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Convergence of a Three-step Iteration Scheme to the Common Fixed Points of Mixed-Type Total Asymtotically Nonexpansive Mappings in Uniformly Convex Banach Spaces

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Abstract. We propose a three-step iteration scheme of hybrid mixed-type for three total asymptotically nonexpansive self mappings and three total asymptotically nonexpansive nonself mappings. In addition, we establish some weak convergence theorems of the scheme to the common fixed point of the mappings in uniformly convex Banach spaces. Our results extend and generalize numerous results currently in literature.

1. Introduction

Let K be a nonempty subset of a real Banach space E. Let $T : K \longrightarrow K$ be a nonlinear mapping, we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of six mappings S_1, S_2, S_3, T_1, T_2 *and* T_3 will be denoted by $\mathcal{F} = \cap_{i=1}^{3}(F(T_i) \cap F(S_i))$.

Definition 1.1. *A mapping $T : K \longrightarrow K$ is said to asymptotically nonexpansive [6] if there exists a sequence $\{k_n\}$ in $[1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that*

$$\|T^n(x) - T^n(y)\| \leq k_n\|x - y\|, \forall x, y \in \mathbb{N} \tag{1.1}$$

. In 1972, the class of asymptotically nonexpansive mapping was introduced by Goebel and Kirk [6]. They proved that if K is a nonempty closed convex subset of a uniformly convex Banach space and $T$is an asymptotically nonexpansive mapping of K, then T has a fixed point.

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Definition 1.2. *A mapping T is said to be total asymptotically nonexpansive [1] if*

$$\|T^n(x) - T^n(y)\| \leq \|x - y\| + \mu_n\phi(\|x - y\|) + \nu_n, \forall x, y \in K, \forall n \in \mathbb{N}, \tag{1.2}$$

where $\{\mu_n\}$and $\{\nu_n\}$ are nonnegative real sequences such that $\mu_n \to 0$ and $\nu_n \to 0$ as $n \to \infty$ and ϕ is a strictly increasing continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$.

From the above definitions, we see that the class of total asymptotically nonexpansive mappings includes the class of asymptotically nonexpansive mapping as a special case; see [4] for more details. Each asymptotically nonexpansive mapping is total asymptotically nonexpansive mapping with $\nu_n = 0, \mu_n = k_n - 1 \; for \; all \; n \geq 1, \phi(t) = t, t \geq 0$.

Definition 1.3. *A subset K of a Banach space E is said to be a retract of E if there exists a continuous mapping $P : E \longrightarrow K$ $(called\ retraction)$ such that $P(x) = x$ for all $x \in K$. If, in addition P is nonexpansive, then P is said to be nonexpansive retraction of E. If $P : E \longrightarrow K$ is a retraction, then $P^2 = P$. A retract of a Hausdorff space must be a closed subset. Every closed convex subset of a uniformly convex Banach space is a retract.*

In 2012, Yolacan and Kiziltune [18] defined the following:

Definition 1.4. *Let K be a nonempty and closed convex subset of a Banach space E. A nonself mapping $T : K \to E$ is said to be total asymptotically nonexpansive mapping if there exist sequences $k_n^{(1)}$ and $k_n^{(2)}$ in $[0, \infty)$ with $k_n^{(1)} \to 0$ and $k_n^{(2)} \to 0$ as $n \to \infty$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that*

$$\|T(PT)^{n-1}(x) - T(PT)^{n-1}(y)\| \leq \|x - y\| + k^1_{(n)}\phi(\|x - y\|) + k_n^{(2)}, \forall x, y \in K, n \in \mathbb{N}. \tag{1.3}$$

Chidume et al. [3] studied the following iterative scheme in 2004:

$$\begin{aligned} x_1 &= x \in K \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1}x_n + (1 - \alpha_n)x_n), n \geq 1, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, K is a nonempty closed convex subset of of a real uniformly convex Banach space E, P is a nonexpansive retraction of E onto K, and proved some strong and weak convergence theorems for asymptotically nonexpansive nonself mappings in the intermediate sense in the framework of uniformly convex Banach spaces.
In 2006, Wang [17] generalised the iteration process (1.4) as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1 - \alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n), \\ y_n &= P((1 - \beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n), n \geq 1, \end{aligned} \tag{1.5}$$

where $T_1, T_2 : K \longrightarrow E$ are two asymptotically nonexpansive nonself mappings, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0, 1)$, and proved some weak and strong convergence theorems for asymptotically nonexpansive nonself mappings.
In 2012, Guo et al [8] generalised the iteration process (1.5) as follows:

$$\begin{aligned} x_1 &= x \in K, \\ x_{n+1} &= P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1} y_n), \\ y_n &= P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1} x_n), n \geq 1, \end{aligned} \tag{1.6}$$

where $T_1, T_2 : K \longrightarrow E$ are two asymptotically nonexpansive nonself mappings, $S_1, S_2 : K \longrightarrow E$ are two asymptotically nonexpansive self mappings and $\{\alpha_n\}, \{\beta_n\}$ are real sequences in $[0, 1)$, and proved some strong and weak convergence theorems for mixed-type asymptotically nonexpansive mappings.

Hybrid Mixed-Type Iteration Scheme

Let E be a real uniformly convex Banach space, K a nonempty closed convex subset of E and $P : E \longrightarrow K$ a nonexpansive retraction of E onto K. Let $S_1, S_2, S_3 : K \longrightarrow K$ be three total asymptotically nonexpansive self mappings and $T_1, T_2, T_3 : K \longrightarrow E$ be three total asymptotically nonexpansive nonself mappings. Then, the hybrid iteration scheme for the above mentioned mappings is as follows:

$$\begin{cases} x_1 = x \in K; \\ x_{n+1} = P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1} y_n); \\ y_n = P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1} z_n); \\ z_n = P((1-\gamma_n)S_3^n x_n + \gamma_n T_3(PT_3)^{n-1} x_n), \end{cases} \tag{1.7}$$

where $\{\alpha_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ are real sequences in $[0, 1)$.
The aim of this paper is to study this new hybrid mixed-type iteration scheme (1.7), prove demiclosedness principle for total asymptotically nonexpansive nonself map and establish some convergence theorems for mixed-type mappings in the setting of uniformly convex Banach spaces.

2. Preliminary

For the sake of convenience, we restate the following concepts and results:
Let E be a Banach space with its dimension greater than or equal to 2. The modulus of convexity of E is a function $\delta_E(\varepsilon) : (0, 2] \longrightarrow (0, 2]$ defined by

$$\delta_E(\varepsilon) = \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1, \|y\| = 1, \varepsilon = \|x-y\|\}.$$

A Banach space E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$, for all $\varepsilon \in (0, 2]$.
We recall the following:

Definition 2.1. *(see [19]: Let $\varrho = \{x \in E : \|x\| = 1\}$ and let $E^\star$ be the dual of E. The space E has Gateaux differentiable norm if $\lim_{n\to\infty} \frac{\|x+ty\|-\|x\|}{t}$ exists $\forall x, y \in \varrho$.*

Definition 2.2. *(see [19]: The space E has Frechet differentiable norm [15] if for each $x \in \varrho$, the limit of the norm above exists and is attained uniformly for all $y \in \varrho$, and in this case, it is also well known that*

$$\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 \le \frac{1}{2}\|x+h\|^2 \le \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|x\|), \tag{2.1}$$

$\forall x, y \in E$, where J is the Frechet derivative of the functional $\frac{1}{2}\|\cdot|^2$ at $x \in E, \langle\cdot\rangle$ is the pairing between E and $E^\star$ and b is an increasing function defined on $[0, \infty)$ such that $\lim_{t\to\infty} \frac{b(t)}{t} = 0$.

Definition 2.3. : *The space E has Opial condition [10] if for any sequence $\{x_n\}$ in E, x_n converges to x weakly, then it follows that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $x \neq y$.*

Examples of Banach spaces satisfying Opial conditions are Hilbert spaces and all spaces $l^p(1 < p < \infty)$. On the other hand, $L^p[0, \pi]$ with $1 < p \neq 2$ fails to satify Opial condition.

Definition 2.4. *: A mapping $T : K \longrightarrow K$ is said to be demiclosed at 0, if for any sequence $\{x_n\}$ in K, the condition that x_n converges weakly to $x \in K$ and Tx_n converges strongly to 0 implies $Tx = 0$.*

Definition 2.5. *: A Banach space has the Kadec-Klec property [14] if for every sequence x_n in $E, x_n \to x$ weakly and $\|x_n\| \to \|x\|$, then it follows that $\|x_n - x\| \to 0$.*

Next, we state the following useful lemmas which will be needed in order to prove our main results.

Lemma 2.1. *(see [16]): Let $\{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty$ and $\{\gamma_n\}_{n=1}^\infty$ be sequences of nonnegative numbers satisfying the inequality:*

$$\alpha_{n+1} \le (1 + \beta_n)\alpha_n + \gamma_n, \forall n \ge 1. \tag{2.2}$$

If $\sum_{n=1}^\infty \beta_n < \infty$ and $\sum_{n=1}^\infty \gamma_n < \infty$, then

(1) $\lim_{n\to\infty} \alpha_n$ *exists*
(2) *In particular, if $\{\alpha_n\}_{n=1}^\infty$ has a subsequence which converges strongly to 0, then $\lim_{n\to\infty} \alpha_n = 0$.*

Lemma 2.2. *(see [14]): Let E be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for each $n \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences in E such that*

$$\limsup_{n\to\infty} \|x_n\| \leq r, \limsup_{n\to\infty} \|y_n\| \leq r \text{ and } \lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r, \tag{2.3}$$

hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

Lemma 2.3. *(see [14]): Let E be a real reflexive Banach space such that its dual $E^\star$ has the Kadec-Klec property. Let $\{x_n\}$ be a bounded sequence in E and $p, q \in \omega_\omega(x_n)$ (where $\omega_\omega(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$). Suppose $\lim_{n\to\infty} \|tx_n + (1-t)p - q\|$ exists for all $t \in [0,1]$. Then, $p = q$.*

Lemma 2.4. *(see [14]): Let K be a nonempty convex subset of a uniformly convex Banach space E. Then, there exists a strictly incraesing continous convex function $\phi : [0,\infty) \to [0,\infty)$ $with$ $\phi(0) = 0$ such that for each Lipshitizian mapping $T : C \longrightarrow C$ with the Lipschiz constant L,*

$$\|tTx - (1-t)Ty - T(tx - (1-t)y)\| \leq L\phi^{-1}(\|x-y\| - \frac{1}{L}\|Tx - Ty\|) \tag{2.4}$$

for all $x, y \in K$ and for all $t \in [0.1]$.

Lemma 2.5. *(see [2]) Let E be a uniformly convex Banach space, K a nonempty bounded close convex subset of E. Then, there exists a strictly increasing continous convex function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ such that for any Lipschitizian mapping $T : K \longrightarrow E$ with Lipschitz constant $L \geq 1$ and elements $\{x_n\}_{j=i}^n$ in K and any nonnegative numbers $\{t_j\}_{j=1}^n$ with $\sum_{j=1}^n t_j = 1$, the following inequality holds:*

$$\|T(\sum_{j=1}^n t_j x_j) - \sum_{j=1}^n t_j Tx_j\| \leq L\phi^{-1}\{max_{1\leq j,k\leq n}(\|x_j - x_k\| - L^{-1}\|Tx_j - Tx_k\|)\}$$

Lemma 2.6. *(see [21]) If the sequence $\{x_n\}_{n=1}^\infty$ converges weakly to x, then there exists a sequence of convex combination $y_j = \sum_{k=1}^{n(j)} \lambda_k^{(j)} x_{k+j}$, $\lambda_k^{(j)} \geq 0$ and $\sum_{k=1}^{n(j)} \lambda^{(j)} = 1$, such that $\|y_j - x\| \to 0$. as $n \to \infty$.*

3. Main Results

Lemma 3.1. (*$Demiclosedness$ $Principle$ for $Nonself$ $Total$ $Asymptotically$ $Nonexpansive$ $Maps$*) *Let K be a nonempty closed convex and bounded subset of a uniformly convex Banach space E and $T : K \longrightarrow E$ be L-Lipschitz continuous and total asymptotically nonexpansive mapping with the function $\phi : [0,\infty) \longrightarrow [0,\infty)$ (such that $\phi(0) = 0$) and nonnegative sequences $\{k_n^{(1)}\}, \{k_n^{(2)}\}$ such that $k_n^{(1)}, k_n^{(2)} \to 0$ as $n \to \infty$. Then, $I - T$ is demiclosed at 0.*

Proof. Let $\{x_n\}$ converge weakly to $\omega \in K$ and $\{x_n - Tx_n\}$ converge strongly to 0. We prove that $(I - T)\omega = 0$. Clearly, $\{x_n\}$ is bounded. So, there exists $\rho > 0$ such that $\{x_n\} \subset C = K \cap \overline{B_\rho}(0)$, where $\overline{B_\rho}(0)$ is a closed ball in E with centre 0 and radius ρ. Thus, C is nonempty, closed , bounded and convex subset in K.

Claim: $T(PT)^{n-1}\omega \to \omega$ as $n \to \infty$. In fact, since $\{x_n\}$ converges weakly to ω, by Lemma 6(see [21]), we have for all $n > 1$, there exists a convex combination

$$y_n = \sum_{i=1}^{m(n)} t_i^{(n)} x_{i+n},\ t_i^{(n)} \geq 0 \text{ and } \sum_{i=1}^{m(n)} t_i^{(n)} = 1 \text{ such that } \|y_n - \omega\| \to 0 \text{ as } n \to \infty. \tag{3.1}$$

Also, since $\{x_n - Tx_n\}$ converges to 0, then for any $\epsilon > 0$ and a positive integer $m \geq 1$, there exists $N_1 = N(\epsilon) > 0$ such that

$$\|(I - T)x_n\| < \frac{\epsilon}{1+m}, \forall n \geq N_1. \tag{3.2}$$

Hence, $\forall n \geq N_1$, using Definition 1.4 and the fact that P is nonexpansive , we have the following estimates:

For arbitrary but fixed $j \geq 1$, we have

$$\begin{aligned}
\|x_n - T(PT)^{(j-1)}x_n\| &\leq \|(I-T)x_n\| + \|(T - T(PT))x_n\| \\
&\quad + \|(T(PT) - T(PT)^2)x_n\| \\
&\quad + \|(T(PT)^2 - T(PT)^3)x_n\| \\
&\quad + \cdots + \|(T(PT)^{j-2} - T(PT)^{j-1}))x_n\| \\
&\leq \|(I-T)x_n\| + (\|(I-T)x_n\| + \mu_n^{(1)}\phi(\|(I-T)x_n\|) \\
&\quad + \xi_n^{(1)}) + (\|(I-T)x_n\| + \mu_n^{(2)}\phi(\|(I-T)x_n\|) + \xi_n^{(2)}) \\
&\quad + (\|(I-T)x_n\| + \mu_n^{(3)}\phi(\|(I-T)x_n\|) + \xi_n^{(3)}) \\
&\quad + \cdots + (\|(I-T)x_n\| + \mu_n^{(j-1)}\phi(\|(I-T)x_n\|) + \xi_n^{(j-1)}) \\
&= \|(I-T)x_n\| + \sum_{j=1}^{m-1} \|(I-T)x_n\| + \sum_{j=1}^{m-1} \mu_n^{(j)}\phi(\|(I-T)x_n\|) \\
&\quad + \sum_{j=1}^{m-1} \xi_n^{(j)} \\
&\leq m\|x_n - Tx_n\| + m\mu_n\phi(\|(I-T)x_n\|) + m\xi_n,
\end{aligned} \tag{3.3}$$

where $\mu_n = \max_{1\leq j\leq m-1}\{\mu_n^{(j)}\}$ and $\xi_n = \max_{1\leq j\leq m-1}\{\xi_n^{(j)}\}$.

From (3.2) and (3.3), we get

$$\|x_n - T(PT)^{j-1}x_n\| < \epsilon. \tag{3.4}$$

Now, since $T : K \longrightarrow E$ is L-Lipschitizian and total asymptotically nonexpansive , so is $T : C \longrightarrow E$. Therefore, $\forall j \geq 1, T(PT)^{j-1} : C \longrightarrow E$ is Lipschitizian mapping with the Lipschitz constant $\mu_j \geq 1$.

In addition,

$$\begin{aligned}
\|T(PT)^{j-1}y_n - y_n\| &= \|T(PT)^{j-1}y_n - \sum_{i=1}^{m(n)} t_i^{(n)}T(PT)^{j-1}x_{i+n} + \sum_{i=1}^{m(n)} t_i^{(n)}T(PT)^{j-1}x_{i+n} \\
&\quad - \sum_{i=1}^{m(n)} t_i^{(n)}x_{i+n}\| \\
&\leq \|T(PT)^{j-1}y_n - \sum_{i=1}^{m(n)} t_i^{(n)}T(PT)^{j-1}x_{i+n}\| \\
&\quad + \sum_{i=1}^{m(n)} t_i^{(n)}\|T(PT)^{j-1}x_{i+n} - x_{i+n}\|. \qquad (3.5)
\end{aligned}$$

Using (3.4), we get

$$\sum_{i=1}^{m(n)} t_i^{(n)}\|T(PT)^{j-1}x_{i+n} - x_{i+n}\| < \epsilon, \forall n \geq N. \qquad (3.6)$$

Furthermore, by Lemma 2.5, there exists a strictly increasing continous function $\phi : [0, \infty) \longrightarrow [0, \infty)$ with $\phi(0) = 0$ such that for all $n \geq N$, we have

$$\begin{aligned}
\|T(PT)^{j-1}y_n - \sum_{i=1}^{m(n)} t_i^{(n)}T(PT)^{j-1}x_{i+n}\| &= \|T(PT)^{j-1}(\sum_{i=1}^{m(n)} t_i^{(n)}x_{i+n}) - \sum_{i=1}^{m(n)} t_i^{(n)}T(PT)^{j-1}x_{i+n}\| \\
&\leq \mu_j\phi^{-1}\{max_{1\leq j,k\leq n}(\|x_{i+n} - x_{i+k}\| \\
&\quad -\mu_j^{-1}\|T(PT)^{J-1}x_{i+n} - T(PT)^{J-1}x_{k+n}\|)\} \\
&= \mu_j\phi^{-1}\{max_{1\leq j,k\leq n}(\|x_{i+n} - T(PT)^{J-1}x_{i+n} \\
&\quad +T(PT)^{J-1}x_{i+n} - T(PT)^{J-1}x_{k+n} \\
&\quad +T(PT)^{J-1}x_{k+n} - x_{i+k}\| \\
&\quad -\mu_j^{-1}\|T(PT)^{J-1}x_{i+n} - T(PT)^{J-1}x_{k+n}\|)\} \\
&\leq \mu_j\phi^{-1}\{max_{1\leq j,k\leq n}(\|x_{i+n} - T(PT)^{J-1}x_{i+n}\| \\
&\quad +\|T(PT)^{J-1}x_{i+n} - T(PT)^{J-1}x_{k+n}\| \\
&\quad +\|T(PT)^{J-1}x_{k+n} - x_{i+k}\| \\
&\quad -\mu_j^{-1}\|T(PT)^{J-1}x_{i+n} - T(PT)^{J-1}x_{k+n}\|)\} \\
&\leq \mu_j\phi^{-1}\{max_{1\leq j,k\leq n}(\epsilon + \epsilon + (1 - \mu_j^{-1}) \\
&\quad \times\|T(PT)^{J-1}x_{i+n} - T(PT)^{J-1}x_{k+n}\|)\} \\
&\leq \mu_j\phi^{-1}\{max_{1\leq j,k\leq n}(\epsilon + \epsilon + (1 - \mu_j^{-1})\mu_j \\
&\quad \times\|x_{i+n} - x_{k+n}\|\} \\
&\leq \mu_j\phi^{-1}\{max_{1\leq j,k\leq n}(\epsilon + \epsilon + (1 - \mu_j^{-1})\mu_j \\
&\quad \times(\|x_{i+n}\| + \|x_{k+n}\|\}.
\end{aligned}$$

Thus,

$$\|T(PT)^{j-1}y_n - \sum_{i=1}^{m(n)} t_i^{(n)} T(PT)^{j-1}x_{i+n}\| \leq \mu_j \phi^{-1}(\epsilon + \epsilon + 2r(1-\mu_j^{-1})\mu_j), \tag{3.7}$$

since x_{i+n} and x_{k+n} are both in C.

Also, (3.5), (3.6) and (3.7) imply that

$$\|T(PT)^{j-1}y_n - y_n\| \leq \mu_j \phi^{-1}(\epsilon + \epsilon + 2r(1-\mu_j^{-1})\mu_j). \tag{3.8}$$

Taking $\limsup_{n\to\infty}$ on both sides of (3.8) and noting that $\epsilon > 0$ is arbitrary, we have that

$$\limsup_{n\to\infty} \|T(PT)^{j-1}y_n - y_n\| \leq \mu_j \phi^{-1}(2r(1-\mu_j^{-1})\mu_j). \tag{3.9}$$

On the other hand, for any $j \geq 1$, it follows from (3.1) that

$$\begin{aligned} \|T(PT)^{j-1}\omega - \omega\| &\leq \|T(PT)^{j-1}\omega - T(PT)^{j-1}y_n\| + \|T(PT)^{j-1}y_n - y_n\| + \|y_n - \omega\| \\ &\leq \mu_j\|y_n - \omega\| + \|T(PT)^{j-1}y_n - y_n\| + \|y_n - \omega\|. \end{aligned} \tag{3.10}$$

Taking $\limsup_{n\to\infty}$ on both sides of the above inequality and using (3.1) and (3.9), we have

$$\|T(PT)^{j-1}\omega - \omega\| \leq \mu_j \phi^{-1}(2r(1-\mu_j^{-1})\mu_j).$$

Again, taking $\limsup_{j\to\infty}$ on both sides of the above inequality, we have

$$\limsup_{j\to\infty} \|T(PT)^{j-1}\omega - \omega\| \leq \phi^{-1}(0) = 0,$$

which implies that $\|T(PT)^{j-1}\omega - \omega\| \to 0$ as $j \to \infty$, and hence proving our claim. By continuity of TP, we have that

$$\lim_{j\to\infty} TP(T(PT)^{j-1}\omega) = TP\omega = T\omega = \omega.$$

This completes the proof. □

Lemma 3.2. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E. Let $S_1, S_2, S_3 : K \longrightarrow K$ be three total asymptotically nonexpansive self mapping with sequences $\{k_n^{(1)}\}, \{k_n^{(2)}\}, \{k_n^{(3)}\} \in [1,\infty)$,*
$\{w_n^{(1)}\}, \{w^{(2)\}_n}, \{w_n^{(3)}\} \in [1,\infty)$ and $T_1, T_2, T_3 : K \longrightarrow E$ are three total asymptotically nonexpansive nonself mappings with sequences $\{\mu_n^{(1)}\}, \{\mu_n^{(2)}\}, \{\mu_n^{(3)}\} \in [1,\infty)$, $\{\nu_n^{(1)}\}, \{\nu_n^{(2)}\}$,
$\{\nu_n^{(3)}\} \in [1,\infty)$. Let $\{x_n\}$ be the sequence defined by (1.7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences $\in [0,1)$. Suppose $\mathcal{F} = (F(T_i) \cap F(S_i)) \neq \emptyset$. If the following conditions hold:

i. $\sum_{n=1}^{\infty} k_n^{(1)} < \infty, \sum_{n=1}^{\infty} k_n^{(2)} < \infty, \sum_{n=1}^{\infty} k_n^{(3)} < \infty, \sum_{n=1}^{\infty} \mu_n^{(1)} < \infty, \sum_{n=1}^{\infty} \mu_n^{(2)} < \infty,$
$\sum_{n=1}^{\infty} \mu_n^{(3)} < \infty, \sum_{n=1}^{\infty} \nu_n^{(1)} < \infty, \sum_{n=1}^{\infty} \nu_n^{(2)} < \infty, \sum_{n=1}^{\infty} \nu_n^{(3)} < \infty,$

ii. *There exists a constant $M > 0$ such that$\Psi(t) = \phi(t) \leq Mt, t \leq 0$.*

Then, $\lim_{n\infty} \|x_n - q\|$ and $\lim_{n\infty} d(x_n - F)$both exist for all $q \in F$.

Proof. Set $h_n = \max(k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, \mu_n^{(1)}, \mu_n^{(2)}, \mu_n^{(3)})$, $M = \max(M_1, M_2, M_3, M_4, M_5, M_6)$ and $\theta_n = \max(\nu_n^{(1)}, \nu_n^{(2)}, \nu_n^{(3)}, \omega_n^{(1)}, \omega_n^{(2)}, \omega_n^{(3)})$. Then, $\sum_{n=1}^{\infty} h_n < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$. For any $q \in F$, it follows from (3.1) that

$$\begin{aligned}
\|z_n - q\| &= |P((1-\beta_n)S_3^n x_n + \beta_n T_3(PT_3)^{n-1}x_n) - P(q)\| \\
&\leq \|(1-\beta_n)S_3^n x_n + \beta_n T_3(PT_3^{n-1}x_n - q\| \\
&= \|(1-\beta_n)S_3^n x_n + \beta_n q - q - \beta_n q + \beta_n T_3(PT_3)^{n-1}x_n\| \\
&= \|(1-\beta_n)S_3^n x_n - (1-\beta_n)q + \beta_n(T_3(PT_3)^{n-1}x_n - q)\| \\
&= \|(1-\beta_n)(S_3^n x_n - q) + \beta_n(T_3(PT_3)^{n-1}x_n - q)\| \qquad (3.11) \\
&\leq (1-\beta_n)\|S_3^n x_n - q\| + \beta_n\|T_3(PT_3)^{n-1}x_n - q\| \\
&\leq (1-\beta_n)[\|x_n - q\| + k_n^{(3)}\Psi(\|x_n - q\|) + \omega_n^{(3)}] + \beta_n[\|x_n - q\| + \mu_n^{(3)}\phi(\|x_n - q\|) \\
&\quad + \nu_n^{(3)}] \\
&= (1-\beta_n)\|x_n - q\| + (1-\beta_n)h_n\Psi(\|x_n - q\|) + (1-\beta_n)\theta_n + \beta_n\|x_n - q\| \\
&\quad + \beta_n h_n \phi(\|x_n - q\|) + \beta_n\theta_n \\
&\leq (1-\beta_n)(1 + h_n M_5)\|x_n - q\| + \beta_n(1 + h_n M_6)\|x_n - q\| + \theta_n \\
&\leq (1-\beta_n)(1 + h_n M)\|x_n - q\| + \beta_n(1 + h_n M)\|x_n - q\| + \theta_n \\
&\leq (1 + h_n M)\|x_n - q\| + \theta_n. \qquad (3.12)
\end{aligned}$$

Also, form (1.7), we get

$$\begin{aligned}
\|y_n - q\| &= |P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}z_n) - P(q)\| \\
&\leq \|(1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}x_n - q\| \\
&= \|(1-\beta_n)S_2^n x_n + \beta_n q - q - \beta_n q + \beta_n T_2(PT_2)^{n-1}z_n\| \\
&= \|(1-\beta_n)S_2^n x_n - (1-\beta_n)q + \beta_n(T_2(PT_2)^{n-1}z_n - q)\| \qquad (3.13) \\
&= \|(1-\beta_n)(S_2^n x_n - q) + \beta_n(T_2(PT_2)^{n-1}z_n - q)\| \\
&\leq (1-\beta_n)\|S_2^n x_n - q\| + \beta_n\|T_2(PT_2)^{n-1}z_n - q\| \\
&\leq (1-\beta_n)[\|x_n - q\| + k_n^{(2)}\Psi(\|x_n - q\|) + \omega_n^{(2)}] + \beta_n[\|z_n - q\| + \mu_n^{(2)}\phi(\|x_n - q\|) \\
&\quad + \nu_n^{(2)}] \\
&= (1-\beta_n)\|x_n - q\| + (1-\beta_n)h_n\Psi(\|x_n - q\|) + (1-\beta_n)\theta_n + \beta_n\|x_n - q\| \\
&\quad + \beta_n h_n \phi(\|z_n - q\|) + \beta_n\theta_n \\
&\leq (1-\beta_n)(1 + h_n M_3)\|x_n - q\| + \beta_n(1 + h_n M_4)\|z_n - q\| + \theta_n \\
&\leq (1-\beta_n)(1 + h_n M)\|x_n - q\| + \beta_n(1 + h_n M)\|z_n - q\| + \theta_n. \qquad (3.14)
\end{aligned}$$

Putting (3.12) into (3.14), we have

$$\begin{aligned} \|y_n - q\| &\leq (1-\beta_n)(1+h_nM)\|x_n - q\| + \beta_n(1+h_nM)[(1+h_nM)\|x_n - q\| + \theta_n] + \theta_n \\ &= (1+h_nM)[(1-\beta_n)\|x_n - q\| + \beta_n((1+h_nM)\|x_n - q\| + \theta_n)] + \theta_n \\ &= (1+h_nM)[(1-\beta_n+\beta_n+\beta_n h_nM))\|x_n - q\| + \theta_n)] + \theta_n \\ &\leq (1+h_nM)[1+h_nM))\|x_n - q\| + \theta_n)] + \theta_n \\ &= (1+h_nM)^2\|x_n - q\| + (2+h_nM)\theta_n. \end{aligned} \tag{3.15}$$

Again, using (1.7), we have

$$\begin{aligned} \|x_{n+1} - q\| &= |P((1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n) - P(q)\| \\ &\leq \|(1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n - q\| \\ &= \|(1-\alpha_n)S_1^n x_n + \alpha_n q - q - \alpha_n q + \alpha_n T_1(PT_1)^{n-1}y_n\| \\ &= \|(1-\alpha_n)S_1^n x_n - (1-\alpha_n)q + \alpha_n(T_1(PT_1)^{n-1}y_n - q)\| \\ &= \|(1-\alpha_n)(S_1^n x_n - q) + \alpha_n(T_1(PT_1)^{n-1}y_n - q)\| \qquad (3.16) \\ &\leq (1-\alpha_n)\|S_1^n x_n - q\| + \alpha_n\|T_1(PT_1)^{n-1}y_n - q\| \\ &\leq (1-\alpha_n)[\|x_n - q\| + k_n^{(1)}\Psi(\|x_n - q\|) + \omega_n^{(1)}] + \alpha_n[\|y_n - q\| \\ &\quad + \mu_n^{(1)}\phi(\|y_n - q\|) + \nu_n^{(1)}] \\ &\leq (1-\alpha_n)\|x_n - q\| + (1-\alpha_n)h_n\Psi(\|x_n - q\|) + (1-\alpha_n)\theta_n + \alpha_n\|y_n - q\| \\ &\quad + \alpha_n h_n\phi(\|y_n - q\|) + \alpha_n\theta_n \\ &\leq (1-\alpha_n)(1+h_nM_1)\|x_n - q\| + \alpha_n(1+h_nM_2)\|y_n - q\| + \theta_n \\ &\leq (1-\alpha_n)(1+h_nM)\|x_n - q\| + \alpha_n(1+h_nM)\|y_n - q\| + \theta_n. \qquad (3.17) \end{aligned}$$

Putting (3.15) into (3.17), we obtain

$$\begin{aligned} \|x_{n+1} - q\| &\leq (1-\alpha_n)(1+h_nM)\|x_n - q\| + \alpha_n(1+h_nM)[(1+h_nM)^2\|x_n - q\| \\ &\quad + (2+h_nM)\theta_n] + \theta_n] \\ &= (1+h_nM)\|x_n - q\| - \alpha_n(1+h_nM)\|x_n - q\| + \alpha_n(1+h_nM)^3\|x_n - q\| \\ &\quad + \alpha_n(1+h_nM)(2+h_nM)\theta_n + \theta_n \\ &\leq [1+(3+3h_nM+h_n^2M^2)h_nM]\|x_n - q\| + [1+(1+h_nM)(2+h_nM]\theta_n \\ &= (1+\delta_n)\|x_n - q\| + \rho_n. \end{aligned} \tag{3.18}$$

where $\delta_n = 1+(3+3h_nM+h_n^2M^2)h_nM$ and $\rho_n = [1+(1+h_nM)(2+h_nM]\theta_n$. Since $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty}\rho_n < \infty$, it follows from lemma 2.1 that $\lim_{n\to\infty}\|x_n - q\|$ exists.

Now taking the infimum over all $q \in F$ in (3.18), we get

$$d(x_{n+1}, F) \leq (1+\delta_{n)} d(x_n, F) + \rho_n, \forall n \in N. \tag{3.19}$$

Again, since $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} \rho_n < \infty$, it follows from lemma 2.1 and (3.19) that $\lim_{n\to\infty} d(x_n, F)$ exists. This completes the proof. □

Lemma 3.3. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E. Let $S_1, S_2, S_3 : K \longrightarrow K$ be three total asymptotically nonexpansive self mapping with sequences $\{k_n^{(1)}\}, \{k_n^{(2)}\}, \{k_n^{(3)}\} \in [1, \infty), \{w_n^{(1)}\}, \{w^{(2)\}_n}, \{w_n^{(3)}\} \in [1, \infty)$ and $T_1, T_2, T_3 : K \longrightarrow E$ are three total asymptotically nonexpansive nonself mappings with sequences $\{\mu_n^{(1)}\}, \{\mu_n^{(2)}\}, \{\mu_n^{(3)}\} \in [1, \infty), \{\nu_n^{(1)}\}, \{\nu_n^{(2)}\}, \{\nu_n^{(3)}\} \in [1, \infty)$. Let $\{x_{n\}}$ be the sequence defined by (1.7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences $\in [0, 1)$. Suppose $\mathcal{F} = (F(T_i) \cap F(S_i)) \neq \emptyset$. If the following conditions hold:*

i. $\sum_{n=1}^{\infty} k_n^{(1)} < \infty, \sum_{n=1}^{\infty} k_n^{(2)} < \infty, \sum_{n=1}^{\infty} k_n^{(3)} < \infty, \sum_{n=1}^{\infty} \mu_n^{(1)} < \infty, \sum_{n=1}^{\infty} \mu_n^{(2)} < \infty,$ $\sum_{n=1}^{\infty} \mu_n^{(3)} < \infty, \sum_{n=1}^{\infty} \nu_n^{(1)} < \infty, \sum_{n=1}^{\infty} \nu_n^{(2)} < \infty, \sum_{n=1}^{\infty} \nu_n^{(3)} < \infty,$

ii. $\|x - T_1(PT_1)^{n-1}y\| \leq \|S_1^n x - T_1(PT_1)^{n-1}y\|, \|x - T_2(PT_2)^{n-1}y\| \leq \|S_2^n x - T_2(PT_2)^{n-1}y\|,$ $\|x - T_3(PT_3)^{n-1}y\| \leq \|S_3^n x - T_3(PT_3)^{n-1}y\|$

iii. *There exists a constant* $M_1, M_2 > 0$ *such that* $\Psi(t) \leq M_1 t, \ \phi(t) \leq M_2 t, \ t \geq 0.$

Then, $\lim_{n\infty} \|x_n - S_i x_n\| = 0$ *and* $\lim_{n\infty} \|x_n - T_i x_n\| = 0$*, for* $i = 1, 2, , 3.$

Proof. Set $h_n = \max(k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, \mu_n^{(1)}, \mu_n^{(2)}, \mu_n^{(3)})$, $M = \max(M_1, M_2, M_3, M_4, M_5, M_6)$ and $\theta_n = \max(\nu_n^{(1)}, \nu_n^{(2)}, \nu_n^{(3)}, \omega_n^{(1)}, \omega_n^{(2)}, \omega_n^{(3)})$. Then, $\sum_{n=1}^{\infty} h_n < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$. for any given $q \in F$, $\lim_{n\infty} \|x_n - q\|$ exists by lemma 3.2. Now, assume that $\lim_{n\infty} \|x_n - q\| = c$. it follows from (3.15), (3.16) and the fact that $\sum_{n=1}^{\infty} h_n < \infty$ and $\sum_{n=1}^{\infty} \theta_n < \infty$ that

$$\lim \|(1-\alpha_n)(S_1^n x_n - q) + \alpha_n T_1(PT_1)^{n-1} y_n - q)\| = c. \tag{3.20}$$

Also, we have

$$\begin{aligned}
\|S_1^n x_n - q\| &\leq \|x_n - q\| + k_n^{(1)} \Psi(\|x_n - q\|) + \omega_n^{(1)} \\
&\leq \|x_n - q\| + k_n^{(1)} M \|x_n - q\|) + \omega_n^{(1)} \\
&\leq (1 + k_n^{(1)} M)\|x_n - q\| + \omega_n^{(1)} \\
&\leq (1 + h_n M)\|x_n - q\| + \theta_n \\
\Rightarrow \limsup \|S_1^n x_n - q\| &\leq \limsup[(1 + h_n M)\|x_n - q\| + \theta_n] = c. \qquad (3.21)
\end{aligned}$$

Furthermore,

$$\begin{aligned}\|T_1(PT_1)y_n - q\| &\leq \|y_n - q\| + \mu_n^{(1)}\phi(\|y_n - q\|) + \nu_n^{(1)} \\ &\leq \|y_n - q\| + \mu_n^{(1)}M\|y_n - q\|) + \nu_n^{(1)} \\ &\leq (1 + \mu_n^{(1)}M)\|y_n - q\| + \nu_n^{(1)} \\ &\leq (1 + h_nM)\|y_n - q\| + \theta_n\end{aligned}$$

Taking limsup on both sides of (3.15), we obtain

$\limsup \|y_n - q\| \leq c$ and so $\limsup \|T_1(PT_1)y_n - q\| \leq \limsup[(1+h_nM)\|y_n - q\| + \theta_n] \leq c$. Thus,

$$\limsup \|T_1(PT_1)y_n - q\| \leq \limsup[(1 + h_nM)\|y_n - q\| + \theta_n] = c. \tag{3.22}$$

Using lemma 2.2, we get

$$\lim_{n\to\infty} \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| = 0. \tag{3.23}$$

By condition (ii), it follows that

$\|x_n - T_1(PT_1)^{n-1}y_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\|,$

and so from (3.23), we have

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1}y_n\| = 0. \tag{3.24}$$

Also, we have

$$\begin{aligned}\|S_2^n x_n - q\| &\leq \|x_n - q\| + k_n^{(2)}\Psi(\|x_n - q\|) + \omega_n^{(2)} \\ &\leq \|x_n - q\| + k_n^{(2)}M\|x_n - q\|) + \omega_n^{(2)} \\ &\leq (1 + k_n^{(2)}M)\|x_n - q\| + \omega_n^{(2)} \\ &\leq (1 + h_nM)\|x_n - q\| + \theta_n \\ \Rightarrow \limsup \|S_2^n x_n - q\| &\leq \limsup[(1 + h_nM)\|x_n - q\| + \theta_n] = c. \end{aligned} \tag{3.25}$$

Furthermore,

$$\begin{aligned}\|T_2(PT_2)z_n - q\| &\leq \|z_n - q\| + \mu_n^{(2)}\phi(\|z_n - q\|) + \nu_n^{(2)} \\ &\leq \|z_n - q\| + \mu_n^{(2)}M\|z_n - q\|) + \nu_n^{(2)} \\ &\leq (1 + \mu_n^{(2)}M)\|z_n - q\| + \nu_n^{(2)} \\ &\leq (1 + h_nM)\|z_n - q\| + \theta_n\end{aligned}$$

Taking lim sup on both sides of (3.12), we obtain $\limsup_{n\to\infty}\|z_n - q\| \le c$ and so

$$\limsup \|T_2(PT_1)z_n - q\| \le \limsup[(1 + h_n M)\|z_n - q\| + \theta_n] \le c. \tag{3.26}$$

(3.13), (3.25), (3.26) and lemma 2.2 imply

$$\lim_{n\to\infty} \|S_2^n x_n - T_2(PT_2)^{n-1} z_n\| = 0. \tag{3.27}$$

(3.27) and condition (ii) yields

$$\lim_{n\to\infty} \|x_n - T_2(PT_2)^{n-1} z_n\| = 0. \tag{3.28}$$

From (3.11), using the same argument as was used in obtaining (3.27) above, we get

$$\lim_{n\to\infty} \|S_3^n x_n - T_3(PT_3)^{n-1} x_n\| = 0. \tag{3.29}$$

Now, we prove that

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1} x_n\| = \lim_{n\to\infty} \|x_n - T_2(PT_2)^{n-1} x_n\| \lim_{n\to\infty} \|x_n - T_3(PT_3)^{n-1} x_n\| = 0.$$

Indeed, since $\|x_n - T_3(PT_3)^{n-1}x_n\| \le \|S_3^n x_n - T_3(PT_3)^{n-1}x_n\|$, (by condition (ii)), it follows from (3.29) that

$$\lim_{n\to\infty} \|x_n - T_3(PT_3)^{n-1} x_n\| = 0. \tag{3.30}$$

Since, $P(S^n x_n) = S^n x_n$ and $P : E \longrightarrow K$ is a nonexpansive retraction of E onto K, we get

$$\begin{aligned}
\|z_n - S_3^n x_n\| &= \|P((1-\gamma_n)S_3^n x_n + \gamma_n T_3(PT_3)^{n-1}x_n) - S_3^n x_n\| \\
&\le \|(1-\gamma_n)S_3^n x_n + \gamma_n T_3(PT_3)^{n-1}x_n - S_3^n x_n\| \\
&= \| - \gamma_n(S_3^n x_n - \gamma_n T_3(PT_3)^{n-1}x_n)\| \\
&= \gamma_n\|(S_3^n x_n - \gamma_n T_3(PT_3)^{n-1}x_n)\|,
\end{aligned}$$

which by (3.29) gives

$$\lim_{n\to\infty} \|z_n - S_3^n x_n\| = 0. \tag{3.31}$$

Observe that

$$\begin{aligned}
\|z_n - x_n\| &= \|z_n - S_3^n x_n + S_3^n x_n - T_3(PT_3)^{n-1}x_n + T_3(PT_3)^{n-1}x_n - x_n\| \\
&\le \|z_n - S_3^n x_n\| + \|S_3^n x_n - T_3(PT_3)^{n-1}x_n\| \\
&\quad + \|T_3(PT_3)^{n-1}x_n - x_n\|.
\end{aligned} \tag{3.32}$$

Thus, it follows from (3.29), (3.30),(3.31) and (3.32) that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.33}$$

Again, observe that

$$\begin{aligned}\|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| &\leq \|S_2^n x_n - T_2(PT_2)^{n-1}z_n\| + \|T_2(PT_2)^{n-1}z_n - T_2(PT_2)^{n-1}x_n\| \\ &\leq \|S_2^n x_n - T_2(PT_2)^{n-1}z_n\| + (\|z_n - x_n\| + k_n^{(2)}\phi(\|z_n - x_n\|) + \nu_n^{(2)} \\ &\leq \|S_2^n x_n - T_2(PT_2)^{n-1}z_n\| + \|z_n - x_n\| + Mh_n(\|z_n - x_n\|) + \theta_n \\ &= \|S_2^n x_n - T_2(PT_2)^{n-1}z_n\| + (1 + Mh_n)\|z_n - x_n\| + \theta_n. \end{aligned} \tag{3.34}$$

From (3.27),(3.33), (3.34) and the fact that $\sum_{n=1}^{\infty}\theta_n < \infty$, we get

$$\lim_{n\to\infty} \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.35}$$

Since $\|x_n - T_2(PT_2)^{n-1}x_n\| \leq \|S_2^n x_n - T_2(PT_2)^{n-1}x_n\|$ (by condition (ii), it follows from (3.35) that

$$\lim_{n\to\infty} \|x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.36}$$

Also, since $P(S^n x_n) = S^n x_n$ and $P : E \longrightarrow K$ is a nonexpansive retraction of E onto K, we get

$$\begin{aligned}\|y_n - S_2^n x_n\| &= \|P((1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}z_n) - S_2^n x_n\| \\ &\leq \|(1-\beta_n)S_2^n x_n + \beta_n T_2(PT_2)^{n-1}z_n - S_2^n x_n\| \\ &= \| - \beta_n(S_2^n x_n - \beta_n T_2(PT_2)^{n-1}z_n)\| \\ &= \beta_n\|(S_2^n x_n - \beta_n T_2(PT_2)^{n-1}z_n)\|, \end{aligned}$$

which by (3.27) gives

$$\lim_{n\to\infty} \|y_n - S_2^n x_n\| = 0. \tag{3.37}$$

Moreover, since

$$\begin{aligned}\|y_n - x_n\| &= \|y_n - S_2^n x_n + S_2^n x_n - T_2(PT_2)^{n-1}z_n + T_2(PT_2)^{n-1}x_n - z_n\| \\ &\leq \|y_n - S_2^n x_n\| + \|S_2^n x_n - T_2(PT_2)^{n-1}z_n\| + \|T_2(PT_2)^{n-1}z_n - x_n\|, \end{aligned}$$

it follows from (3.27), (3.28) and (3.37) that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.38}$$

Observe that

$$\begin{aligned}\|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| &\leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - T_1(PT_1)^{n-1}x_n\| \\ &\leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + (\|y_n - x_n\| + k_n^{(1)}\Psi(\|y_n - x_n\|) + \nu_n^{(1)} \\ &\leq \|S_2^n x_n - T_2(PT_2)^{n-1}z_n\| + \|y_n - x_n\| + Mh_n(\|z_n - x_n\|) + \theta_n \\ &= \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + (1 + Mh_n)\|y_n - x_n\| + \theta_n. \end{aligned} \tag{3.39}$$

From (3.23), (3.38), (3.39) and the fact that $\sum_{n=1}^{\infty}\theta_n < \infty$

$$\lim_{n\to\infty} \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\| = 0. \tag{3.40}$$

Now, since $\|x_n - T_1(PT_1)^{n-1}x_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1}x_n\|$ (by condition (ii), it follows from (3.40) that

$$\lim_{n\to\infty} \|x_n - T_1(PT_1)^{n-1}x_n\| = 0. \tag{3.41}$$

From

$$\begin{aligned} \|x_{n+1} - S_1^n x_n\| &= \|P[(1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n] - S_1^n x_n\| \\ &\leq \|(1-\alpha_n)S_1^n x_n + \alpha_n T_1(PT_1)^{n-1}y_n - S_1^n x_n\| \\ &= \| - \alpha_n(S_1^n x_n - T_1(PT_1)^{n-1}y_n])\| \\ &= \alpha_n\|S_1^n x_n - T_1(PT_1)^{n-1}y_n]\| \end{aligned}$$

and (3.23), we obtain

$$\lim_{n\to\infty} \|x_{n+1} - S_1^n x_n\| = 0. \tag{3.42}$$

From

$$\|x_{n+1} - T_1(PT_1)^{n-1}y_n\| \leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\|,$$

(3.23) and (3.42), we get

$$\lim_{n\to\infty} \|x_{n+1} - T_1(PT_1)^{n-1}y_n\| = 0. \tag{3.43}$$

Also, from (3.23), (3.24) and the inequality

$$\|S_1^n x_n - x_n\| \leq \|S_1^n x_n - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - x_n\|,$$

we have

$$\lim_{n\to\infty} \|S_1^n x_n - x_n\| = 0. \tag{3.44}$$

Again, from (3.41), (3.44) and the inequality

$$\|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| \leq \|S_1^n x_n - x_n\| + \|x_n - T_2(PT_2)^{n-1}x_n\|,$$

we have

$$\lim_{n\to\infty} \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| = 0. \tag{3.45}$$

Since

$$\begin{aligned}\|x_{n+1} - T_2(PT_2)^{n-1}y_n\| &\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| \\ &\quad + \|T_2(PT_2)^{n-1}x_n - T_2(PT_2)^{n-1}y_n\| \\ &\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| + (\|x_n - y_n\| \\ &\quad + k_n^{(2)}\phi(\|x_n - y_n\|) + \nu_n^{(2)}) \\ &\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| + \|x_n - y_n\| \\ &\quad + Mh_n\|x_n - y_n\|) + \theta_n \\ &= \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_2(PT_2)^{n-1}x_n\| \\ &\quad + (1 + Mh_n)\|x_n - y_n\|) + \theta_n,\end{aligned}$$

it follows from (3.38), (3.42), (3.45) and the fact that $\sum_{n=1}^{\infty} \theta_n < \infty$ that

$$\lim_{n\to\infty} \|x_{n+1} - T_2(PT_2)^{n-1}y_n\| = 0. \tag{3.46}$$

Now, from (3.30), (3.41) and the inequality

$$\|S_1^n x_n - T_3(PT_3)^{n-1}x_n\| \leq \|S_1^n x_n - x_n\| + \|x_n - T_3(PT_3)^{n-1}x_n\|,$$

we obtain

$$\lim_{n\to\infty} \|S_1^n x_n - T_3(PT_3)^{n-1}x_n\| = 0. \tag{3.47}$$

Since

$$\begin{aligned}\|x_{n+1} - T_3(PT_3)^{n-1}y_n\| &\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_3(PT_3)^{n-1}x_n\| \\ &\quad + \|T_3(PT_3)^{n-1}x_n - T_3(PT_3)^{n-1}y_n\| \\ &\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_3(PT_3)^{n-1}x_n\| + (\|x_n - y_n\| \\ &\quad + k_n^{(3)}\phi(\|x_n - y_n\|) + \nu_n^{(3)}) \\ &\leq \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_3(PT_3)^{n-1}x_n\| + \|x_n - y_n\| \\ &\quad + Mh_n\|x_n - y_n\|) + \theta_n \\ &= \|x_{n+1} - S_1^n x_n\| + \|S_1^n x_n - T_3(PT_3)^{n-1}x_n\| \\ &\quad + (1 + Mh_n)\|x_n - y_n\|) + \theta_n\end{aligned}$$

it follows from (3.38), (3.42), (3.47) and the fact that $\sum_{n=1}^{\infty} \theta_n < \infty$ that

$$\lim_{n\to\infty} \|x_{n+1} - T_3(PT_3)^{n-1}y_n\| = 0. \tag{3.48}$$

Again, since $(PTi)(PTI)^{n-2}y_{n-1}, x_n \in K$ for $i = 1, 2, 3$ and T_1, T_2, T_3 are three total asymptotically nonexpansive nonself mappings, we have

$$\begin{aligned}
\|T_i(PT_i)^{n-1}y_{n-1} - T_ix_n\| &= \|T_i(PT_i)(PT_i)^{n-2}y_{n-1} - T_i(Px_n)\| \\
&\leq \|(PT_i)(PT_i)^{n-2}y_{n-1} - P(x_n)\| \\
&\quad +k_n^{(i)}\phi(\|(PT_i)(PT_i)^{n-2}y_{n-1} - P(x_n)\|) + \nu_n^{(i)} \\
&\leq \|(PT_i)(PT_i)^{n-2}y_{n-1} - P(x_n)\| \\
&\quad +Mh_n\|(PTi)(PTi)^{n-2}y_{n-1} - P(x_n)\| + \theta_n \\
&= (1 + Mh_n)\|(PT_i)(PT_i)^{n-2}y_{n-1} - P(x_n)\| + \theta_n \\
&= (1 + Mh_n)\|T_i(PT_i)^{n-2}y_{n-1} - x_n\| + \theta_n. \qquad (3.49)
\end{aligned}$$

For $i = 1.2.3,$, it follows from (3.43), (3.46) and (3.48) that

$$\lim_{n\to\infty} \|Ti(PTi)^{n-1}y_{n-1} - Tix_n\| = 0. \qquad (3.50)$$

Observe that

$$\|x_{n+1} - y_n\| \leq \|x_{n+1} - T_1(PT_1)^{n-1}y_n\| + \|T_1(PT_1)^{n-1}y_n - x_n\| + \|x_n - y_n\|,$$

so that, by (3.24), (3.38) and (3.43), we get

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \qquad (3.51)$$

Next,observe, for $i = 1, 2, 3$, that

$$\begin{aligned}
\|x_n - T_ix_n\| &\leq \|x_n - T_i(PT_i)^{n-1}x_n\| + \|T_i(PT_i)^{n-1}x_n - T_i(PT_i)^{n-1}y_{n-1}\| \\
&\quad +\|T_i(PT_i)^{n-1}y_{n-1} - T_ix_n\| \\
&\leq \|x_n - T_i(PT_i)^{n-1}x_n\| + [\|x_n - y_{n-1}\| + k_n^{(i)}\phi(\|x_n - y_{n-1}\|) \\
&\quad +\nu_n^{(i)}] + \|T_i(PT_i)^{n-1}y_{n-1} - T_ix_n\| \\
&\leq \|x_n - T_i(PT_i)^{n-1}x_n\| + \|x_n - y_{n-1}\| + k_n^{(i)}M\|x_n - y_{n-1}\| \\
&\quad +\nu_n^{(i)} + \|T_i(PT_1)^{n-1}y_{n-1} - T_ix_n\| \\
&= \|x_n - T_i(PT_i)^{n-1}x_n\| + (1 + k_n^{(i)}M)\|x_n - y_{n-1}\| + \nu_n^{(i)}] \\
&\quad +\|T_i(PT_i)^{n-1}y_{n-1} - T_ix_n\| \\
&\leq \|x_n - T_i(PT_i)^{n-1}x_n\| + max[sup_{n\geq 1}(1 + k_n^{(i)}M)]\|x_n - y_{n-1}\| \\
&\quad +max[sup_{n\geq 1}]\nu_n^{(i)}] + \|T_i(PT_i)^{n-1}y_{n-1} - T_ix_n\|
\end{aligned}$$

Thus, it follows from (3.30), (3.36), (3.41), (3.50) and (3.51) that $\lim_{n\to\infty} \|x_n - T_ix_n\| = 0$, for $i = 1, , 2, 3$.

Finally, we prove that $\lim_{n\to\infty}\|x_n - S_i^n x_n\| = 0$, for $i = 1, , 2, 3$.

Infact, by condition (ii), we have for $i = 1, 2, 3$, that

$$\|x_n - S_i^n x_n \leq \|x_n - T_i(PT_i)^{n-1}x_n\| + \|S_i^n x_n - T_i(PT_i)^{n-1}x_n\|$$

Thus, it follows from (3.29), (3.30), (3.36), (3.40), (3.41) and (3.45) that

$$\lim_{n\to\infty}\|x_n - S_i^n x_n\| = 0,\ \ for\ i = 1, 2, 3. \tag{3.52}$$

This completes the proof of Lemma 3.3. □

Lemma 3.4. *Under the assumption of Lemma 3.2, for all $p_1, p_2 \in \cap_{i1}^3 (F(S_i) \cap F(T_i))$, the limit $\lim_{n\to\infty}\|x_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0, 1]$, where $\{x_n\}$ is the sequence defined by (1.7).*

Proof. By lemma 3.2, $\lim_{n\to\infty}\|x_n - q\|$ exists for all $q \in F$ and therefor $\{x_n\}$ is bounded. Let $a_n(t) = \|x_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0, 1]$. Then, $\lim_{n\to\infty} a_{(0)} = \|p_1 - p_2\|$ and $\lim_{n\to\infty} a_{(1)} = \|x_n - p_2\|$ exist by Lemma 3.2. It remains therefor to prove Lemma 3.4 for $t \in (0, 1)$. For all $x \in K$, we define the mapping

$$\begin{cases} R_n(x) = P[(1-\gamma_n)S_3^n + \gamma T_3(PT_3)^{n-1}x_n]; \\ W_n(x) = P[(1-\beta_n)S_2^n + \beta T_2(PT_2)^{n-1}x_n]; \\ V_n(x) = P[(1-\alpha_n)S_1^n + \alpha T_1(PT_1)^{n-1}x_n], n \geq 1. \end{cases} \tag{3.53}$$

Then, it follows that $x_{n+1} = V_n x_n, V_n p = p, \forall p \in F$. Now, from (3.12), (3.15) and (3.18) of Lemma 3.2, we see that

$$\begin{cases} \|R_{n(x)} - R_n(y)\| \leq (1+h_n)M\|x-y\| + \theta_n; \\ \|W_{n(x)} - W_n(y)\| \leq (1+r_n)M\|x-y\| + \delta_n\theta_n; \\ \|V_{n(x)} - V_n(y)\| \leq (1+e_n)M\|x-y\| + \theta_n = f_n\|x-y\| + g_n, \end{cases} \tag{3.54}$$

where $r_n = 2h_n + h_n^2M^2, \delta_n = 2 + h_nM$, $e_n = 3h_nM + 3h_n^2M^2 + h_n^3M^3$ and $g_n = (1+h_nM)(2+h_nM)\theta_n$ with $\sum_{n=1}^{\infty} e_n < \infty, \sum_{n=1}^{\infty} g_n < \infty$ and $f_n = 1 + e_n$. Since $\sum_{n=1}^{\infty} e_n < \infty$, it follows that $f_n \to 1$ as $n \to \infty$. Set

$$\begin{cases} S_{n,m} = V_{n+m-1}V_{n+m-2}\cdots V_n,\ \ m \in N; \\ b_{n,m} = \|S_{n,m}(tx_n + (1-t)p_1) - S_{n.m}(tx_m + (1-t)p_2\|. \end{cases} \tag{3.55}$$

From (3.54) and (3.55), we have

$$\begin{aligned}
\|S_{n,m}(x)-S_{n,m}(y)\| &= \|V_{n+m-1}V_{n+m-2}\cdots V_n(x)-V_{n+m-1}V_{n+m-2}\cdots V_n(y)\| \\
&\leq f_{n+m-1}\|V_{n+m-2}V_{n+m-3}\cdots V_n(x)-V_{n+m-2}V_{n+m-3}\cdots V_n(y)\| \\
&\quad +g_{n+m-1} \\
&\leq (f_{n+m-1})(f_{n+m-2})\|V_{n+m-3}V_{n+m-4}\cdots V_n(x) \\
&\quad -V_{n+m-3}V_{n+m-4}\cdots V_n(y)\|+g_{n+m-1}+g_{n+m-2} \\
&\vdots \\
&\leq (\prod_{i=n}^{n+m-1} f_i)\|x-y\|+\sum_{i=n}^{n+m-1} g_i \\
&= B_n\|x-y\|+\sum_{i=n}^{n+m-1} g_i,
\end{aligned} \tag{3.56}$$

for all $x,y\in K$, where $B_n=\prod_{i=n}^{n+m-1} f_i$, $S_{n,m}x_n=x_n$ and $S_{n,m}p=p$ for all $p\in F$. Thus,

$$\begin{aligned}
a_{n+m}(t) &= \|tx_n+(1-t)p_1-p_2\| \\
&= \|S_{n,m}(tx_n+(1-t)p_1-p_2\| \\
&\leq b_{n,m}+\|S_{n,m}(tx_n+(1-t)p_1-p_2\|.
\end{aligned} \tag{3.57}$$

By using Theorem 2.3 in [5], we have

$$\begin{aligned}
b_{n,m} &\leq \psi^{-1}(\|(x_n-u\|-\|x_{n+1}-S_{n,m}u\|) \\
&= \psi^{-1}(\|(x_n-u\|-\|x_{n+1}-u+u-S_{n,m}u\|) \\
&\leq \psi^{-1}(\|(x_n-u\|-(\|x_{n+1}-u\|+\|S_{n,m}u-u\|)),
\end{aligned} \tag{3.58}$$

so that the sequence $\{b_{n.m}\}$ converges uniformly to 0, i.e, $b_{n,m}\to 0$ as $n\to\infty$. Since $\lim_{n\to} B_n=1$ and $\lim_{n\to\infty} b_{n,m}=0$, it follows from (3.57) that $\limsup_{n\to\infty} a_n(t)\leq \liminf_{b\to\infty} b_{n.m}\leq \liminf_{n\to\infty} a_n(t)$. This shows that $\lim_{n\to\infty} a_n(t)$ exists, i.e, $\lim_{n\to\infty}\|tx_n+(1-t)p_1-p_2\|$ exists for all $t\in[0,1]$. This completes the proof Lemma 3.4. □

Lemma 3.5. *Under the assumption of Lemma 3.2, if E has Frechet differentiable norm,then for all $p_1, Jp_2\in\mathcal{F}=\cap_{i=1}^{3}(F(T_i)\cap F(S_i))$, the $\lim_{n\to\infty}(\langle x_n, J(p_1-p_2)\rangle$ exists, where $\{x_n\}$ is the sequence defined by (1.7). If $\omega_\omega(x_n)$ denotes the set of all weak subsequential limits of $\{x_n\}$, then $\langle q_1-q_2, J(p_1-p_2\rangle=0$ for all $p_1,p_2\in F$ and for all $q_1,q_2\in\omega_\omega(x_n)$.*

Proof. Suppose that $x=p_1-p_2$ with $p_1\neq p_2$ and $h=t(x_n-p_1)$ in(2.1). Then, we have

$$\begin{aligned}
t(\langle x_n, J(p_1-p_2)\rangle+\frac{1}{2}\|p_1-p_2\|^2 &\leq \frac{1}{2}\|tx_n+(1-t)p_1-p_2\|^2 \\
&\leq t(\langle x_n, J(p_1-p_2)\rangle+\frac{1}{2}\|p_1-p_2\|^2+b(t\|x_n-p_1\|)
\end{aligned}$$

Since $\sup_{n\geq 1}\|x_n - p\| \leq Q$ for some $Q > 0$, we have

$$\begin{aligned} t\lim_{n\to\infty} sup(\langle x_n, J(p_1-p_2)\rangle + \frac{1}{2}\|p_1-p_2\|^2 &\leq \frac{1}{2}\lim_{n\to\infty} sup\|tx_n+(1-t)p_1-p_2\|^2 \\ &\leq t\lim_{n\to\infty} inf(\langle x_n, J(p_1-p_2)\rangle + \frac{1}{2}\|p_1-p_2\|^2 \\ &\quad +b(tQ) \end{aligned}$$

That is, $t\lim_{n\to\infty} sup(\langle x_n, J(p_1-p_2)\rangle \leq t\liminf_{n\to\infty}(\langle x_n, J(p_1-p_2)\rangle + b(tQ)$. If $t\to 0$, then $\lim_{n\to\infty}\langle x_n - p_1, J(p_1-p_2)\rangle$ exists for all $p_1, p_2 \in F$ and for all $q_2, q_2 \in \omega_\omega(x_n)$; in particular, $(\langle q_1-q_2, J(p_1-p_2)\rangle = 0$ for all $q_2, q_2 \in \omega_\omega(x_n)$. This completes the proof Lemma 3.5. □

Theorem 3.6. *Under the assumption of Lemma* 3.2*, if E has Frechet differentiable norm, then the sequence $\{x_n\}$ defined by* (1.7) *converges weakly to a common fixed point in $\mathcal{F} = \cap_{i=1}^3 F(T_i)\cap F(S_i)$.*

Proof. By Lemma 3.5, $(\langle q_1-q_2, J(p_1-p_2)\rangle = 0$ for all $q_2, q_2 \in \omega_\omega(x_n)$. Therefore, $\|q^\star - p\star\|^2 = \langle q^\star - p^\star, J(q^\star - p^\star)\rangle = 0$. This implies that $p^\star = q^\star$. Consequently, $\{x_n\}$ converges to a common fixed point of $\mathcal{F} = \cap_{i=1}^3 F(T_i)\cap F(S_i)$. This completes the proof Theorem 3.6. □

Theorem 3.7. *Under the assumption of Lemma* 3.2*, if the dual space $E^\star$ of E has the Kadec Klec (KK) property and the mappings $I - S_i$ and $I - T_i$ for $i = 1, 2, 3$, where I denotes the identity mapping, are demiclosed at zero, then the sequence $\{x_n\}$ defined by* (1.7) *converges weakly to a common fixed point in $\mathcal{F} = \cap_{i=1}^3(F(T_i)\cap F(S_i))$.*

Proof. By Lemma 3.2 $\{x_n\}$ is bounded and since E is reflexive, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q^\star \in K$. By Lemma 3.3, we have $\lim_{n\to\infty}\|x_{n_k} - S_i x_{n_k}\| = 0$ and $\lim_{n\to\infty}\|x_{n_k} - T_i x_{n_k}\| = 0$ for $i = 1, 2, 3$. Since by hypothesis, the mappings $I - S_i$ and $I - T_i$ for $i = 1, 2, 3$, where I denotes the identity mapping, are demiclosed at zero, $S_i q^\star = q^\star$ and $T_i q^\star = q^\star$ for $i = 1, 2, 3$.; which means $q^\star \in \mathcal{F} = \cap_{i=1}^3(F(T_i)\cap F(S_i))$. Now, we show that $\{x_n\}$ converges weakly to $q^\star$. Suppose $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ which converges weakly to $p^\star \in K$. By the same method as above, we have $p^\star \in F$ and $q^\star \in \omega_\omega(x_n)$. By Lemma 3.4, the limit $\lim_{n\to\infty}\|tx_n + (1-t)q^\star - p^\star\|$ exists for all $t \in [0, 1]$ and so $q^\star = p^\star$. Thus, the sequence $\{x_n\}$ converges weakly to $q^\star \in F$. This completes the proof. □

Theorem 3.8. *Under the assumption of Lemma* 3.2*, if E satisfies Opial's condition and the mappings $I - S_i$ and $I - T_i$ for $i = 1, 2, 3$, where I denotes the identity mapping, are demiclosed at zero, then the sequence $\{x_n\}$ defined by* (1.7) *converges weakly to a common fixed point in $\mathcal{F} = \cap_{i=1}^3(F(T_i)\cap F(S_i))$.*

Proof. Let $q\star \in F$. From Lemma 3.2, the squence $\{\|x_n - p\star\|\}$ is convergent and hence bounded. Since, E is uniformly convex , every bounded subset of E is weakly compact. Thus, the exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some $q^\star \in K$. By Lemma 3.3, we have

$\lim_{n\to\infty}\|x_{n_k}-S_ix_{n_k}\|=0$ and $\lim_{n\to\infty}\|x_{n_k}-T_ix_{n_k}\|=0$ for $i=1,2,3$. Since by hypothesis, the mappings $I-S_i$ and $I-T_i$ for $i=1,2,3$, where I denotes the identity mapping, are demiclosed at zero, $S_iq^\star=q^\star$ and $T_iq^\star=q^\star$ for $i=1,2,3.$; which means $q^\star\in\mathcal{F}=\cap_{i=1}^3(F(T_i)\cap F(S_i))$. Finally, we show that $\{x_n\}$ converges weakly to $q^\star$. Suppose on the contrary that $\{x_{n_j}\}$ is another subsequence of $\{x_n\}$ which converges weakly to $p^\star\in K$ and $q^\star\neq p^\star$ By Lemma 3.2, $\lim_{n\to\infty}\|x_n-q^\star\|$ and $\lim_{n\to\infty}\|x_n-p^\star\|$ exist. By virtue of Opial's condition on E, we obtain

$$\begin{aligned}\lim_{n\to\infty}\|x_n-q^\star\| &= \lim_{n\to\infty}\|x_{n_k}-q^\star\|\\ &< \lim_{n\to\infty}\|x_{n_k}-p^\star\|\\ &= \lim_{n\to\infty}\|x_n-p^\star\|\\ &= \lim_{n\to\infty}\|x_{n_j}-p^\star\|\\ &< \lim_{n\to\infty}\|x_{n_j}-q^\star\|\\ &= \lim_{n\to\infty}\|x_n-q^\star\|,\end{aligned} \tag{3.59}$$

which is a contradiction, so $q^\star=p^\star$ Therefore, the sequence $\{x_n\}$ defined by (1.7) converges weakly to $q^\star\in F$. This completes the proof. □

Corollary 3.9. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E. Let $S_1,S_2,S_3:K\longrightarrow K$ be three generalize asymptotically nonexpansive self mapping with sequences $\{k_n^{(1)}\},\{k_n^{(2)}\},\{k_n^{(3)}\}\in[1,\infty)$,*
$\{w_n^{(1)}\},\{w^{(2)}\}_n,\{w_n^{(3)}\}\in[1,\infty)$ and $T_1,T_2,T_3:K\longrightarrow E$ are three generalize asymptotically nonexpansive nonself mappings with sequences $\{\mu_n^{(1)}\},\{\mu_n^{(2)}\},\{\mu_n^{(3)}\}\in[1,\infty)$, $\{\nu_n^{(1)}\},\{\nu_n^{(2)}\}$,
$\{\nu_n^{(3)}\}\in[1,\infty)$. Let $\{x_n\}$ be the sequence defined by (1.7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences $\in[0,1)$.. Suppose $\mathcal{F}=\cap_{i=1}^3(F(T_i)\cap F(S_i))\neq 0$. If the following conditions hold:

i. $\sum_{n=1}^\infty k_n^{(1)}<\infty$, $\sum_{n=1}^\infty k_n^{(2)}<\infty$, $\sum_{n=1}^\infty k_n^{(3)}<\infty$, $\sum_{n=1}^\infty \mu_n^{(1)}<\infty$, $\sum_{n=1}^\infty \mu_n^{(2)}<\infty$, $\sum_{n=1}^\infty \mu_n^{(3)}<\infty$, $\sum_{n=1}^\infty \nu_n^{(1)}<\infty$, $\sum_{n=1}^\infty \nu_n^{(2)}<\infty$, $\sum_{n=1}^\infty \nu_n^{(3)}<\infty$,
ii. *There exists a constant $M>0$ such that $\Psi(t)=\phi(t)\leq Mt, t\leq 0$.*

Then, $\lim_{n\infty}\|x_n-q\|$ and $\lim_{n\infty}d(x_n-F)$ both exist for all $q\in F$.

Corollary 3.10. *Let E be a uniformly convex Banach space and K a nonempty closed convex subset of E. Let $S_1,S_2,S_3:K\longrightarrow K$ be three generalize asymptotically nonexpansive self mapping with sequences $\{k_n^{(1)}\},\{k_n^{(2)}\},\{k_n^{(3)}\}\in[1,\infty)$,*
$\{w_n^{(1)}\},\{w^{(2)}\}_n,\{w_n^{(3)}\}\in[1,\infty)$ and $T_1,T_2,T_3:K\longrightarrow E$ are three generalize asymptotically nonexpansive nonself mappings with sequences $\{\mu_n^{(1)}\},\{\mu_n^{(2)}\},\{\mu_n^{(3)}\}\in[1,\infty)$, $\{\nu_n^{(1)}\},\{\nu_n^{(2)}\}$,
$\{\nu_n^{(3)}\}\in[1,\infty)$. Let $\{x_n\}$ be the sequence defined by (1.7), where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences $\in[0,1)$. Suppose $\mathcal{F}=\cap_{i=1}^3(F(T_i)\cap F(S_i))\neq 0$. If the following conditions hold:

i. $\sum_{n=1}^{\infty} k_n^{(1)} < \infty, \sum_{n=1}^{\infty} k_n^{(2)} < \infty, \sum_{n=1}^{\infty} k_n^{(3)} < \infty, \sum_{n=1}^{\infty} \mu_n^{(1)} < \infty, \sum_{n=1}^{\infty} \mu_n^{(2)} < \infty, \sum_{n=1}^{\infty} \mu_n^{(3)} < \infty, \sum_{n=1}^{\infty} \nu_n^{(1)} < \infty, \sum_{n=1}^{\infty} \nu_n^{(2)} < \infty, \sum_{n=1}^{\infty} \nu_n^{(3)} < \infty,$

ii. $\|x - T_1(PT_1)^{n-1}y\| \leq \|S_1^n x - T_1(PT_1)^{n-1}y\|, \|x - T_2(PT_2)^{n-1}y\| \leq \|S_2^n x - T_2(PT_2)^{n-1}y\|,$ $\|x - T_3(PT_3)^{n-1}y\| \leq \|S_3^n x - T_3(PT_3)^{n-1}y\|$

iii. *There exists a constant* $M_1, M_2 > 0$ *such that* $\Psi(t) \leq M_1 t, \ \phi(t) \leq M_2 t, \ t \geq 0.$

Then, $\lim_{n\infty} \|x_n - S_i x_n\| = 0$ *and* $\lim_{n\infty} \|x_n - T_i x_n\| = 0$*, for* $i = 1, 2, , 3.$

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Not applicable

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Availability of Data and Material

Not applicable

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IKA and NCU wrote the paper while DII suggested the idea and did the analysis. The three authors read and approved the final manuscript.

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