A Comparative Numerical Study between Minorant Functions and Line Search Methods in Penalty Methods for Linear Optimization

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ABSTRACT

The aim of this paper is to present a comparative numerical study between the minorant functions and line search methods in computing the step size in the penalty method for linear optimization. The minorant functions were confirmed by many interesting numerical experimentations to be more beneficial than the classical line search methods.

Keywords-linear optimization; penalty methods; line search; minorant function

I. INTRODUCTION

Optimization is an especially interesting topic in data science. Linear Optimization (LO) or Linear Programming (LP) problems form an important class of optimization problems, aiming to find the feasible region and optimize the solution in order to have the highest or lowest value of a function. It is now well established as an important and very active branch of applied mathematics. The wide applicability of LO models and its richness as a mathematical theory that underlines these models and the methods developed to solve them have been the driving forces behind the rapid and continuing evolution of the subject. The template is general enough to express many different problems in engineering, industry, commerce, economics, business administration, physical sciences, and mathematics, or in any other area where decisions (in a board sense) must be taken in some complex (or conflicting) situation that can be represented by a mathematical model. There are two classes of methods for the resolution of LO problems, simplex method and interior point methods. In this study, we are interested in interior point methods. These are efficient methods developed to solve LO and NP problems. Several algorithms have been proposed to solve LO problems. Some fundamental classes of interior point methods are the affine method, the projective method with the potential reduction of Karmarkar and its alternatives [5, 9], central trajectory methods, and penalty/barrier methods [6, 13]. Our work is based on the latter type of interior point methods for solving LO problems.

In this paper, we propose a logarithmic barrier interior-point method for solving LO problems. In fact, the main difficulty to be anticipated in establishing iterations in such a method will come from the determination and computation of the step-size. The aim of this paper is to present a comparative numerical study between the line search methods and the minorant function to compute the step-size along the direction in barrier logarithmic methods.

II. PROBLEM FORMULATION

We consider the following LO problem

$$\begin{cases} \min_x b^T x \\ A^T x \geq c, x \in R^m. \end{cases} \quad (1)$$

where $A \in R^{m\times n}$, such that $rangA = m < n$, $c \in R^n$ and $b \in R^m$. The problem (1) is the dual of the following linear program:

$$\begin{cases} \max_y c^T y \\ A^T y = b \\ y \in R^n,\ y \geq 0. \end{cases} \quad (2)$$

The problem (1) can be written in the following standard form:

$$\begin{cases} \min_x b^T x \\ A^T x - c = s \\ x \in R^m,\ s \in R^n,\ s \geq 0. \end{cases} \quad (3)$$

One of the advantages of problem (1) with respect to its dual problem (2) is that the variable of the objective function is a vector instead of a matrix. Furthermore, under certain convenient hypothesis, the resolution of problem (1) is equivalent to (2) in the sense that the optimal solution of one of the two problems can be reduced directly from the other through the application of the theorem of the sladeness complementary [15]. In the rest of the paper, the following are denoted:

- $X = \{x \in R^m \ : A^T x - c \geq 0\}$, the set of feasible solutions of (1).
- $\hat{X} = \{x \in R^m \ : A^T x - c > 0\}$, the set of strictly feasible solutions of (1).
- $F = \{y \in R^n : Ay = b, y \geq 0\}$, the set of feasible solutions of (2).
- $\hat{F} = \{y \in R^n : Ay = b, y > 0\}$, the set of strictly feasible solutions of (2).

Let $u,\ v \in R^n$. Their scalar product is defined by:

$\langle u, v\rangle = u^T v = \sum_{i=1}^{n} u_i v_i$

We suppose that the sets $\hat{X}$ and $\hat{F}$ are not empty.

A. *The Perturbed Problem of* (1)

Problem (1) is approximated by the following perturbed problem:

$$\begin{cases} \min_x f_\eta(x) \\ x \in R^m. \end{cases} \quad (4)$$

where $\eta > 0$ is the the penalty parameter and f_η is the barrier function defined by:

$$f_\eta(x) = \begin{cases} b^T x + n\eta \ln\eta - \eta \ln \sum_{i=1}^{n} \langle e_i, A^T x - c\rangle & \text{if } A^T x - c > 0 \\ +\infty & \text{if not} \end{cases}$$

where $(e_1, e_2, \dots, e_n)$ is the canonical base in R^n. We are interested in solving the problem (4).

The idea of this new approach consists to introduce one original process to calculate the step-size based on minorant functions. The main advantage of (4) resides in the strict convexity of its objective function and its feasible domain. Consequently, the conditions of optimality are necessary and sufficient. This fosters theoretical and numerical studies of the problem. In the next section, the existence and uniqueness of the optimal solution of (4) is proved and we show its convergence to (1), in particular the behavior of its optimal value and its primal solutions when $\eta \to 0$, then $\lim_{\eta\to 0} x_\eta = x^*$ is an optimal solution of (1). In Section III, we propose an interior point algorithm based on the Newton's approach which allows us to solve the nonlinear system resulting from the optimality conditions. The iteration of this algorithm is of descent type, defined by: $x_{k+1} = x_k + \alpha_k d_k$, where d_k is the descent direction and α_k is the step-size. Also, we present different steps-sizes by minimizing a minorant function which approximates the one-dimensional function $\theta(\alpha_k) = \min_{\alpha > 0} f(x + \alpha d)$. The last Section is dedicated to the presentation of comparative numerical tests to illustrate the effectiveness of our approach and to determine the most efficient algorithm.

The main advantage of (4) resides in the strict convexity of its objective function and its feasible domain. Consequently, the conditions of optimality are necessary and sufficient. This fosters theoretical and numerical studies of the problem. Before this, it is necessary to show that (4) has at least an optimal solution.

B. *Convergence of the Perturbed Problem*

Firstly, we give the following definition:

Definition: Let f be a function defined from R^m to $R \cup \{\infty\}$. f is called inf-compact if for all $\eta > 0$, the set $X_\eta(f) = \{x \in R^m :\ f(x) \leq \eta\}$ is compact, which comes in particular to say that its cone of recession is reduced to zero.

To prove that (4) has an optimal solution, we show that f_η is inf-compact. For that, it is enough to prove that the cone of recession $\hat{X}\left(\left(f_\eta\right)_\infty\right) = \left\{d \in R^n :\ \left(f_\eta\right)_\infty(d) \leq 0\right\}$, is reduced to the origin, i.e. $\left(\left(f_\eta\right)_\infty(d) \leq 0\right) \Rightarrow$ (d = 0), where $\left(f_\eta\right)_\infty$ is defined by:

$$\left(f_\eta\right)_\infty(d) = \lim_{\alpha\to+\infty} \frac{f_\eta(x + \alpha d) - f_\eta(x)}{\alpha} = b^T d.$$

This needs the following proposition:

Proposition: $d = 0$ whenever $b^T d \leq 0$ and $A^T d \in \hat{X}$.

Then, the problem (4) has an optimal solution. We know that the Hessian matrix $H = \nabla^2 f_\eta(x)$ is positive definite, then the problem (4) is strictly convex, and if it has an optimal solution, then it is unique. We have:

$$f_\eta(x) = b^T x + n\eta \ln\eta - \eta \ln \sum_{i=1}^{n} \langle e_i, A^T x - c\rangle$$

Then:

$$\nabla f_\eta(x) = b - \eta \sum_{i=1}^{n} \frac{Ae_i}{\langle e_i, A^T x - c\rangle}$$

and:

$$\nabla^2 f_\eta(x) = \eta \sum_{i=1}^{n} \frac{Ae_i (Ae_i)^T}{\langle e_i, A^T x - c\rangle^2}$$

As f_η is inf-compact and strictly convex, therefore the problem (4) admits a unique optimal solution. We denote by $x(\eta)$ or x_η the unique optimal solution of (4).

C. *Convergence of the Perturbed Problem to the Initial Problem*

For $x \in \hat{X}$, let's introduce the symmetrical definite positive matrix B_i of rank $m, i = 1, \dots, n$ and the lower triangular matrix L, such that $B_i = Ae_i (Ae_i)^T = LL^T$, which implies that H is a positive definite matrix. In what follows, we will be interested in the behavior of the optimal value and the optimal solution $x(\eta)$ of problem (4).

Proposition: For $\eta > 0$, let x_η an optimal solution of the problem (4) then there is an optimal solution of (1) $x \in X$, such that, $\lim_{\eta\to 0} x_\eta = x$.

Remark: We know that if one of the problems (1) and (2) has an optimal solution, and the values of their objective functions are equal and finite, the other problem has an optimal solution.

III. RESOLUTION OF THE PERTURBED PROBLEM

In this part, we are interested in the numerical solution of problem (4). We use a logarithmic barrier interior point method. This method types are based on the optimality conditions which are necessary and sufficient, and consist of constructing a sequence of iterate $x_{k+1} = x_k + \alpha_k d_k$, where x_η is an optimal solution of (4) if it satisfies the following condition:

$$\nabla f_\eta(x_\eta) = 0 \quad (5)$$

To solve (5) we use the Newton's approach which means to find in each iteration a vector $x_{\eta k} + d_k$ checking the following linear system:

$$H_k d_k = -\nabla f_\eta(x_{\eta k}). \quad (6)$$

As $H_k = \nabla^2 f_\eta(x_{\eta k})$ is a symmetric positive definite matrix, the Cholesky methods and the conjugate gradient methods are the best convenient for solving (6). To ensure the convergence of the algorithm towards an optimal solution x^* of (4), it should be made sure that all the iterations $x_{\eta k} + d_k$ remain strictly feasible. For that, we introduce a step-size α_k checking the condition:

$$A^T(x_{\eta k} + \alpha_k d_k) - c > 0$$

A. Effectual Computation of the Step Size

There are two main techniques used for computing the displacement step α_k.

1) Line Search

These methods try out a sequence of candidate values of α_k, stopping to accept one of these values when some conditions are satisfied. An ideal choice of the step-size is the global minimization of the one-dimensional function $\theta(\cdot)$ defined by:

$$\theta(\alpha_k) = \min_{\alpha>0} f(x + \alpha d)$$

The most used line search methods are Wolfe, Goldstein—Armijo, Fibonacci, etc. Unfortunately, these methods have big computational cost.

2) Principle of Approximate Function

A minorant function $\tilde{\varphi}$ must be close to:

$$\varphi(\alpha) = \frac{1}{\eta}\left[f_\eta(x + \alpha d) - f_\eta(x)\right]$$

which must give the $\min_\alpha \tilde{\varphi}(\alpha)$ in $[0, \hat{\alpha}]$ by a simple and easy manner, which permits the computation of the step-size at each iteration in a relatively short time and with a smaller number of instructions in contrast to line search technique.

Authors in [12] gave a simple form for the function, which is presented in the following proposition:

Proposition [12]: Let $\hat{\alpha} = sup\{\alpha: 1 + z_i\alpha\}$ with $z_i = \frac{\langle e_i, A^T d\rangle}{\langle e_i, A^T x - c\rangle}, \forall i = 1, \ldots, n$. For all $\alpha \in [0, \hat{\alpha}]$, the following function $\varphi(\alpha)$ is well defined:

$$\varphi(\alpha) = n(\textstyle\sum_{i=1}^n z_i)\alpha - \|z\|^2\alpha - \sum_{i=1}^n \ln(1 + z_i\alpha)$$

such that, $\varphi(\alpha)$verifies the following properties:

$$\|z\|^2 = n(\bar{z}^2 + \sigma_z^2) = \varphi''(0) = -\varphi'(0), \varphi(0) = 0$$

B. Minorant Function

Authors in [10] proposed 3 minorant functions in 2019. In this paper, we are interested in their best minorant function defined as:

$$\tilde{\varphi}_1(\alpha) = \delta\alpha - (n-1)\ln(1 + \beta\alpha) - \ln(1 + \gamma\alpha)$$

with:

$$\delta = n\bar{z} - \|z\|^2$$
$$\beta = \bar{z} - \frac{\sigma_z}{\sqrt{n-1}}$$
$$\gamma = \bar{z} + \sigma_z\sqrt{n-1}.$$

In addition, they proved that the minorant function $\tilde{\varphi}_1$ is defined and convex on $[0, \hat{\alpha}]$, $\varphi(\alpha) > \tilde{\varphi}_1(\alpha)$ ($\tilde{\varphi}_1$ minorant function on $[0, \hat{\alpha}]$), and the function $\tilde{\varphi}_1$ verifies the following properties:

$$\|z\|^2 = n(\bar{z}^2 + \sigma_z^2) = \tilde{\varphi}''(0) = -\tilde{\varphi}'(0), \tilde{\varphi}(0) = 0$$

The minimum of $\tilde{\varphi}_1$ is obtained in $\bar{\alpha}_i = \alpha_{opt}$, such that, ${\tilde{\varphi}_1}'(0) = 0$. We are then coming back to solve the second order following equation: $\alpha^2 - 2b\alpha + c = 0$, with:

$$b = \frac{1}{2}\left(\frac{n}{\delta} - \frac{1}{\beta} - \frac{1}{\gamma}\right) \text{ and } c = \frac{-\|z\|^2}{\beta\gamma\delta}$$

The roots of this equation are of the type $\bar{\alpha} = b \pm \sqrt{b^2 - c}$. Let's take one root of the two that belong to $[0, \hat{\alpha}[$. Thus, the $\bar{\alpha}$ is explicitly computed, then, we consider it belongs to the interval $(0, \hat{\alpha} - \varepsilon)$ and $\varphi'(\alpha) < 0$, with $\varepsilon > 0$ being a fixed precision.

Remark: The calculation of $\bar{\alpha}$ is performed by a dichotomous procedure, in the cases where $\bar{\alpha}_i \notin (0, \hat{\alpha} - \varepsilon)$, and $\varphi'(\alpha) > 0$, as follows:

Put $\alpha = 0$ and $b = \hat{\alpha} - \varepsilon$ while $|b - a| > 10^{-4}$. If $\varphi\left(\frac{a+b}{2}\right) < 0$, then $b = \frac{a+b}{2}$. Else $a = \frac{a+b}{2}$, so $\bar{\alpha} = b$. This calculation guarantees a better approximation of the minimizer of $\tilde{\varphi}'(\alpha)$ while remaining in the domain of φ.

Proposition [10]**:** Let x_{k+1} and x_k be two strictly feasible solutions of (4) obtained respectively at the $k+1$ and k iterations, so we have $f_\eta(x_{k+1}) < f_\eta(x_k)$.

IV. ALGORITHM DESCRIPTION AND NUMERICAL RESULTS

In this section, we present the algorithm of our approach to obtain an optimal solution $\bar{x}$ to problem (1) and some numerical tests.

A. The Algorithm

In this section, we present the algorithm of our approach to obtain an optimal solution $\bar{x}$ to the problem (1).

For simplicity, we consider x_k instead of $x_{\eta k}$and x instead of x_η.

```
Begin algorithm
Initialization
x_0 is a strictly feasible solution of
(1), d_0 ∈ R^m and ε > 0 is a given precision.
Iteration
While |b^T d_k| > ε do
Solve the system H_k d_k = −∇f_η(x_ηk).
Compute the step-size using the strategy
of minorant function or line search
method.
Take the new iterate x_{k+1} = x_k + α_k d_k.
Take k = k + 1.
End while
End algorithm
```

This approach tries to reduce the number of iterations and the time of calculation. Some examples are presented below.

B. Numerical Results

To measure the numerical performance of the proposed methods, we present a numerical comparison of the results obtained by the proposed algorithm using the minorant function given in [10] to compute the step-size and those obtained by using the line search Wolfe's method. We use examples with fixed and variable sizes to carry out the numerical tests.

The following examples are taken from the literature [1, 4, 8] and were implemented in MATLAB. We took $\varepsilon = 1.0e - 006$. In the result table:

- (size) represents the size of the example.
- (itrat) represents the number of iterations necessary to obtain an optimal solution.
- (time) represents the time of computation in seconds (s).
- (mf st) represents the strategy that uses minorant function.
- (lr st) represents the strategy that uses Wolfe's line search.

Recall that the considered problem (1) is:

$$\begin{cases} \min_x b^T x \\ A^T x \geq c, x \in R^n \end{cases}$$

We note that the matrices used in the numerical tests are full matrices.

1) Examples with Fixed Size

Example 01:

$$A = \begin{bmatrix} 2 & 3 & 1 & 2 \\ 3 & 0 & -2 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \text{ and } c = [4 \quad 1 \quad 2 \quad 0]^T$$

- The initial strictly feasible point is $x^0 = [1 \quad 1.5 \quad 1 \quad 1]^T$.
- The optimal solution is $x^0 = [0 \quad 0.67 \quad 0 \quad 0]^T$.

Example 02:

$$A = \begin{bmatrix} 2 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and $c = [3 \quad -1 \quad 1 \quad 0 \quad 0 \quad 0]^T$

- The initial strictly feasible point is $x^0 = [1 \quad 1 \quad 2]^T$.
- The optimal solution is $x^* = [0.5 \quad 0.0713 \quad 0.5]^T$.

Example 03:

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$$

and $c = [4 \quad -2 \quad -2 \quad 0 \quad 0 \quad 0]^T$

- The initial strictly feasible point is $x^0 = [0.5 \quad 1 \quad 1]^T$.
- The optimal solution is $x^0 = [0 \quad 0 \quad 0]^T$.

Example 04:

$$A = \begin{bmatrix} 1 & 0 & -4 & 3 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 3 & 1 & 0 & -1 & 3 & 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 5 & -3 & 3 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 1 & -5 & 0 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 1 & 1 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & -3 & 2 & -1 & 4 & 5 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$b = [1 \quad 4 \quad 4 \quad 5 \quad 7 \quad 5]^T$

$c = [-4 \quad -5 \quad -1 \quad -3 \quad 5 \quad -8 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$

- The initial strictly feasible point is: $x^0 = [-2 \quad 4 \quad 1 \quad 1 \quad 1 \quad 1]^T$
- The optimal solution is $x^* = [0.5 \quad 1.5 \quad 0 \quad 0 \quad 1.5 \quad 0]^T$

Example 05:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 5 & 4 & 3 & 2 & 1 & 1 & 0 & 0 & 0 & 0 \\ 6 & 7 & 8 & 9 & 10 & 5 & 2 & 8 & 3 & 1 & 0 & 1 & 0 & 0 & 0 \\ 11 & 12 & 13 & 14 & 15 & 6 & 7 & 80 & 90 & 10 & 0 & 0 & 1 & 0 & 0 \\ 1 & 10 & 20 & 30 & 40 & 50 & 60 & 80 & 90 & 10 & 0 & 0 & 0 & 1 & 0 \\ 3 & 9 & 27 & 60 & 45 & 60 & 75 & 8 & 9 & 46 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$b_i = 10^4, i = 1, \ldots, 5,$

$c = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T$

- The initial strictly feasible point is $x^0 = [1 \quad 1 \quad 1 \quad 1 \quad 1]^T$
- The optimal solution is $x^0 = [0 \quad 0 \quad 0.0888 \quad 0 \quad 0.0078]^T$

The results of the last examples are given in Table I.

TABLE I. EXAMPLES WITH FIXED SIZE

Size	mf st		ls st	
	itrat	time	itrat	time
2×4	5	0.032	13	0.130
3×6	6	0.044	33	0.128
3×6	7	0.051	34	0.132
6×12	9	0.055	34	0.306
5×15	8	0.048	22	0.170

2) Example with Variable Size

$n = 2m,$ For $i, j = 1, \dots, m,$

$A[i,j] = 0$ if $i \neq j$ or $(i+1) \neq j$

$A[i,j] = A[i, i+m] = 1, \quad b[i] = 2.$

- The initial strictly feasible point is $x^0 = [1 \quad 1 \quad \dots \quad 1]^T$.
- The optimal solution is $x^* = [0 \quad 0 \quad \dots \quad 0]^T$.

Table II resumes the obtained results.

TABLE II. EXAMPLES WITH FIXED SIZE

Size	mf st		ls st	
	itrat	time	itrat	time
50×100	1	0.031	49	8.5512
100×200	1	0.053	50	32.6145
200×400	2	0.088	51	91.6524
400×800	3	0.096	52	161.4374
500×1000	3	0.12	52	411.8901

V. CONCLUSION

The conducted numerical study clearly shows that the strategy of the minorant function seems to be more efficient than that of the line search in time and number of iterations. Our future work will consider further improving the computational time of the logarithmic barrier algorithm by proposing another, better, approximate function. But the extensions would be envisaged to the nonlinear, and not necessarily relevant to the LO problem.

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