \eta_{\Upsilon_\mathcal{G}} \cup \eta_{\Upsilon_\mathcal{H}})$ and $\Gamma_\mathcal{G} \cup \Gamma_\mathcal{H} = (\mu_{\Gamma_\mathcal{G}} \cup \mu_{\Gamma_\mathcal{H}}, \nu_{\Gamma_\mathcal{G}} \cup \nu_{\Gamma_\mathcal{H}}, \eta_{\Gamma_\mathcal{G}} \cup \eta_{\Gamma_\mathcal{H}})$ s. t.,

$(\mu_{\Upsilon_\mathcal{G}} \cup \mu_{\Upsilon_\mathcal{H}})(\xi)$

$= \mu_{\Upsilon_\mathcal{G}}(\xi)$, if $\xi \in V_G - V_H$

$= \mu_{\Upsilon_\mathcal{H}}(\xi)$, if $\xi \in V_H - V_G$

$$= \frac{\mu_{\Upsilon_{\mathcal{G}}}(\xi)\mu_{\Upsilon_{\mathcal{H}}}(\xi)}{\mu_{\Upsilon_{\mathcal{G}}}(\xi)+\mu_{\Upsilon_{\mathcal{H}}}(\xi)-\mu_{\Upsilon_{\mathcal{G}}}(\xi)\mu_{\Upsilon_{\mathcal{H}}}(\xi)}, \text{ if } \xi \in V_G \cap V_H$$

$(\nu_{\Upsilon_{\mathcal{G}}} \cup \nu_{\Upsilon_{\mathcal{H}}})(\xi)$

$$= \nu_{\Upsilon_{\mathcal{G}}}(\xi), \text{ if } \xi \in V_G - V_H$$

$$= \nu_{\Upsilon_{\mathcal{H}}}(\xi), \text{ if } \xi \in V_H - V_G$$

$$= \frac{\nu_{\Upsilon_{\mathcal{G}}}(\xi)\nu_{\Upsilon_{\mathcal{H}}}(\xi)}{\nu_{\Upsilon_{\mathcal{G}}}(\xi)+\nu_{\Upsilon_{\mathcal{H}}}(\xi)-\nu_{\Upsilon_{\mathcal{G}}}(\xi)\nu_{\Upsilon_{\mathcal{H}}}(\xi)}, \text{ if } \xi \in V_G \cap V_H.$$

$(\eta_{\Upsilon_{\mathcal{G}}} \cup \eta_{\Upsilon_{\mathcal{H}}})(\xi)$

$$= \eta_{\Upsilon_{\mathcal{G}}}(\xi), \text{ if } \xi \in V_G - V_H$$

$$= \eta_{\Upsilon_{\mathcal{H}}}(\xi), \text{ if } \xi \in V_H - V_G$$

$$= \frac{\eta_{\Upsilon_{\mathcal{G}}}(\xi)+\eta_{\Upsilon_{\mathcal{H}}}(\xi)-2\eta_{\Upsilon_{\mathcal{G}}}(\xi)\eta_{\Upsilon_{\mathcal{H}}}(\xi)}{1-\eta_{\Upsilon_{\mathcal{G}}}(\xi)\eta_{\Upsilon_{\mathcal{H}}}(\xi)}, \text{ if } \xi \in V_G \cap V_H.$$

$(\mu_{\Gamma_{\mathcal{G}}} \cup \mu_{\Gamma_{\mathcal{H}}})(ab)$

$$= \mu_{\Gamma_{\mathcal{G}}}(ab), \text{ if } (ab) \in E_G - E_H$$

$$= \mu_{\Gamma_{\mathcal{H}}}(ab), \text{ if } (ab) \in E_H - E_G$$

$$= \frac{\mu_{\Gamma_{\mathcal{G}}}(ab)\mu_{\Gamma_{\mathcal{H}}}(ab)}{\mu_{\Gamma_{\mathcal{G}}}(ab)+\mu_{\Gamma_{\mathcal{H}}}(ab)-\mu_{\Gamma_{\mathcal{G}}}(ab)\mu_{\Gamma_{\mathcal{H}}}(ab)}, \text{ if } \xi \in E_G \cap E_H$$

$(\nu_{\Gamma_{\mathcal{G}}} \cup \nu_{\Gamma_{\mathcal{H}}})(ab)$

$$= \nu_{\Gamma_{\mathcal{G}}}(ab), \text{ if } (ab) \in E_G - E_H$$

$$= \nu_{\Gamma_{\mathcal{H}}}(ab), \text{ if } (ab) \in E_H - E_G$$

$$= \frac{\nu_{\Gamma_{\mathcal{G}}}(ab)\nu_{\Gamma_{\mathcal{H}}}(ab)}{\nu_{\Gamma_{\mathcal{G}}}(ab)+\nu_{\Gamma_{\mathcal{H}}}(ab)-\nu_{\Gamma_{\mathcal{G}}}(ab)\nu_{\Gamma_{\mathcal{H}}}(ab)}, \text{ if } (ab) \in E_G \cap E_H$$

$(\eta_{\Gamma_{\mathcal{G}}} \cup \eta_{\Gamma_{\mathcal{H}}})(ab)$

$$= \eta_{\Gamma_{\mathcal{G}}}(ab), \text{ if } (ab) \in E_G - E_H$$

$$= \eta_{\Gamma_{\mathcal{H}}}(ab), \text{ if } (ab) \in E_H - E_G$$

$$= \frac{\eta_{\Gamma_{\mathcal{G}}}(ab)+\eta_{\Gamma_{\mathcal{H}}}(ab)-2\eta_{\Gamma_{\mathcal{G}}}(ab)\eta_{\Gamma_{\mathcal{H}}}(ab)}{1-\eta_{\Gamma_{\mathcal{G}}}(ab)\eta_{\Gamma_{\mathcal{H}}}(ab)}, \text{ if } (ab) \in E_G \cap E_H.$$

Example 1: We consider two PDFG's $A = (\Upsilon_A, \Gamma_A)$ (Shown in Fig. 1(a)) and $B = (\Upsilon_B, \Gamma_B)$ (Shown in Fig. 1(b))of the graphs $A' = (V_A, E_A)$ and $B' = (V_B, E_B)$ respectively, where $V_A = \{x, y, z\}$, $E_A = \{xy, yz, zx\}$, $V_B = \{y, z, w\}$ and $E_B = \{yz, yw, zw\}$. Then the union of A and B are shown in Figure 1(c).

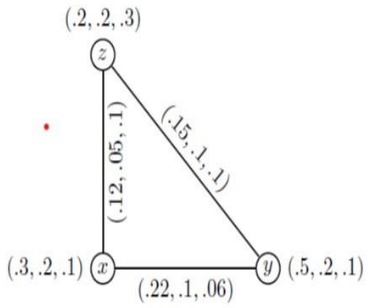

Figure 1(a). PDFG A

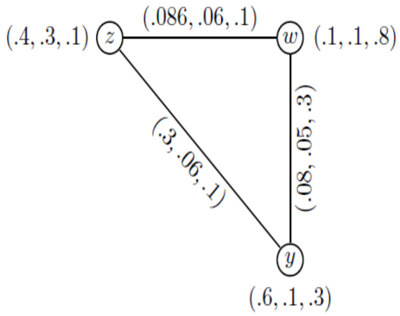

Figure 1(b). PDFG B

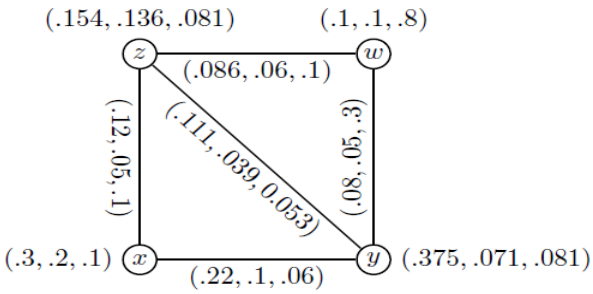

Figure 1(c). PDFG $A \cup B$

Join

The join of two PDFG's $\mathcal{G} = (V_{\mathcal{G}}, \Upsilon_{\mathcal{G}}, \Gamma_{\mathcal{G}})$ and $\mathcal{H} = (V_{\mathcal{H}}, \Upsilon_{\mathcal{H}}, \Gamma_{\mathcal{H}})$ of the graphs $G' = (V_G, E_G)$ and $H' = (V_H, E_H)$ respectively, is denoted by $\mathcal{G} + \mathcal{H}$ and is defined as $(V_{\mathcal{G}} \cup V_{\mathcal{H}}, E, \Upsilon_{\mathcal{G}} + \Upsilon_{\mathcal{H}}, \Gamma_{\mathcal{G}} + \Gamma_{\mathcal{H}})$, where $\Upsilon_{\mathcal{G}} + \Upsilon_{\mathcal{H}} = (\mu_{\Upsilon_{\mathcal{G}}} + \mu_{\Upsilon_{\mathcal{H}}}, \nu_{\Upsilon_{\mathcal{G}}} + \nu_{\Upsilon_{\mathcal{H}}}, \eta_{\Upsilon_{\mathcal{G}}} + \eta_{\Upsilon_{\mathcal{H}}})$, $\Gamma_{\mathcal{G}} + \Gamma_{\mathcal{H}} = (\mu_{\Gamma_{\mathcal{G}}} + \mu_{\Gamma_{\mathcal{H}}}, \nu_{\Gamma_{\mathcal{G}}} + \nu_{\Gamma_{\mathcal{H}}}, \eta_{\Gamma_{\mathcal{G}}} + \eta_{\Gamma_{\mathcal{H}}})$, $V_G \cap V_H = \phi$, $E = E_G \cup E_H \cup E'$ ($E' =$ set of all edges joining the nodes of V_G and V_H) s. t.,

$$(\mu_{\Upsilon_{\mathcal{G}}} + \mu_{\Upsilon_{\mathcal{H}}})(\xi) = (\mu_{\Upsilon_{\mathcal{G}}} \cup \mu_{\Upsilon_{\mathcal{H}}})(\xi), \text{ if } \xi \in V_G \cup V_H$$

$$(\nu_{\Upsilon_{\mathcal{G}}} + \nu_{\Upsilon_{\mathcal{H}}})(\xi) = (\nu_{\Upsilon_{\mathcal{G}}} \cup \nu_{\Upsilon_{\mathcal{H}}})(\xi), \text{ if } \xi \in V_G \cup V_H$$

$$(\nu_{\Upsilon_{\mathcal{G}}} + \nu_{\Upsilon_{\mathcal{H}}})(\xi) = (\nu_{\Upsilon_{\mathcal{G}}} \cup \nu_{\Upsilon_{\mathcal{H}}})(\xi), \text{ if } \xi \in V_G \cup V_H$$

$$(\mu_{\Gamma_{\mathcal{G}}} + \mu_{\Gamma_{\mathcal{H}}})(ab)$$

$$= (\mu_{\Gamma_{\mathcal{G}}} \cup \mu_{\Gamma_{\mathcal{H}}})(ab), \text{ if } (ab) \in E_G \cup E_H$$

$$= \frac{\mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{H}}}(b)}{\mu_{\Upsilon_{\mathcal{G}}}(a) + \mu_{\Upsilon_{\mathcal{H}}}(b) - \mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{H}}}(b)}, \text{ if } (ab) \in E'$$

$$(\nu_{\Gamma_{\mathcal{G}}} + \nu_{\Gamma_{\mathcal{H}}})(ab)$$

$$= (\nu_{\Gamma_{\mathcal{G}}} \cup \nu_{\Gamma_{\mathcal{H}}})(ab), \text{ if } (ab) \in E_G \cup E_H$$

$$= \frac{\nu_{\Upsilon_{\mathcal{G}}}(a)\nu_{\Upsilon_{\mathcal{H}}}(b)}{\nu_{\Upsilon_{\mathcal{G}}}(a) + \nu_{\Upsilon_{\mathcal{H}}}(b) - \nu_{\Upsilon_{\mathcal{G}}}(a)\nu_{\Upsilon_{\mathcal{H}}}(b)}, \text{ if } (ab) \in E'.$$

$$(\eta_{\Gamma_{\mathcal{G}}} + \eta_{\Gamma_{\mathcal{H}}})(ab)$$

$$= (\eta_{\Gamma_{\mathcal{G}}} \cup \eta_{\Gamma_{\mathcal{H}}})(ab), \text{ if } (ab) \in E_G \cup E_H$$

$$= \frac{\eta_{\Upsilon_{\mathcal{G}}}(a) + \eta_{\Upsilon_{\mathcal{H}}}(b) - 2\eta_{\Upsilon_{\mathcal{G}}}(a)\eta_{\Upsilon_{\mathcal{H}}}(b)}{1 - \eta_{\Upsilon_{\mathcal{G}}}(a)\eta_{\Upsilon_{\mathcal{H}}}(b)}, \text{ if } (ab) \in E'$$

Theorem 1: The Join of two PDFG's is a PDFG.

Composition

The composition of two PDFG's $\mathcal{G} = (V_{\mathcal{G}}, \Upsilon_{\mathcal{G}}, \Gamma_{\mathcal{G}})$ and $\mathcal{H} = (V_{\mathcal{H}}, \Upsilon_{\mathcal{H}}, \Gamma_{\mathcal{H}})$ of the graphs $G' = (V_G, E_G)$ and $H' = (V_H, E_H)$ respectively, is denoted by $\mathcal{G} \circ \mathcal{H}$ and is defined as $(V_G \times V_H, E, \Upsilon_{\mathcal{G}} \circ \Upsilon_{\mathcal{H}}, \Gamma_{\mathcal{G}} \circ \Gamma_{\mathcal{H}})$, where $\Upsilon_{\mathcal{G}} \circ \Upsilon_{\mathcal{H}} = (\mu_{\Upsilon_{\mathcal{G}}} \circ \mu_{\Upsilon_{\mathcal{H}}}, \nu_{\Upsilon_{\mathcal{G}}} \circ \nu_{\Upsilon_{\mathcal{H}}}, \eta_{\Upsilon_{\mathcal{G}}} \circ \eta_{\Upsilon_{\mathcal{H}}})$, $\Gamma_{\mathcal{G}} \circ \Gamma_{\mathcal{H}} = (\mu_{\Gamma_{\mathcal{G}}} \circ \mu_{\Gamma_{\mathcal{H}}}, \nu_{\Gamma_{\mathcal{G}}} \circ \nu_{\Gamma_{\mathcal{H}}}, \eta_{\Gamma_{\mathcal{G}}} \circ \eta_{\Gamma_{\mathcal{H}}})$ and $E = \{((s_0, t_0)(s_0, t_1)) : s_0 \in V_G, (t_0 t_1) \in E_H\} \cup \{((s_0, t_0)(s_1, t_0)) : (s_0 s_1) \in E_G, t_0 \in V_H\} \cup \{((s_0, t_0)(s_1, t_1)) : (s_0 s_1) \in E_G, t_0 \neq t_1\}$ s. t.,

$$(\mu_{\Upsilon_{\mathcal{G}}} \circ \mu_{\Upsilon_{\mathcal{H}}})(\alpha, \beta) = \frac{\mu_{\Upsilon_{\mathcal{G}}}(\alpha)\mu_{\Upsilon_{\mathcal{H}}}(\beta)}{\mu_{\Upsilon_{\mathcal{G}}}(\alpha) + \mu_{\Upsilon_{\mathcal{H}}}(\beta) - \mu_{\Upsilon_{\mathcal{G}}}(\alpha)\mu_{\Upsilon_{\mathcal{H}}}(\beta)}$$

$$(\nu_{\Upsilon_{\mathcal{G}}} \circ \nu_{\Upsilon_{\mathcal{H}}})(\alpha, \beta) = \frac{\nu_{\Upsilon_{\mathcal{G}}}(\alpha)\nu_{\Upsilon_{\mathcal{H}}}(\beta)}{\nu_{\Upsilon_{\mathcal{G}}}(\alpha) + \nu_{\Upsilon_{\mathcal{H}}}(\beta) - \nu_{\Upsilon_{\mathcal{G}}}(\alpha)\nu_{\Upsilon_{\mathcal{H}}}(\beta)}$$

$$(\eta_{\Upsilon_{\mathcal{G}}} \circ \eta_{\Upsilon_{\mathcal{H}}})(\alpha, \beta) = \frac{\eta_{\Upsilon_{\mathcal{G}}}(\alpha) + \eta_{\Upsilon_{\mathcal{H}}}(\beta) - 2\eta_{\Upsilon_{\mathcal{G}}}(\alpha)\eta_{\Upsilon_{\mathcal{H}}}(\beta)}{1 - \eta_{\Upsilon_{\mathcal{G}}}(\alpha)\eta_{\Upsilon_{\mathcal{H}}}(\beta)}$$

$\forall \gamma \in V_{\mathcal{G}}$ and $\forall (\alpha, \beta) \in E_{\mathcal{H}}$,

$$(\mu_{\Gamma_{\mathcal{G}}} \circ \mu_{\Gamma_{\mathcal{H}}})((\gamma, \alpha)(\gamma, \beta)) = \frac{\mu_{\Upsilon_{\mathcal{G}}}(\gamma)\mu_{\Gamma_{\mathcal{H}}}(\alpha\beta)}{\mu_{\Upsilon_{\mathcal{G}}}(\gamma) + \mu_{\Gamma_{\mathcal{H}}}(\alpha\beta) - \mu_{\Upsilon_{\mathcal{G}}}(\gamma)\mu_{\Gamma_{\mathcal{H}}}(\alpha\beta)}$$

$$(\nu_{\Gamma_{\mathcal{G}}} \circ \nu_{\Gamma_{\mathcal{H}}})((\gamma, \alpha)(\gamma, \beta)) = \frac{\nu_{\Upsilon_{\mathcal{G}}}(\gamma)\nu_{\Gamma_{\mathcal{H}}}(\alpha\beta)}{\nu_{\Upsilon_{\mathcal{G}}}(\gamma) + \nu_{\Gamma_{\mathcal{H}}}(\alpha\beta) - \nu_{\Upsilon_{\mathcal{G}}}(\gamma)\nu_{\Gamma_{\mathcal{H}}}(\alpha\beta)}$$

$$(\eta_{\Gamma_{\mathcal{G}}} \circ \eta_{\Gamma_{\mathcal{H}}})((\gamma, \alpha)(\gamma, \beta)) = \frac{\eta_{\Upsilon_{\mathcal{G}}}(\gamma) + \eta_{\Gamma_{\mathcal{H}}}(\alpha\beta) - 2\eta_{\Upsilon_{\mathcal{G}}}(\gamma)\eta_{\Gamma_{\mathcal{H}}}(\alpha\beta)}{1 - \eta_{\Upsilon_{\mathcal{G}}}(\gamma)\eta_{\Gamma_{\mathcal{H}}}(\alpha\beta)}$$

$\forall \gamma \in V_{\mathcal{H}}$ and $\forall (\alpha, \beta) \in E_{\mathcal{G}}$,

$$(\mu_{\Gamma_{\mathcal{G}}} \circ \mu_{\Gamma_{\mathcal{H}}})((\alpha, \gamma)(\beta, \gamma)) = \frac{\mu_{\Upsilon_{\mathcal{H}}}(\gamma)\mu_{\Gamma_{\mathcal{G}}}(\alpha\beta)}{\mu_{\Upsilon_{\mathcal{H}}}(\gamma) + \mu_{\Gamma_{\mathcal{G}}}(\alpha\beta) - \mu_{\Upsilon_{\mathcal{H}}}(\gamma)\mu_{\Gamma_{\mathcal{G}}}(\alpha\beta)}$$

$$(\nu_{\Gamma_{\mathcal{G}}} \circ \nu_{\Gamma_{\mathcal{H}}})((\alpha, \gamma)(\beta, \gamma)) = \frac{\nu_{\Upsilon_{\mathcal{H}}}(\gamma)\nu_{\Gamma_{\mathcal{G}}}(\alpha\beta)}{\nu_{\Upsilon_{\mathcal{H}}}(\gamma) + \nu_{\Gamma_{\mathcal{G}}}(\alpha\beta) - \nu_{\Upsilon_{\mathcal{H}}}(\gamma)\nu_{\Gamma_{\mathcal{G}}}(\alpha\beta)}$$

$$(\eta_{\Gamma_{\mathcal{G}}} \circ \eta_{\Gamma_{\mathcal{H}}})((\alpha, \gamma)(\beta, \gamma)) = \frac{\eta_{\Upsilon_{\mathcal{H}}}(\gamma) + \eta_{\Gamma_{\mathcal{G}}}(\alpha\beta) - 2\eta_{\Upsilon_{\mathcal{H}}}(\gamma)\eta_{\Gamma_{\mathcal{G}}}(\alpha\beta)}{1 - \eta_{\Upsilon_{\mathcal{H}}}(\gamma)\eta_{\Gamma_{\mathcal{G}}}(\alpha\beta)}$$

$\forall (\alpha, \beta) \in E_G$, and $\gamma \neq \delta \in V_H$,

$$(\mu_{\Gamma_{\mathcal{G}}} \circ \mu_{\Gamma_{\mathcal{H}}})((\alpha, \gamma)(\beta, \delta)) = \frac{\mu_{\Gamma_{\mathcal{G}}}(\alpha\beta)\mu_{\Upsilon_{\mathcal{H}}}(\gamma)\mu_{\Upsilon_{\mathcal{H}}}(\delta)}{\begin{pmatrix} \mu_{\Gamma_{\mathcal{G}}}(\alpha\beta)\mu_{\Upsilon_{\mathcal{H}}}(\gamma) + \mu_{\Gamma_{\mathcal{G}}}(\alpha\beta)\mu_{\Upsilon_{\mathcal{H}}}(\delta) \\ +\mu_{\Upsilon_{\mathcal{H}}}(\delta)\mu_{\Upsilon_{\mathcal{H}}}(\gamma) - 2\mu_{\Gamma_{\mathcal{G}}}(\alpha\beta)\mu_{\Upsilon_{\mathcal{H}}}(\gamma)\mu_{\Upsilon_{\mathcal{H}}}(\delta) \end{pmatrix}}$$

$$(\nu_{\Gamma_{\mathcal{G}}} \circ \nu_{\Gamma_{\mathcal{H}}})((\alpha, \gamma)(\beta, \delta)) = \frac{\mu_{\Gamma_{\mathcal{G}}}(\alpha\beta)\mu_{\Upsilon_{\mathcal{H}}}(\gamma)\mu_{\Upsilon_{\mathcal{H}}}(\delta)}{\begin{pmatrix} \mu_{\Gamma_{\mathcal{G}}}(\alpha\beta)\mu_{\Upsilon_{\mathcal{H}}}(\gamma) + \mu_{\Gamma_{\mathcal{G}}}(\alpha\beta)\mu_{\Upsilon_{\mathcal{H}}}(\delta) + \\ \mu_{\Upsilon_{\mathcal{H}}}(\delta)\mu_{\Upsilon_{\mathcal{H}}}(\gamma) - 2\mu_{\Gamma_{\mathcal{G}}}(\alpha\beta)\mu_{\Upsilon_{\mathcal{H}}}(\gamma)\mu_{\Upsilon_{\mathcal{H}}}(\delta) \end{pmatrix}}$$

$$(\eta_{\Gamma_{\mathcal{G}}} \circ \nu_{\Gamma_{\mathcal{H}}})((\alpha,\gamma)(\beta,\delta)) = \frac{\begin{pmatrix} \eta_{\Gamma_{\mathcal{G}}}(\alpha\beta)+\eta_{\Upsilon_{\mathcal{H}}}(\gamma)+\eta_{\Upsilon_{\mathcal{H}}}(\delta)-2\eta_{\Gamma_{\mathcal{G}}}(\alpha\beta) \\ \eta_{\Upsilon_{\mathcal{H}}}(\gamma)-2\eta_{\Gamma_{\mathcal{G}}}(\alpha\beta)\eta_{\Upsilon_{\mathcal{H}}}(\delta)-2\eta_{\Upsilon_{\mathcal{H}}}(\delta) \\ \eta_{\Upsilon_{\mathcal{H}}}(\gamma)+4\eta_{\Gamma_{\mathcal{G}}}(\alpha\beta)\eta_{\Upsilon_{\mathcal{H}}}(\gamma)\eta_{\Upsilon_{\mathcal{H}}}(\delta) \end{pmatrix}}{\begin{pmatrix} 1-\eta_{\Gamma_{\mathcal{G}}}(\alpha\beta)\eta_{\Upsilon_{\mathcal{H}}}(\gamma)-\eta_{\Gamma_{\mathcal{G}}}(\alpha\beta) \\ \eta_{\Upsilon_{\mathcal{H}}}(\delta)-\eta_{\Upsilon_{\mathcal{H}}}(\delta)\eta_{\Upsilon_{\mathcal{H}}}(\gamma)+2\eta_{\Gamma_{\mathcal{G}}}(\alpha\beta)\eta_{\Upsilon_{\mathcal{H}}} \end{pmatrix}}$$

Example 2: We consider two PDFG's $A=(\Upsilon_A,\Gamma_A)$ and $B=(\Upsilon_B,\Gamma_B)$ of the graphs $A'=(V_A,E_A)$ and $B'=(V_B,E_B)$ respectively, where $V_A=\{x,y,z\}$, $E_A=\{xy,yz\}$, $V_B=\{a,b\}$ and $E_B=\{ab\}$. Then the composition of A and B are shown in Fig. 2(c).

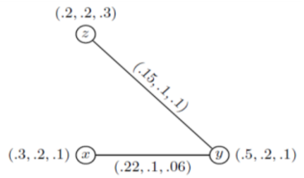

Figure 2(a). PDFG A

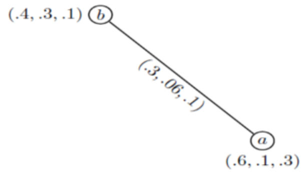

Figure 2(b). PDFG B

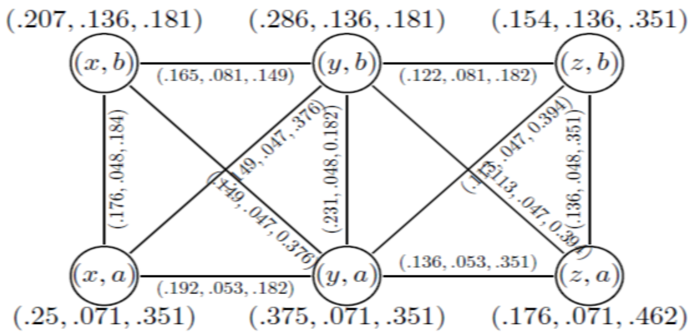

Figure 2(c). PDFG $A \circ B$

Cartesian Product

The cartesian product of two PDFG's $\mathcal{G}=(V_{\mathcal{G}},\Upsilon_{\mathcal{G}},\Gamma_{\mathcal{G}})$ and $\mathcal{H}=(V_{\mathcal{H}},\Upsilon_{\mathcal{H}},\Gamma_{\mathcal{H}})$ of the graphs $G'=(V_G,E_G)$ and $H'=(V_H,E_H)$ respectively, is denoted by $\mathcal{G}\times\mathcal{H}$ and is defined as $(V_G\times V_H,\Upsilon_{\mathcal{G}}\times\Upsilon_{\mathcal{H}},\Gamma_{\mathcal{G}}\times\Gamma_{\mathcal{H}})$, where $\Upsilon_{\mathcal{G}}\times\Upsilon_{\mathcal{H}}=(\mu_{\Upsilon_{\mathcal{G}}}\times\mu_{\Upsilon_{\mathcal{H}}},\nu_{\Upsilon_{\mathcal{G}}}\times\nu_{\Upsilon_{\mathcal{H}}},\eta_{\Upsilon_{\mathcal{G}}}\times\eta_{\Upsilon_{\mathcal{H}}})$ and $\Gamma_{\mathcal{G}}\times\Gamma_{\mathcal{H}}=(\mu_{\Gamma_{\mathcal{G}}}\times\mu_{\Gamma_{\mathcal{H}}},\nu_{\Gamma_{\mathcal{G}}}\times\nu_{\Gamma_{\mathcal{H}}},\eta_{\Gamma_{\mathcal{G}}}\times\eta_{\Gamma_{\mathcal{H}}})$ s. t.,

$\forall(\alpha,\beta)\in V_{\mathcal{G}}\times V_{\mathcal{H}}$,

$$(\mu_{\Upsilon_{\mathcal{G}}}\times\mu_{\Upsilon_{\mathcal{H}}})(\alpha,\beta)=\frac{\mu_{\Upsilon_{\mathcal{G}}}(\alpha)\mu_{\Upsilon_{\mathcal{H}}}(\beta)}{\mu_{\Upsilon_{\mathcal{G}}}(\alpha)+\mu_{\Upsilon_{\mathcal{H}}}(\beta)-\mu_{\Upsilon_{\mathcal{G}}}(\alpha)\mu_{\Upsilon_{\mathcal{H}}}(\beta)}$$

$$(\nu_{\Upsilon_{\mathcal{G}}}\times\nu_{\Upsilon_{\mathcal{H}}})(\alpha,\beta)=\frac{\nu_{\Upsilon_{\mathcal{G}}}(\alpha)\nu_{\Upsilon_{\mathcal{H}}}(\beta)}{\nu_{\Upsilon_{\mathcal{G}}}(\alpha)+\nu_{\Upsilon_{\mathcal{H}}}(\beta)-\nu_{\Upsilon_{\mathcal{G}}}(\alpha)\nu_{\Upsilon_{\mathcal{H}}}(\beta)}$$

$$(\eta_{\Upsilon_{\mathcal{G}}}\times\eta_{\Upsilon_{\mathcal{H}}})(\alpha,\beta)=\frac{\eta_{\Upsilon_{\mathcal{G}}}(\alpha)+\eta_{\Upsilon_{\mathcal{H}}}(\beta)-2\eta_{\Upsilon_{\mathcal{G}}}(\alpha)\eta_{\Upsilon_{\mathcal{H}}}(\beta)}{1-\eta_{\Upsilon_{\mathcal{G}}}(\alpha)\eta_{\Upsilon_{\mathcal{H}}}(\beta)}$$

$\forall\gamma\in V_{\mathcal{G}}$ and $\forall(\alpha,\beta)\in E_{\mathcal{H}}$,

$$(\mu_{\Gamma_{\mathcal{G}}} \times \mu_{\Gamma_{\mathcal{H}}})((\gamma,\alpha)(\gamma,\beta)) = \frac{\mu_{\Upsilon_{\mathcal{G}}}(\gamma)\mu_{\Gamma_{\mathcal{H}}}(\alpha\beta)}{\mu_{\Upsilon_{\mathcal{G}}}(\gamma) + \mu_{\Gamma_{\mathcal{H}}}(\alpha\beta) - \mu_{\Upsilon_{\mathcal{G}}}(\gamma)\mu_{\Gamma_{\mathcal{H}}}(\alpha\beta)}$$

$$(\nu_{\Gamma_{\mathcal{G}}} \times \nu_{\Gamma_{\mathcal{H}}})((\gamma,\alpha)(\gamma,\beta)) = \frac{\nu_{\Upsilon_{\mathcal{G}}}(\gamma)\nu_{\Gamma_{\mathcal{H}}}(\alpha\beta)}{\nu_{\Upsilon_{\mathcal{G}}}(\gamma) + \nu_{\Gamma_{\mathcal{H}}}(\alpha\beta) - \nu_{\Upsilon_{\mathcal{G}}}(\gamma)\nu_{\Gamma_{\mathcal{H}}}(\alpha\beta)}$$

$$(\eta_{\Gamma_{\mathcal{G}}} \times \eta_{\Gamma_{\mathcal{H}}})((\gamma,\alpha)(\gamma,\beta)) = \frac{\eta_{\Upsilon_{\mathcal{G}}}(\gamma) + \eta_{\Gamma_{\mathcal{H}}}(\alpha\beta) - 2\eta_{\Upsilon_{\mathcal{G}}}(\gamma)\eta_{\Gamma_{\mathcal{H}}}(\alpha\beta)}{1 - \eta_{\Upsilon_{\mathcal{G}}}(\gamma)\eta_{\Gamma_{\mathcal{H}}}(\alpha\beta)}$$

$\forall \gamma \in V_{\mathcal{H}}$ and $\forall (\alpha,\beta) \in E_{\mathcal{G}}$,

$$(\mu_{\Gamma_{\mathcal{G}}} \times \mu_{\Gamma_{\mathcal{H}}})((\alpha,\gamma)(\beta,\gamma)) = \frac{\mu_{\Upsilon_{\mathcal{H}}}(\gamma)\mu_{\Gamma_{\mathcal{G}}}(\alpha\beta)}{\mu_{\Upsilon_{\mathcal{H}}}(\gamma) + \mu_{\Gamma_{\mathcal{G}}}(\alpha\beta) - \mu_{\Upsilon_{\mathcal{H}}}(\gamma)\mu_{\Gamma_{\mathcal{G}}}(\alpha\beta)}$$

$$(\nu_{\Gamma_{\mathcal{G}}} \times \nu_{\Gamma_{\mathcal{H}}})((\alpha,\gamma)(\beta,\gamma)) = \frac{\nu_{\Upsilon_{\mathcal{H}}}(\gamma)\nu_{\Gamma_{\mathcal{G}}}(\alpha\beta)}{\nu_{\Upsilon_{\mathcal{H}}}(\gamma) + \nu_{\Gamma_{\mathcal{G}}}(\alpha\beta) - \nu_{\Upsilon_{\mathcal{H}}}(\gamma)\nu_{\Gamma_{\mathcal{G}}}(\alpha\beta)}$$

$$(\eta_{\Gamma_{\mathcal{G}}} \times \eta_{\Gamma_{\mathcal{H}}})((\alpha,\gamma)(\beta,\gamma)) = \frac{\eta_{\Upsilon_{\mathcal{H}}}(\gamma) + \eta_{\Gamma_{\mathcal{G}}}(\alpha\beta) - 2\eta_{\Upsilon_{\mathcal{H}}}(\gamma)\eta_{\Gamma_{\mathcal{G}}}(\alpha\beta)}{1 - \eta_{\Upsilon_{\mathcal{H}}}(\gamma)\eta_{\Gamma_{\mathcal{G}}}(\alpha\beta)}$$

$\forall (\alpha,\beta)(\gamma,\delta) \in (V_{\mathcal{G}} \times V_{\mathcal{H}}) - E$,

$(\mu_{\Gamma_{\mathcal{G}}} \times \mu_{\Gamma_{\mathcal{H}}})((\alpha,\beta)(\gamma,\delta)) = 0$, $(\nu_{\Gamma_{\mathcal{G}}} \times \nu_{\Gamma_{\mathcal{H}}})((\alpha,\beta)(\gamma,\delta)) = 0$,

$(\eta_{\Gamma_{\mathcal{G}}} \times \eta_{\Gamma_{\mathcal{H}}})((\alpha,\beta)(\gamma,\delta)) = 0$.

Remark 1: The cartesian product of two PDFG's is not necessarily a DFG.

Complement of a PDFG

Let $\mathcal{G} = (V_G, \Upsilon_{\mathcal{G}}, \Gamma_{\mathcal{G}})$ be a PDFG of the graph $G = (V_G, E_G)$. Then the complement of $\mathcal{G}$ is represented as $\mathcal{G}^c = (V_G, \Upsilon_{\mathcal{G}^c}, \Gamma_{\mathcal{G}^c})$ and is defined as follows:

$\mu_{\Upsilon_{\mathcal{G}}} = \mu_{\Upsilon_{\mathcal{G}^c}}$, $\nu_{\Upsilon_{\mathcal{G}}} = \nu_{\Upsilon_{\mathcal{G}^c}}$ and $\eta_{\Upsilon_{\mathcal{G}}} = \eta_{\Upsilon_{\mathcal{G}^c}}$

$$\mu_{\Gamma_{\mathcal{G}^c}}(ab) = \frac{\mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{G}}}(b)}{\mu_{\Upsilon_{\mathcal{G}}}(a) + \mu_{\Upsilon_{\mathcal{G}}}(b) - \mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{G}}}(b)}, \text{ if } \mu_{\Gamma_{\mathcal{G}}}(ab) = 0$$

$$= \frac{\mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{G}}}(b)}{\mu_{\Upsilon_{\mathcal{G}}}(a) + \mu_{\Upsilon_{\mathcal{G}}}(b) - \mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{G}}}(b)} - \mu_{\Gamma_{\mathcal{G}}}(ab), \text{ if } 0 < \mu_{\Gamma_{\mathcal{G}}}(ab) \le 1$$

$$\nu_{\Gamma_{\mathcal{G}^c}}(ab) = \frac{\nu_{\Upsilon_{\mathcal{G}}}(a)\nu_{\Upsilon_{\mathcal{G}(b)}}}{\nu_{\Upsilon_{\mathcal{G}}}(a) + \nu_{\Upsilon_{\mathcal{G}}}(b) - \nu_{\Upsilon_{\mathcal{G}}}(a)\nu_{\Upsilon_{\mathcal{G}(b)}}}, \text{ if } \nu_{\Gamma_{\mathcal{G}}}(ab) = 0$$

$$= \frac{\nu_{\Upsilon_{\mathcal{G}}}(a)\nu_{\Upsilon_{\mathcal{G}}}(b)}{\nu_{\Upsilon_{\mathcal{G}}}(a) + \nu_{\Upsilon_{\mathcal{G}}}(b) - \nu_{\Upsilon_{\mathcal{G}}}(a)\nu_{\Upsilon_{\mathcal{G}(b)}}} - \nu_{\Gamma_{\mathcal{G}}}(ab), \text{ if } 0 < \nu_{\Gamma_{\mathcal{G}}}(ab) \le 1$$

$$\eta_{\Gamma_{\mathcal{G}^c}}(ab) = \frac{\eta_{\Upsilon_{\mathcal{G}}}(a) + \eta_{\Upsilon_{\mathcal{G}}}(b) - 2\eta_{\Upsilon_{\mathcal{G}}}(a)\eta_{\Upsilon_{\mathcal{G}}}(b)}{1 - \eta_{\Upsilon_{\mathcal{G}}}(a)\eta_{\Upsilon_{\mathcal{G}}}(b)}, \text{ if } \eta_{\Gamma_{\mathcal{G}}}(ab) = 0$$

$$= \eta_{\Gamma_{\mathcal{G}}}(ab) - \frac{\eta_{\Upsilon_{\mathcal{G}}}(a) + \eta_{\Upsilon_{\mathcal{G}}}(b) - 2\eta_{\Upsilon_{\mathcal{G}}}(a)\eta_{\Upsilon_{\mathcal{G}}}(b)}{1 - \eta_{\Upsilon_{\mathcal{G}}}(a)\eta_{\Upsilon_{\mathcal{G}}}(b)}, \text{ if } 0 < \eta_{\Gamma_{\mathcal{G}}}(ab) \le 1$$

Example 3: We consider the PDFG $\mathcal{G}=(\Upsilon_A,\Gamma_A)$ of the graph $G'=(V_G,E_G)$ where $V_G=\{x,y,z\}$, $E_G=\{yz\}$. Then complement $\mathcal{G}^c$ of $\mathcal{G}$ shown in Fig. 3(a) and Fig. 3(b) respectively.

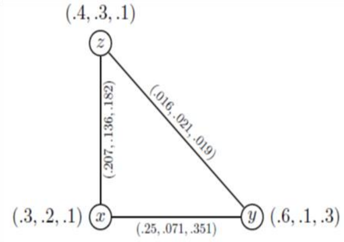

Figure 3(a). PDFG $\mathcal{G}$

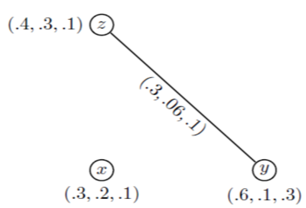

Figure 3(b). $\mathcal{G}^c$

Theorem 2: Let $\mathcal{G}=(V_G,\Upsilon_{\mathcal{G}},\Gamma_{\mathcal{G}})$ be a PDFG of the graph $G=(V_G,E_G)$. Then $(\mathcal{G}^c)^c=\mathcal{G}$.

Homomorphism, Isomorphism, Weak isomorphism, Co-weak isomorphism

Let us consider two PDFG's $\mathcal{G}=(V_{\mathcal{G}},\Upsilon_{\mathcal{G}},\Gamma_{\mathcal{G}})$ and $\mathcal{H}=(V_{\mathcal{H}},\Upsilon_{\mathcal{H}},\Gamma_{\mathcal{H}})$ of the graphs $G'=(V_G,E_G)$ and $H'=(V_H,E_H)$, where $\Upsilon_{\mathcal{G}}=(\mu_{\Upsilon_{\mathcal{G}}},\nu_{\Upsilon_{\mathcal{G}}},\eta_{\Upsilon_{\mathcal{G}}})$, $\Upsilon_{\mathcal{H}}=(\mu_{\Upsilon_{\mathcal{H}}},\nu_{\Upsilon_{\mathcal{H}}},\eta_{\Upsilon_{\mathcal{H}}})$, $\Gamma_{\mathcal{G}}=(\mu_{\Gamma_{\mathcal{G}}},\nu_{\Gamma_{\mathcal{G}}},\eta_{\Gamma_{\mathcal{G}}})$ and $\Gamma_{\mathcal{H}}=(\mu_{\Gamma_{\mathcal{H}}},\nu_{\Gamma_{\mathcal{H}}},\eta_{\Gamma_{\mathcal{H}}})$.

(Homomorphism)

A mapping $\phi:\mathcal{G}\to\mathcal{H}$ is said to be a homomorphism, if

$\forall\xi\in V_G$ $\mu_{\Upsilon_{\mathcal{G}}}(\xi)\le\mu_{\Upsilon_{\mathcal{H}}}(\phi(\xi))$, $\nu_{\Upsilon_{\mathcal{G}}}(\xi)\le\nu_{\Upsilon_{\mathcal{H}}}(\phi(\xi))$ and $\eta_{\Upsilon_{\mathcal{G}}}(\xi)\ge\eta_{\Upsilon_{\mathcal{H}}}(\phi(\xi))$;

$\forall(ab)\in E_G$ $\mu_{\Gamma_{\mathcal{G}}}(ab)\le\mu_{\Gamma_{\mathcal{H}}}(\phi(ab))$, $\nu_{\Gamma_{\mathcal{G}}}(ab)\le\mu_{\Gamma_{\mathcal{H}}}(\phi(ab))$ and $\eta_{\Gamma_{\mathcal{G}}}(ab)\ge\mu_{\Gamma_{\mathcal{H}}}(\phi(ab))$.

(Isomorphism)

A mapping $\phi:\mathcal{G}\to\mathcal{H}$ is said to be an isomorphism, if

$\forall\xi\in V_G$ $\mu_{\Upsilon_{\mathcal{G}}}(\xi)=\mu_{\Upsilon_{\mathcal{H}}}(\phi(\xi))$, $\nu_{\Upsilon_{\mathcal{G}}}(\xi)=\nu_{\Upsilon_{\mathcal{H}}}(\phi(\xi))$ and $\eta_{\Upsilon_{\mathcal{G}}}(\xi)=\eta_{\Upsilon_{\mathcal{H}}}(\phi(\xi))$;

$\forall(ab)\in E_G$

$\mu_{\Gamma_{\mathcal{G}}}(ab)=\mu_{\Gamma_{\mathcal{H}}}(\phi(ab))$, $\nu_{\Gamma_{\mathcal{G}}}(ab)=\mu_{\Gamma_{\mathcal{H}}}(\phi(ab))$ and $\eta_{\Gamma_{\mathcal{G}}}(ab)=\mu_{\Gamma_{\mathcal{H}}}(\phi(ab))$.

If $\mathcal{G}$ and $\mathcal{H}$ are isomorphism, then we write $\mathcal{G}\cong\mathcal{H}$.

(Weak-isomorphism)

A mapping $\phi:\mathcal{G}\to\mathcal{H}$ is said to be a weak isomorphism, if

ϕ homomorphism;

$\forall\xi\in V_G$ $\mu_{\Upsilon_{\mathcal{G}}}(\xi)=\mu_{\Upsilon_{\mathcal{H}}}(\phi(\xi))$, $\nu_{\Upsilon_{\mathcal{G}}}(\xi)=\nu_{\Upsilon_{\mathcal{H}}}(\phi(\xi))$ and $\eta_{\Upsilon_{\mathcal{G}}}(\xi)=\eta_{\Upsilon_{\mathcal{H}}}(\phi(\xi))$.

(Co-weak isomorphism)

A mapping $\phi:\mathcal{G}\to\mathcal{H}$ is said to be a co-weak isomorphism, if

ϕ is a homomorphism;

$\forall(ab)\in E_G$ $\mu_{\Gamma_{\mathcal{G}}}(ab)=\mu_{\Gamma_{\mathcal{H}}}(\phi(ab))$, $\nu_{\Gamma_{\mathcal{G}}}(ab)=\mu_{\Gamma_{\mathcal{H}}}(\phi(ab))$ and $\eta_{\Gamma_{\mathcal{G}}}(ab)=\mu_{\Gamma_{\mathcal{H}}}(\phi(ab))$.

Self-complementary

Let $\mathcal{G}=(V_G,\Upsilon_{\mathcal{G}},\Gamma_{\mathcal{G}})$ be a PDFG of the graph $G=(V_G,E_G)$. Then $\mathcal{G}$ is said to be self-complementary if $\mathcal{G}\cong\mathcal{G}^c$.

Theorem 3: Let $\mathcal{G}=(V_G,\Upsilon_{\mathcal{G}},\Gamma_{\mathcal{G}})$ be a self-complementary PDFG of the graph $G=(V_G,E_G)$. Then

$$\sum_{s_0\neq t_0}\mu_{\Gamma_{\mathcal{G}}}(s_0t_0)=\frac{1}{2}\sum_{s_0\neq t_0}\frac{\mu_{\Upsilon_{\mathcal{G}}}(s_0)\mu_{\Upsilon_{\mathcal{G}}}(t_0)}{\mu_{\Upsilon_{\mathcal{G}}}(s_0)+\mu_{\Upsilon_{\mathcal{G}}}(t_0)-\mu_{\Upsilon_{\mathcal{G}}}(s_0)\mu_{\Upsilon_{\mathcal{G}}}(t_0)},$$

$$\sum_{s_0\neq t_0}\nu_{\Gamma_{\mathcal{G}}}(s_0t_0)=\frac{1}{2}\sum_{s_0\neq t_0}\frac{\nu_{\Upsilon_{\mathcal{G}}}(s_0)\nu_{\Upsilon_{\mathcal{G}}}(t_0)}{\nu_{\Upsilon_{\mathcal{G}}}(s_0)+\nu_{\Upsilon_{\mathcal{G}}}(t_0)-\nu_{\Upsilon_{\mathcal{G}}}(s_0)\nu_{\Upsilon_{\mathcal{G}}}(t_0)}$$

$$\sum_{s_0\neq t_0}\eta_{\Gamma_{\mathcal{G}}}(s_0t_0)=\frac{1}{2}\sum_{s_0\neq t_0}\frac{\eta_{\Upsilon_{\mathcal{G}}}(s_0)+\mu_{\Upsilon_{\mathcal{G}}}(t_0)-2\eta_{\Upsilon_{\mathcal{G}}}(s_0)\eta_{\Upsilon_{\mathcal{G}}}(t_0)}{1-\mu_{\Upsilon_{\mathcal{G}}}(s_0)\mu_{\Upsilon_{\mathcal{G}}}(t_0)}.$$

Proof:

Let $\mathcal{G}$ be a self-complementary graph. So, $\exists$ an isomorphism $\phi:\mathcal{G}\to\mathcal{G}^c$ s. t.,

$\forall\xi\in V_G$ $\mu_{\Upsilon_{\mathcal{G}}}(\xi)=\mu_{\Upsilon_{\mathcal{G}^c}}(\phi(\xi))$, $\nu_{\Upsilon_{\mathcal{G}}}(\xi)=\nu_{\Upsilon_{\mathcal{G}^c}}(\phi(\xi))$

$\forall(ab)\in E_G$, $\mu_{\Gamma_{\mathcal{G}}}(ab)=\mu_{\Gamma_{\mathcal{G}^c}}(\phi(ab))$, $\nu_{\Gamma_{\mathcal{G}}}(ab)=\mu_{\Gamma_{\mathcal{G}^c}}(\phi(ab))$ and $\eta_{\Gamma_{\mathcal{G}}}(ab)=\mu_{\Gamma_{\mathcal{G}^c}}(\phi(ab))$.

Now, we know that,

$$\mu_{\Gamma_{\mathcal{G}^c}}(\phi(a)\phi(b))=\frac{\mu_{\Upsilon_{\mathcal{G}^c}}(\phi(a))\mu_{\Upsilon_{\mathcal{G}^c}}(\phi(b))}{\mu_{\Upsilon_{\mathcal{G}^c}}(\phi(a))+\mu_{\Upsilon_{\mathcal{G}^c}}(\phi(b))-\mu_{\Upsilon_{\mathcal{G}^c}}(\phi(a))\mu_{\Upsilon_{\mathcal{G}^c}}(\phi(b))}-\mu_{\Gamma_{\mathcal{G}}}(\phi(a)\phi(b))$$

Or, $\mu_{\Gamma_{\mathcal{G}}}(ab))=\dfrac{\mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{G}}}(b)}{\mu_{\Upsilon_{\mathcal{G}}}(a)+\mu_{\Upsilon_{\mathcal{G}}}(b)-\mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{G}}}(b)}-\mu_{\Gamma_{\mathcal{G}}}(\phi(a)\phi(b))$

Or, $\sum_{a\neq b}\mu_{\Gamma_{\mathcal{G}}}(ab))+\sum_{a\neq b}\mu_{\Gamma_{\mathcal{G}}}(\phi(a)\phi(b))=\sum_{a\neq b}\dfrac{\mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{G}}}(b)}{\mu_{\Upsilon_{\mathcal{G}}}(a)+\mu_{\Upsilon_{\mathcal{G}}}(b)-\mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{G}}}(b)}$

Or, $2\sum_{a\neq b}\mu_{\Gamma_{\mathcal{G}}}(ab))=\sum_{a\neq b}\dfrac{\mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{G}}}(b)}{\mu_{\Upsilon_{\mathcal{G}}}(a)+\mu_{\Upsilon_{\mathcal{G}}}(b)-\mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{G}}}(b)}$

Or, $\sum_{a\neq b}\mu_{\Gamma_{\mathcal{G}}}(ab))=\dfrac{1}{2}\sum_{a\neq b}\dfrac{\mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{G}}}(b)}{\mu_{\Upsilon_{\mathcal{G}}}(a)+\mu_{\Upsilon_{\mathcal{G}}}(b)-\mu_{\Upsilon_{\mathcal{G}}}(a)\mu_{\Upsilon_{\mathcal{G}}}(b)}$.

In similar way we can proof the remaining two results. This completes the proof.

4. Conclusion

In this paper, we have introduced the new concept of Picture dombi fuzzy graph. We have proposed some operators of union, join, composition and cartesian product of any two dombi picture fuzzy graphs and investigate many interesting properties of Dombi picture fuzzy graph. Finally, we define the complement Picture dombi fuzzy graph and the isomorphic properties on it. The concept of picture dombi fuzzy graphs can be used to model in several areas of expert systems, transportation, artificial neural networks, pattern recognition and computer networks.

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