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A Note on Some Isomorphic Properties in Projective Tensor Products

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Abstract: A Banach space X is sequentially Right (resp. weak sequentially Right) if every Right subset of X^* is relatively weakly compact (resp. weakly precompact). A Banach space X has the L-limited (resp. the wL-limited) property if every L-limited subset of X^* is relatively weakly compact (resp. weakly precompact). We study Banach spaces with the weak sequentially Right and the wL-limited properties. We investigate whether the projective tensor product of two Banach spaces X and Y has the sequentially Right property when X and Y have the respective property.

Key words: R-sets, L-limited sets, sequentially Right spaces, L-limited property.

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1. Introduction

A bounded subset A of a Banach space X is called a *Dunford-Pettis* (DP) (resp. *limited*) subset of X if every weakly null (resp. w^*-null) sequence (x_n^*) in X^* tends to 0 uniformly on A; i.e.,

$$\lim_n \left(\sup\{|x_n^*(x)| : x \in A\} \right) = 0.$$

A sequence (x_n) is DP (resp. limited) if the set $\{x_n : n \in \mathbb{N}\}$ is DP (resp. limited).

A subset S of X is said to be *weakly precompact* provided that every sequence from S has a weakly Cauchy subsequence. Every DP (resp. limited) set is weakly precompact [37, p. 377], [1] (resp. [4, Proposition]).

An operator $T : X \to Y$ is called *weakly precompact (or almost weakly compact)* if $T(B_X)$ is weakly precompact and *completely continuous (or Dunford-Pettis)* if T maps weakly convergent sequences to norm convergent sequences.

In [35] the authors introduced the *Right topology* on a Banach space X. It is the restriction of the Mackey topology $\tau(X^{**}, X)$ to X and it is also the topology of uniform convergence on absolutely convex $\sigma(X^*, X^{**})$ compact subsets of X^*. Further, $\tau(X^{**}, X)$ can also be viewed as the topology of uniform convergence on relatively $\sigma(X^*, X^{**})$ compact subsets of X^* [26].

A sequence (x_n) in a Banach space X is Right null if and only if it is weakly null and DP (see Proposition 1).

An operator $T : X \to Y$ is called *pseudo weakly compact (pwc)* (or *Dunford-Pettis completely continuous (DPcc)*) if it takes Right null sequences in X into norm null sequences in Y ([35], [25]). Every completely continuous operator $T : X \to Y$ is pseudo weakly compact. If $T : X \to Y$ is an operator with weakly precompact adjoint, then T is a pseudo weakly compact operator ([18, Corollary 5]).

A subset K of X^* is called a *Right set (R-set)* if each Right null sequence (x_n) in X tends to 0 uniformly on K [26]; i.e.,

$$\lim_n \big(\sup\{|x^*(x_n)| : x^* \in K\} \big) = 0.$$

A Banach space X is said to be *sequentially Right (SR)* (has *property (SR)*) if every pseudo weakly compact operator $T : X \to Y$ is weakly compact, for any Banach space Y [35]. Banach spaces with property (V) are sequentially Right ([35, Corollary 15]).

A subset A of a dual space X^* is called an *L-limited set* if every weakly null limited sequence (x_n) in X converges uniformly on A [39]; i.e.,

$$\lim_n \big(\sup\{|x^*(x_n)| : x^* \in A\} \big) = 0.$$

A Banach space X has the *L-limited property* if every L-limited subset of X^* is relatively weakly compact [39]. An operator $T : X \to Y$ is called *limited completely continuous (lcc)* if T maps weakly null limited sequences to norm null sequences [40].

In this paper we introduce the weak sequentially Right (wSR) and wL-limited properties. A Banach space X is said to have the *weak sequentially Right* (wSR) (resp. the *wL-limited*) *property* if every Right (resp. L-limited) subset of X^* is weakly precompact. We obtain some characterizations of these properties with respect to some geometric properties of Banach spaces, such as the Gelfand-Phillips property, the Grothendieck property, and properties (wV) and (wL). We generalize some results from [39]. We also show that property (SR) can be lifted from a certain subspace of X to X.

We study whether the projective tensor product $X \otimes_\pi Y$ has the (SR) (resp. the L-limited) property if $L(X, Y^*) = K(X, Y^*)$, and X and Y have the respective property. We prove that in some cases, if $X \otimes_\pi Y$ has the (wSR) property, then $L(X, Y^*) = K(X, Y^*)$.

2. Definitions and notation

Throughout this paper, X, Y, E, and F will denote Banach spaces. The unit ball of X will be denoted by B_X and X^* will denote the continuous linear dual of X. An operator $T : X \to Y$ will be a continuous and linear function. We will denote the canonical unit vector basis of c_0 by (e_n) and the canonical unit vector basis of ℓ_1 by (e_n^*). The set of all operators, weakly compact operators, and compact operators from X to Y will be denoted by $L(X,Y)$, $W(X,Y)$, and $K(X,Y)$. The projective tensor product of X and Y will be denoted by $X \otimes_\pi Y$.

A bounded subset A of X^* is called an *L-set* if each weakly null sequence (x_n) in X tends to 0 uniformly on A; i.e.,

$$\lim_n \big(\sup\{|x^*(x_n)| : x^* \in A\}\big) = 0.$$

A Banach space X has *the Dunford-Pettis property* (DPP) if every weakly compact operator $T : X \to Y$ is completely continuous, for any Banach space Y. Schur spaces, $C(K)$ spaces, and $L_1(\mu)$ spaces have the DPP. The reader can check [8], [9], and [10] for a guide to the extensive classical literature dealing with the DPP.

A Banach space X has the *Dunford-Pettis relatively compact property (DPrcP)* if every Dunford-Pettis subset of X is relatively compact [14]. Schur spaces have the $DPrcP$. The space X does not contain a copy of ℓ_1 if and only if X^* has the $DPrcP$ if and only if every L-set in X^* is relatively compact ([14, Theorem 1], [13, Theorem 2]).

The space X has the *Gelfand-Phillips (GP)* property if every limited subset of X is relatively compact. The following spaces have the Gelfand-Phillips property: Schur spaces; spaces with w^*-sequential compact dual unit balls (for example subspaces of weakly compactly generated spaces, separable spaces, spaces whose duals have the Radon-Nikodým property, reflexive spaces, and spaces whose duals do not contain ℓ_1); dual spaces X^* whith X not containing ℓ_1; Banach spaces with the separable complementation property, i.e., every separable subspace is contained in a complemented separable subspace (for example $L_1(\mu)$ spaces, where μ is a positive measure) [42, p. 31], [4, Proposition], [12, Theorem 3.1 and p. 384], [11, Proposition 5.2], [13, Corollary 5].

A series $\sum x_n$ in X is said to be *weakly unconditionally convergent (wuc)* if for every $x^* \in X^*$, the series $\sum |x^*(x_n)|$ is convergent. An operator $T : X \to Y$ is called *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent ones.

A bounded subset A of X^* is called a *V-subset* of X^* provided that

$$\lim_n \big(\sup\{\,|x^*(x_n)| : x^* \in A\,\}\big) = 0)$$

for each wuc series $\sum x_n$ in X.

A Banach space X has *property* (V) if every V-subset of X^* is relatively weakly compact [33]. A Banach space X has property (V) if every unconditionally converging operator T from X to any Banach space Y is weakly compact [33, Proposition 1]. $C(K)$ spaces and reflexive spaces have property (V) ([33, Theorem 1, Proposition 7]). A Banach space X has *property* (wV) if every V-subset of X^* is weakly precompact [41].

A Banach space X has the *reciprocal Dunford-Pettis property* $(RDPP)$ if every completely continuous operator T from X to any Banach space Y is weakly compact. The space X has the $RDPP$ if and only if every L-set in X^* is relatively weakly compact [28]. Banach spaces with property (V) have the $RDPP$ [33]. A Banach space X has *property* (wL) if every L-set in X^* is weakly precompact [19].

A topological space S is called *dispersed (or scattered)* if every nonempty closed subset of S has an isolated point. A compact Hausdorff space K is dispersed if and only if $\ell_1 \not\hookrightarrow C(K)$ [34, Main theorem].

The Banach-Mazur distance $d(X,Y)$ between two isomorphic Banach spaces X and Y is defined by $\inf(\|T\|\|T^{-1}\|)$, where the infinum is taken over all isomorphisms T from X onto Y. A Banach space X is called an *$\mathcal{L}_\infty$-space* (resp. *$\mathcal{L}_1$-space*) [5, p. 7] if there is a $\lambda \geq 1$ so that every finite dimensional subspace of X is contained in another subspace N with $d(N, \ell_\infty^n) \leq \lambda$ (resp. $d(N, \ell_1^n) \leq \lambda$) for some integer n. Complemented subspaces of $C(K)$ spaces (resp. $L_1(\mu)$) spaces) are $\mathcal{L}_\infty$-spaces (resp. $\mathcal{L}_1$-spaces) ([5, Proposition 1.26]). The dual of an $\mathcal{L}_1$-space (resp. $\mathcal{L}_\infty$-space) is an $\mathcal{L}_\infty$-space (resp. $\mathcal{L}_1$- space) ([5, Proposition 1.27]). The $\mathcal{L}_\infty$-spaces, $\mathcal{L}_1$-spaces, and their duals have the DPP ([5, Corollary 1.30]).

3. The weak sequentially Right and wL-limited properties

The following result gives a characterization of Right null sequences.

Proposition 1. *A sequence (x_n) in a Banach space X is Right null if and only if it is weakly null and DP.*

Proof. Suppose that (x_n) is a Right null sequence in X. Then (x_n) is weakly null, since the Right topology is stronger than the weak topology.

Let (x_n^*) be a weakly null sequence in X^*. Since $\{x_n^* : n \in \mathbb{N}\}$ is relatively weakly compact in X^* and (x_n) is Right null, (x_n) converges uniformly on $\{x_n^* : n \in \mathbb{N}\}$. Therefore $\lim_n \sup_i |x_i^*(x_n)| = 0$, and thus $\lim_n |x_n^*(x_n)| = 0$. Hence $\{x_n : n \in \mathbb{N}\}$ is a DP set.

Suppose that (x_n) is a weakly null DP sequence. Let K be a relatively weakly compact subset of X^*. Suppose that (x_n) does not converge uniformly on K. Let $\epsilon > 0$ and let (x_n^*) be a sequence in K so that $|x_n^*(x_n)| > \epsilon$ for all n. Without loss of generality suppose that (x_n^*) converges weakly to x^*, $x^* \in X^*$. Since $(x_n^* - x^*)$ is weakly null in X^* and (x_n) is DP, $\lim_n (x_n^* - x^*)(x_n) = 0$. Thus $\lim_n x_n^*(x_n) = 0$, a contradiction. Hence (x_n) converges uniformly to zero on K, and thus (x_n) is Right null. ∎

A Banach space X is sequentially Right if and only if every Right subset of X^* is relatively weakly compact [26, Theorem 3.25]. A Banach space X has the L-limited property if and only if every limited completely continuous operator $T : X \to Y$ is weakly compact, for every Banach space Y [39, Theorem 2.8]. In the next theorem we give elementary operator theoretic characterizations of weak precompactness, relative weak compactness, and relative norm compactness for Right sets and L-limited sets. The argument contains the theorems in [26] and [39] just cited.

We say that a Banach space X is *weak sequentially Right (wSR)* or has the *(wSR) property* (resp. has the *wL-limited property*) if every Right (resp. L-limited) subset of X^* is weakly precompact. If $\ell_1 \not\hookrightarrow X^*$, then X is weak sequentially Right and has the wL-limited property, by Rosenthal's theorem ([8, Ch. XI]).

THEOREM 2. *Let X be a Banach space. The following assertions are equivalent:*

1. (i) *For every Banach space Y, every pseudo weakly compact operator $T : X \to Y$ has a weakly precompact (weakly compact, resp. compact) adjoint.*

   (ii) *Every pseudo weakly compact operator $T : X \to \ell_\infty$ has a weakly precompact (weakly compact, resp. compact) adjoint.*

   (iii) *Every Right subset of X^* is weakly precompact (relatively weakly compact, resp. relatively compact).*

2. (i) *For every Banach space Y, every limited completely continuous operator $T : X \to Y$ has a weakly precompact (weakly compact, resp. compact) adjoint.*
   (ii) *Every limited completely continuous operator $T : X \to \ell_\infty$ has a weakly precompact (weakly compact, resp. compact) adjoint.*
   (iii) *Every L-limited subset of X^* is weakly precompact (relatively weakly compact, resp. relatively compact).*

Proof. We will show that 1.(i)⇒1.(ii)⇒1.(iii)⇒1.(i) in the weakly precompact case as well as 2.(i)⇒2.(ii)⇒2.(iii)⇒2.(i) in the compact case. These two arguments are similar, and the arguments for the remaining implications of the theorem follow the same pattern.

1. (*weakly precompact*) (i)⇒(ii) is clear.

(ii)⇒(iii) Let K be a Right subset of X^* and let (x_n^*) be a sequence in K. Define $T : X \to \ell_\infty$ by $T(x) = (x_i^*(x))$. Let (x_n) be a Right null sequence in X. Since K is a Right set,

$$\lim_n \|T(x_n)\| = \lim_n \sup_i |x_i^*(x_n)| = 0.$$

Therefore T is pseudo weakly compact, and thus $T^* : \ell_\infty^* \to X^*$ is weakly precompact. Hence $(T^*(e_n^*)) = (x_n^*)$ has a weakly Cauchy subsequence.

(iii)⇒(i) Let $T : X \to Y$ be a pseudo weakly compact operator. Let (x_n) be a Right null sequence in X. If $y^* \in B_{Y^*}$, $\langle T^*(y^*), x_n\rangle \le \|T(x_n)\| \to 0$. Then $T^*(B_{Y^*})$ is a Right subset of X^*. Therefore $T^*(B_{Y^*})$ is weakly precompact, and thus T^* is weakly precompact.

2. (*compact*) (i)⇒(ii) is clear.

(ii)⇒(iii) Let K be an L-limited subset of X^* and let (x_n^*) be a sequence in K. Define $T : X \to \ell_\infty$ as above and note that T is limited completely continuous. Thus $T^* : \ell_\infty^* \to X^*$ is compact, and $(T^*(e_n^*)) = (x_n^*)$ has a norm convergent subsequence.

(iii)⇒(i) Let $T : X \to Y$ be a limited completely continuous operator. Let (x_n) be a weakly null limited sequence in X. If $y^* \in B_{Y^*}$, $\langle T^*(y^*), x_n\rangle \le \|T(x_n)\| \to 0$. Then $T^*(B_{Y^*})$ is an L-limited subset of X^*. Therefore $T^*(B_{Y^*})$ is relatively compact, and thus T^* is compact. ∎

COROLLARY 3. *If X is weak sequentially Right (has the wL-limited, resp. the L-limited property), then every quotient space of X has the same property.*

Proof. We only prove the result for the weak sequentially Right property. The proofs for the other properties are similar.

Suppose that X is weak sequentially Right. Let Z be a quotient space of X and $Q : X \to Z$ be a quotient map. Let $T : Z \to E$ be a pseudo weakly compact operator. Then $TQ : X \to E$ is pseudo weakly compact, and thus $(TQ)^*$ is weakly precompact by Theorem 2. Since $Q^*T^*(B_E^*)$ is weakly precompact and Q^* is an isomorphism, $T^*(B_E^*)$ is weakly precompact. Apply Theorem 2. ∎

Corollary 4. *Suppose X is weak sequentially Right and Y is a Banach space. Then an operator $T : X \to Y$ is pseudo weakly compact if and only if $T^* : Y^* \to X^*$ is weakly precompact.*

Proof. If $T : X \to Y$ is pseudo weakly compact, then $T^* : Y^* \to X^*$ is weakly precompact by Theorem 2, since X is weak sequentially Right.

The converse follows from [18, Corollary 5]. ∎

Corollary 5. (i) *If X is weak sequentially Right (resp. has the wL-limited property), then every pseudo weakly compact (resp. limited completely continuous) operator $T : X \to Y$ is weakly precompact.*

(ii) *If X is an infinite dimensional space with the Schur property, then X is not weak sequentially Right (resp. does not have the wL-limited property).*

(iii) *If X is weak sequentially Right (resp. has the wL-limited property), then $\ell_1 \overset{c}{\not\hookrightarrow} X$.*

Proof. (i) Suppose X is weak sequentially right (resp. has the wL-limited property). Let $T : X \to Y$ be pseudo weakly compact (resp. limited completely continuous). Then T^* is weakly precompact by Theorem 2. Hence T is weakly precompact, by [2, Corollary 2].

(ii) Since X has the Schur property, the identity operator $i : X \to X$ is pseudo weakly compact (resp. limited completely continuous). Since X is an infinite dimensional space with the Schur property, i is not weakly precompact. Apply (i).

(iii) Apply Corollary 3 and (ii). ∎

Corollary 6. *A Banach space X has the L-limited property if every separable subspace of X has the same property.*

Proof. Let $T : X \to Y$ be a limited completely continuous operator. Then for every closed subspace Z of X, $T|_Z$ is limited completely continuous. Let

(x_n) be a sequence in B_X and let $Z = [x_n : n \in \mathbb{N}]$ be the closed linear span of (x_n). Since Z is a separable subspace of X, Z has the L-limited property. Since $T|_Z$ is limited completely continuous, it is weakly compact by Theorem 2. Then there is a subsequence (x_{n_k}) of (x_n) so that $(T(x_{n_k}))$ is weakly convergent. Thus T is weakly compact. Apply Theorem 2. ∎

Example. Corollary 6 cannot be reversed. Indeed, consider ℓ_1 as a subspace of ℓ_∞. By [39, Theorem 2.11], ℓ_∞ has the L-limited property. However, ℓ_1 does not have the L-limited property, by [39, Corollary 2.9] (or Corollary 5 (ii)).

Theorem 7. *The Banach space X has the DPP if and only if every Right subset of X^* is an L-set.*

Proof. Suppose X has the DPP. Then every weakly null sequence (x_n) is DP ([9, Theorem 1]). Therefore every Right subset of X^* is an L-set.

Conversely, let $T : X \to Y$ be a pseudo weakly compact operator. Then $T^*(B_{Y^*})$ is a Right subset of X^*, hence an L-set. Therefore T is completely continuous, and thus X has the DPP by [26, Proposition 3.17], [25, Theorem 1.5], [18, Theorem 10]. ∎

Corollary 8. *Suppose that X has the DPP. Then the following are equivalent:*

(i) *X does not contain a copy of ℓ_1.*

(ii) *Every L-set in X^* is relatively compact.*

(iii) *Every Right subset of X^* is relatively compact.*

(iv) *X^* has the Schur property.*

Proof. (i)⇔(ii) by [13, Theorem 2]. (ii)⇔(iii) by Theorem 7. (i)⇔(iv) by [9, p. 23]. ∎

Corollary 9. *X^* has the Schur property if and only if every Right subset of X^* is relatively compact.*

Proof. If X^* has the Schur property, then X has the DPP and X does not contain a copy of ℓ_1 ([9, p. 23]). Hence every Right subset of X^* is relatively compact by Corollary 8.

Conversely, let (x_n^*) be a weakly Cauchy sequence in X^*. Then (x_n^*) is a Right set, by the proof of [26, Corollary 3.26]. Thus (x_n^*) is relatively compact, and X^* has the Schur property. ∎

Corollary 10. (i) *Suppose X has the DPP and Y has the DPrcP. Then any operator $T : X \to Y$ is completely continuous.*

(ii) *The space X has the DPP and the DPrcP if and only if X has the Schur property.*

Proof. (i) Let $T : X \to Y$ be an operator. Since Y has the $DPrcP$, T is pseudo weakly compact. Then $T^*(B_{Y^*})$ is a Right set, thus an L-set in X^* (by Theorem 7). Hence T is completely continuous.

(ii) Suppose X has the DPP and the $DPrcP$. Then the identity operator $i : X \to X$ is completely continuous by (i). Hence X has the Schur property. If X has the Schur property, then X has the DPP and the $DPrcP$. ∎

Corollary 10 (i) generalizes [13, Corollary 6] when Y is a dual space E^* with E not containing ℓ_1 (since E^* has the $DPrcP$ [14, Theorem 1]).

A bounded subset A of X^* is called *w^*- sequentially compact* if every sequence from A has a subsequence which converges to a point in the w^*-topology of X^*.

The following theorem generalizes [39, Theorem 2.2 (b), (c)].

Theorem 11. *If (x_n^*) is a w^*-Cauchy sequence in X^*, then $\{x_n^* : n \in \mathbb{N}\}$ is an L-limited set.*

Proof. Supppse that (x_n^*) is a w^*-Cauchy sequence in X^* and $\{x_n^* : n \in \mathbb{N}\}$ is not an L-limited set. By passing to a subsequence if necessary, there is an $\epsilon > 0$ and a weakly null limited sequence (x_n) in X such that $|x_n^*(x_n)| > \epsilon$ for all n. Let $k_1 = 1$ and choose $k_2 > k_1$ so that $|x_{k_1}^*(x_{k_2})| < \epsilon/2$. We can do this since (x_n) is weakly null. Continue inductively. Choose $k_n > k_{n-1}$ so that $|x_{k_{n-1}}^*(x_{k_n})| < \epsilon/2$ for all n. Then

$$|(x_{k_n}^* - x_{k_{n-1}}^*)(x_{k_n})| = |x_{k_n}^*(x_{k_n}) - x_{k_{n-1}}^*(x_{k_n})| > \epsilon/2.$$

This is a contradiction, since $(x_{k_n}^* - x_{k_{n-1}}^*)$ is w^*-null in X^* and (x_{k_n}) is limited in X. ∎

A Banach space X has the *Grothendieck property* if every w^*- convergent sequence in X^* is weakly convergent [10, p. 179]. A space X is *weakly sequentially complete* if every weakly Cauchy sequence in X is weakly convergent.

Corollary 12. (i) *If X has the L-limited property, then X^* is weakly sequentially complete.*

(ii) *(*[39, Theorem 2.10]*) If X has the L-limited property, then X is a Grothendieck space.*

Proof. (i) Suppose that X has the L-limited property. Let (x_n^*) be a weakly Cauchy sequence in X^*. By Theorem 11, $\{x_n^* : n \in \mathbb{N}\}$ is an L-limited set, and thus relatively weakly compact. Hence (x_n^*) is weakly convergent.

(ii) Let (x_n^*) be a w^*- convergent sequence in X^*. By Theorem 11, (x_n^*) is an L-limited set, thus relatively weakly compact. Hence (x_n^*) is weakly convergent. ∎

Corollary 13. (i) *A Banach space X with the Gelfand-Phillips property has the wL-limited property if and only if X^* contains no copy of ℓ_1.*

(ii) *A Banach space X with the DPrcP has the (wSR) property if and only if X^* contains no copy of ℓ_1.*

(iii) *If X has the wL-limited property, then c_0 is not complemented in X.*

(iv) *(*[39, Corollary 2.9]*) A Banach space X is reflexive if and only if it has the Gelfand-Phillips property and the L-limited property.*

(v) *(*[7, Corollary 17]*) A Banach space X is reflexive if and only if it has the DPrcP and the (SR) property.*

Proof. (i) Suppose that X has the Gelfand-Phillips property and the wL-limited property. Then the identity operator $i : X \to X$ is limited completely continuous (since X has the Gelfand-Phillips property) and $i^* : X^* \to X^*$ is weakly precompact by Theorem 2. Hence X^* contains no copy of ℓ_1, by Rosenthal's ℓ_1 theorem. The converse follows by Rosenthal's ℓ_1 theorem.

(ii) The proof is similar to that of (i).

(iii) Suppose that X has the wL-limited property. Since c_0 is separable, it has the Gelfand-Phillips property [4, Proposition]. By (i), c_0 does not have the wL-limited property. Hence c_0 is not complemented in X by Corollary 3.

(iv) If X is reflexive, then it has the Gelfand-Phillips property [4, Proposition] and the L-limited property. Conversely, X^* contains no copy of ℓ_1 by (i) and X^* is weakly sequentially complete by Corollary 12. Then X^*, thus X, is reflexive.

(v) Suppose X is reflexive. Then X has the (SR) property and X^* does not contain a copy of ℓ_1. Hence X^{**}, thus X, has the $DPrcP$ ([13, Theorem 2]). Conversely, X^* contains no copy of ℓ_1 by (i) and X^* is weakly sequentially complete by [26, Corollary 3.26]. Then X is reflexive. ∎

Example. The converse of Corollary 12 (i) does not hold. Let X be the first Bourgain-Delbaen space [5, p. 25]. Then X has the Schur property and X^* is weakly sequentially complete. Since X has the Schur property, X does not have the L-limited property (by Corollary 13 (iv)).

Corollary 14. (i) *If X has property (wV), then X is weak sequentially Right.*

(ii) *If X has the L-limited (resp. the wL-limited) property, then X is sequentially Right (resp. weak sequentially Right).*

(iii) *If X is sequentially Right (resp. weak sequentially Right), then it has the $RDPP$ (resp. property (wL)).*

(iv) *If X is an infinite dimensional space with the L-limited property, then X^* does not have the Schur property.*

Proof. (i) Suppose X has property (wV). Let $T : X \to Y$ be pseudo weakly compact. Then T is unconditionally converging [35, Proposition 14]. Hence T^* is weakly precompact [19, Theorem 1]. Apply Theorem 2.

(ii) Suppose X has the the L-limited (resp. the wL-limited) property. Let (x_n) be a weakly null limited sequence in X. Then (x_n) is a weakly null DP sequence. Hence every Right subset of X^* is L-limited, thus relatively weakly compact (resp. weakly precompact).

(iii) Suppose X is sequentially Right (resp. weak sequentially Right). Every L-set in X^* is a Right set, thus relatively weakly compact (resp. weakly precompact). Hence X has the $RDPP$ [28] (resp. property (wL)).

(iv) Suppose that X has the L-limited property. Then X has the Grothendieck property, by Corollary 12 (ii). By the Jossefson-Nissezweig theorem, there is a w^*-null sequence (x_n^*) in X^* of norm one. Then (x_n^*) is weakly null and not norm null, and X^* does not have the Schur property. ∎

The fact that a space with property (SR) has the $RDPP$ was obtained in [26, Corollary 3.3].

Example. The converse of Corollary 14 (i) is not true. Let Y be the second Bourgain-Delbaen space [5, p. 25]. The space Y is a non-reflexive $\mathcal{L}_\infty$-space with the DPP that does not contain c_0 or ℓ_1 and such that $Y^* \simeq \ell_1$. The space Y is sequentially Right by Corollary 8. Since Y does not contain c_0, the identity operator $i : Y \to Y$ is unconditionally converging ([8, p. 54]) and $i^* : Y^* \to Y^*$ is not weakly precompact (since $Y^* \simeq \ell_1$). Thus Y does not have property (wV) by [19, Theorem 1].

The converse of Corollary 14 (ii) (strong properties) is not true. The second Bourgain-Delbaen space Y is sequentially Right and does not have the L-limited property (by Corollary 14 (iv)).

The converse of Corollary 14 (iii) (strong properties) is not true. Let J be the original James space [24]. Since J is separable and 1-codimensional in J^{**}, all duals of J are separable and ℓ_1 fails to embed in any of them. Moreover, none of these spaces can be weakly sequentially complete. Thus J and its duals are weak sequentially Right, but none of these spaces are sequentially Right by [26, Corollary 3.26], since their duals are not weakly sequentially complete. Since J does not contain ℓ_1, every completely continuous operator on J is compact (by a result of Odell [37, p. 377]), and thus weakly compact. Hence J has the $RDPP$.

The following theorem shows that the space E has property (SR) if some subspace of it has property (SR).

LEMMA 15. *([23, Theorem 2.7]) Let E be a Banach space, F a reflexive subspace of E (resp. a subspace not containing copies of ℓ_1), and $Q : E \to E/F$ the quotient map. Let (x_n) be a bounded sequence in E such that $(Q(x_n))$ is weakly convergent (resp. weakly Cauchy). Then (x_n) has a weakly convergent (resp. weakly Cauchy) subsequence.*

Let E be a Banach space and F be a subspace of E^*. Let

$$^{\perp}F = \{x \in E : y^*(x) = 0 \text{ for all } y^* \in F\}.$$

THEOREM 16. (i) *Let E be a Banach space and F be a reflexive subspace of E^*. If $^{\perp}F$ has property (SR) (resp. the L-limited property), then E has the same property.*

(ii) *Let E be a Banach space and F be a subspace of E^* not containing copies of ℓ_1. If $^{\perp}F$ has property (wSR) (resp. the wL-limited property), then E has the same property.*

Proof. We only prove (i) for the (SR) property. The other proofs are similar.

Suppose that $^{\perp}F$ has property (SR). Let $Q : E^* \to E^*/F$ be the quotient map and $i : E^*/F \to (^{\perp}F)^*$ be the natural surjective isomorphism ([31, Theorem 1.10.16]). It is known that $iQ : E^* \to (^{\perp}F)^*$ is $w^* - w^*$ continuous, since $iQ(x^*)$ is the restriction of x^* to $^{\perp}F$ ([31, Theorem 1.10.16]). Then there is an operator $S :^{\perp} F \to E$ such that $iQ = S^*$.

Let $T : E \to G$ be a pseudo weakly compact operator. Then $TS :^{\perp} F \to G$ is pseudo weakly compact. Since ${}^{\perp}F$ has property (SR), TS has a weakly compact adjoint, by Theorem 2. Since $S^*T^* = iQT^*$ is weakly compact and i is a surjective isomorphism, QT^* is weakly compact. Let (x_n^*) be a sequence in B_{G^*}. By passing to a subsequence, we can assume that $(QT^*(x_n^*))$ is weakly convergent. Hence $(T^*(x_n^*))$ has a weakly convergent subsequence by Lemma 15. Thus E has property (SR). ∎

The $w^* - w$ continuous operators from X^* to Y will be denoted by $L_{w^*}(X^*, Y)$.

THEOREM 17. *Let X be a Banach space and A be a bounded subset of X^*. The following are equivalent:*

(i) *A is an L-limited set.*

(ii) *Every operator $T \in L_{w^*}(X^*, c_0)$ that is w^*-norm sequentially continuous maps A into a relatively compact set.*

Proof. (i)⇒(ii) Let $T \in L_{w^*}(X^*, c_0)$ be an operator so that T is w^*-norm sequentially continuous. Note that $T^* \in L_{w^*}(\ell_1, X)$, $(x_n) = (T^*(e_n^*))$ is a weakly null sequence in X, and $T(x^*) = (x^*(x_i))_i$. If (x_n^*) is a w^*-null sequence in X^* and $y \in B_{\ell_1}$, then

$$|\langle x_n^*, T^*(y)\rangle| \le \|T(x_n^*)\| \to 0.$$

Hence $T^*(B_{\ell_1})$, thus (x_n), is limited. Since A is an L-limited set, $\sup_{x^* \in A} |x^*(x_n)| \to 0$. Therefore $T(A)$ is relatively compact in c_0, by the characterization of relatively compact subsets of c_0.

(ii)⇒(i) Let (x_n) be a weakly null limited sequence in X. Define $T : X^* \to c_0$ by $T(x^*) = (x^*(x_n))_n$. Note that $T^*(b) = \sum b_n x_n$, $b = (b_n) \in \ell_1$, $T^*(\ell_1) \subseteq X$, and $T \in L_{w^*}(X^*, c_0)$. If (x_n^*) is a w^*-null sequence in X^*, then

$$\|T(x_n^*)\| = \sup_i |x_n^*(x_i)| \to 0,$$

since (x_i) is limited. Hence T is w^*-norm sequentially continuous operator, and $T(A)$ is relatively compact in c_0. By the characterization of relatively compact subsets of c_0, $\sup_{x^* \in A} |x^*(x_n)| \to 0$, and thus A is an L-limited subset of X^*. ∎

An operator $T : X \to Y$ is called *limited* if $T(B_X)$ is a limited subset of Y ([4]). The operator T is limited if and only if $T^* : Y^* \to X^*$ is w^*-norm sequentially continuous.

Corollary 18. *Let X be a Banach space and A be a bounded subset of X^*. The following are equivalent:*

(i) *A is an L-limited set.*

(ii) *For every limited operator $S \in L_{w^*}(\ell_1, X)$, $S^*(A)$ is relatively compact.*

Proof. (i)⇒(ii) Let $S \in L_{w^*}(\ell_1, X)$ be a limited operator. Then $S^* \in L_{w^*}(X^*, c_0)$ and S^* is w^*-norm sequentially continuous. By Theorem 17, $S^*(A)$ is relatively compact.

(ii)⇒(i) Let $T \in L_{w^*}(X^*, c_0)$ be a w^*-norm sequentially continuous operator and let $S = T^*$. Then $S \in L_{w^*}(\ell_1, X)$, S is limited, and $S^*(A)$ is relatively compact. By Theorem 17, A is an L-limited set. ∎

Corollary 19. *Suppose that A is a bounded subset of X^* such that for every $\epsilon > 0$, there is an L-limited subset A_ϵ of X^* such that $A \subseteq A_\epsilon + \epsilon B_{X^*}$. Then A is an L-limited set.*

Proof. Suppose that A satisfies the hypothesis. Let $\epsilon > 0$ and A_ϵ as in the hypothesis. Let $T \in L_{w^*}(X^*, c_0)$ be an operator such that T is w^*-norm sequentially continuous and $\|T\| \leq 1$. Then $T(A) \subseteq T(A_\epsilon) + \epsilon B_{c_0}$, and $T(A_\epsilon)$ is relatively compact by Theorem 17. Then $T(A)$ is relatively compact [8, p. 5], and thus A is an L-limited set by Theorem 17. ∎

4. The (wSR) and wL-limited properties in projective tensor products

In this section we consider the (SR) and L-limited properties in the projective tensor product $X \otimes_\pi Y$. We begin by noting that there are examples of Banach spaces X and Y such that $X \otimes_\pi Y$ has the (SR) and L-limited properties. If $1 < q' < p < \infty$, then $L(\ell_p, \ell_{q'}) = K(\ell_p, \ell_{q'})$ ([36], [10, p. 247]). If q is the conjugate of q', then $\ell_p \otimes_\pi \ell_q$ is reflexive (by [38, Theorem 4.19], [10, p. 248]), and thus has the (SR) and L-limited properties. Then the spaces $X = \ell_p$ and $Y = \ell_q$ are as desired.

If $H \subseteq L(X, Y)$, $x \in X$ and $y^* \in Y^*$, let $H(x) = \{T(x) : T \in H\}$ and $H^*(y^*) = \{T^*(y^*) : T \in H\}$.

In the proofs of Theorems 23 and 25 we will need the following results.

THEOREM 20. *([20, Theorem 1]) Let H be a subset of $K(X,Y)$ such that*

(i) *$H(x)$ is weakly precompact compact for all $x \in X$.*
(ii) *$H^*(y^*)$ is relatively weakly compact for all $y^* \in Y^*$.*

Then H is weakly precompact.

THEOREM 21. *([20, Theorem 3]) Suppose that $L(X,Y) = K(X,Y)$ and H is a subset of $K(X,Y)$ such that:*

(i) *$H(x)$ is relatively weakly compact for all $x \in X$.*
(ii) *$H^*(y^*)$ is relatively weakly compact for all $y^* \in Y^*$.*

Then H is relatively weakly compact.

LEMMA 22. *Suppose $L(X,Y^*) = K(X,Y^*)$. If (x_n) is a weakly null DP sequence in X and (y_n) is a DP sequence in Y, then $(x_n \otimes y_n)$ is a weakly null DP sequence in $X \otimes_\pi Y$.*

Proof. Suppose that (x_n) is weakly null DP in X and $\|y_n\| \le M$ for all $n \in \mathbb{N}$. Let $T \in L(X,Y^*) \simeq (X \otimes_\pi Y)^*$ ([10, p. 230]). Since T is completely continuous,

$$\langle T, x_n \otimes y_n \rangle \le M\|T(x_n)\| \to 0.$$

Thus $(x_n \otimes y_n)$ is weakly null in $X \otimes_\pi Y$.

Let (A_n) be a weakly null sequence in $(X \otimes_\pi Y)^* \simeq L(X,Y^*)$ and let $x^{**} \in X^{**}$. Since the map $\gamma_{x^{**}} : L(X,Y^*) = K(X,Y^*) \to Y^*$, $\gamma_{x^{**}}(T) = T^{**}(x^{**})$ is linear and bounded, $(A_n^{**}(x^{**}))$ is weakly null in Y^*. Therefore

$$\langle x^{**}, A_n^*(y_n) \rangle = \langle A_n^{**}(x^{**}), y_n \rangle \to 0,$$

since (y_n) is DP in Y. Hence $(A_n^*(y_n))$ is weakly null in X^*. Then

$$\langle A_n, x_n \otimes y_n \rangle = \langle A_n^*(y_n), x_n \rangle \to 0,$$

since (x_n) is DP in X. Thus $(x_n \otimes y_n)$ is DP in $X \otimes_\pi Y$. ∎

THEOREM 23. *([7, Theorem 18]) Suppose that $L(X,Y^*) = K(X,Y^*)$. If X and Y are sequentially Right, then $X \otimes_\pi Y$ is sequentially right.*

Proof. Let H be a Right subset of $(X \otimes_\pi Y)^* \simeq L(X, Y^*) = K(X, Y^*)$. We will use Theorem 20. We will verify the conditions (i) and (ii) of this theorem. Let (T_n) be a sequence in H and let $x \in X$. We prove that $\{T_n(x) : n \in \mathbb{N}\}$ is a Right subset of Y^*. Let (y_n) be a Right null sequence in Y. Thus (y_n) is weakly null and DP. For each n,

$$\langle T_n(x), y_n \rangle = \langle T_n, x \otimes y_n \rangle.$$

We show that $(x \otimes y_n)$ is Right null in $X \otimes_\pi Y$. If $T \in (X \otimes_\pi Y)^* \simeq L(X, Y^*)$ ([10, p. 230]), then

$$|\langle T, x \otimes y_n \rangle| = |\langle T(x), y_n \rangle| \to 0,$$

since (y_n) is weakly null. Thus $(x \otimes y_n)$ is weakly null. Let (A_n) be a weakly null sequence in $(X \otimes_\pi Y)^* \simeq L(X, Y^*)$. Since the map $\phi_x : L(X, Y^*) \to Y^*$, $\phi_x(T) = T(x)$ is linear and bounded, $(A_n(x))$ is weakly null in Y^*. Therefore

$$|\langle A_n, x \otimes y_n \rangle| = |\langle A_n(x), y_n \rangle| \to 0,$$

since (y_n) is DP in Y. Thus $(x \otimes y_n)$ is DP and $(x \otimes y_n)$ is Right null. Since (T_n) is a Right set,

$$|\langle T_n, x \otimes y_n \rangle| = |\langle T_n(x), y_n \rangle| \to 0.$$

Thus $\{T_n(x) : n \in \mathbb{N}\}$ is a Right subset of Y^*, hence relatively weakly compact (by Theorem 2). We thus verified (i) of Theorem 20.

Let $y^{**} \in Y^{**}$. We show that $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ is a Right subset of X^*. Let (x_n) be a Right null sequence in X. Thus (x_n) is weakly null and DP. For each n,

$$\langle T_n^*(y^{**}), x_n \rangle = \langle y^{**}, T_n(x_n) \rangle.$$

It is enough to show that $(T_n(x_n))$ is weakly null in Y^*. Let (y_n) be a Right null sequence in Y. By Lemma 22 and Proposition 1, $(x_n \otimes y_n)$ is Right null in $X \otimes_\pi Y$. Since (T_n) is a Right set,

$$|\langle T_n, x_n \otimes y_n \rangle| = |\langle T_n(x_n), y_n \rangle| \to 0.$$

Therefore $(T_n(x_n))$ is a Right subset of Y^*, thus relatively weakly compact (by Theorem 2). By passing to a subsequence, we can assume that $(T_n(x_n))$ is weakly convergent. Let $y \in Y$. An argument similar to the one above shows that $(x_n \otimes y)$ is Right null in $X \otimes_\pi Y$. Then

$$|\langle T_n, x_n \otimes y \rangle| = |\langle T_n(x_n), y \rangle| \to 0,$$

since (T_n) is a Right set. Hence $(T_n(x_n))$ is w^*-null. Since $(T_n(x_n))$ is also weakly convergent, $(T_n(x_n))$ is weakly null. Then $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ is a Right subset of X^*. Hence $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ is relatively weakly compact (by Theorem 2). By Theorem 20, H is weakly precompact. We can assume without loss of generality that (T_n) is weakly Cauchy. Since X and Y are sequentially Right, X^* and Y^* are both weakly sequentially complete [26, Corollary 3.26], and thus $L(X, Y^*) = K(X, Y^*)$ is weakly sequentially complete, by [22, Theorem 3.10]. Then (T_n) is weakly convergent. ∎

Remark. Theorem 23 can also be proved as follows. Let H be a Right subset of $(X \otimes_\pi Y)^* \simeq L(X, Y^*) = K(X, Y^*)$ and let (T_n) be a sequence in H. By the proof of Theorem 23, $\{T_n(x) : n \in \mathbb{N}\}$ and $\{T_n^*(y^{**}) : n \in \mathbb{N}\}$ are relatively weakly compact for all $x \in X$ and $y^{**} \in Y^{**}$. By Theorem 21, H is relatively weakly compact.

LEMMA 24. *Suppose $L(X, Y^*) = K(X, Y^*)$. If (x_n) is a weakly null limited sequence in X and (y_n) is a limited sequence in Y, then $(x_n \otimes y_n)$ is a weakly null limited sequence in $X \otimes_\pi Y$.*

Proof. By Lemma 22, $(x_n \otimes y_n)$ is a weakly null. Let (A_n) be a w^*-null sequence in $(X \otimes_\pi Y)^* \simeq L(X, Y^*)$. Then $(A_n^*(x))$ is a w^*-null sequence in Y^*. If $x \in X$, then $\langle A_n(x), y_n \rangle = \langle A_n^*(y_n), x \rangle \to 0$, since (y_n) is limited in Y. Hence $(A_n^*(y_n))$ is w^*-null in X^*. Since (x_n) is limited,

$$\langle A_n, x_n \otimes y_n \rangle = \langle A_n^*(y_n), x_n \rangle \to 0.$$

Thus $(x_n \otimes y_n)$ is limited in $X \otimes_\pi Y$. ∎

THEOREM 25. ([7, Theorem 25]) *Suppose that $L(X, Y^*) = K(X, Y^*)$. If X and Y have the L-limited property, then $X \otimes_\pi Y$ has the L-limited property.*

Proof. The proof is similar to the proof of Theorem 23 and uses Lemma 24. ∎

Remark. Theorem 25 can also be proved with a method similar to the one in the previous remark.

The fact that the (SR) and L-limited properties are inherited by quotients, immediately implies the following result.

Corollary 26. (i) *Suppose that* $L(X^*, Y^*) = K(X^*, Y^*)$*, and* X^* *and* Y *are sequentially Right. Then the space* $N_1(X, Y)$ *of all nuclear operators from* X *to* Y *is sequentially Right.*

(ii) *Suppose that* $L(X^*, Y^*) = K(X^*, Y^*)$*, and* X^* *and* Y *have the L-limited property. Then the space* $N_1(X, Y)$ *of all nuclear operators from* X *to* Y *has the L-limited property.*

Proof. It is known that $N_1(X, Y)$ is a quotient of $X^* \otimes_\pi Y$ ([38, p. 41]). (i) Apply Theorem 23. (ii) Apply Theorem 25. ∎

Observation 1. If $T : Y \to X^*$ be an operator such that $T^*|_X$ is (weakly) compact, then T is (weakly) compact. To see this, let $T : Y \to X^*$ be an operator such that $T^*|_X$ is (weakly) compact. Let $S = T^*|_X$. Suppose $x^{**} \in B_{X^{**}}$ and choose a net (x_α) in B_X which is w^*- convergent to x^{**}. Then $(T^*(x_\alpha)) \overset{w^*}{\to} T^*(x^{**})$. Now, $(T^*(x_\alpha)) \subseteq S(B_X)$, which is a relatively (weakly) compact set. Then $(T^*(x_\alpha)) \to T^*(x^{**})$ (resp. $(T^*(x_\alpha)) \overset{w}{\to} T^*(x^{**})$). Hence $T^*(B_{X^{**}}) \subseteq \overline{S(B_X)}$, which is relatively (weakly) compact. Therefore $T^*(B_{X^{**}})$ is relatively (weakly) compact, and thus T is (weakly) compact.

It follows that if $L(X, Y^*) = K(X, Y^*)$, then $L(Y, X^*) = K(Y, X^*)$.

The following result improves Corollaries 19 and 21 of [7].

Corollary 27. *If* X *is sequentially Right and* Y^* *has the Schur property (or* Y *is sequentially Right and* X^* *has the Schur property), then* $X \otimes_\pi Y$ *is sequentially Right.*

Proof. Since Y^* has the Schur property, every Right set in Y^* is relatively compact (by Corollary 9). Let $T : X \to Y^*$ be an operator. Then T is pseudo weakly compact (since Y^* has the Schur property), hence compact (by Theorem 2). Apply Theorem 23. ∎

Theorem 28. *Suppose that* $L(X, Y^*) = K(X, Y^*)$*. The following statements are equivalent:*

1. (i) *X and Y are sequentially Right and at least one of them does not contain ℓ_1.*
   (ii) *$X \otimes_\pi Y$ is sequentially Right.*
2. (i) *X and Y have the L-limited property and at least one of them does not contain ℓ_1.*
   (ii) *$X \otimes_\pi Y$ has the L-limited property.*

Proof. We only prove 1. The other proof is similar.

(i)⇒(ii) by Theorem 23.

(ii)⇒(i) Suppose that $X \otimes_\pi Y$ is sequentially Right. Then X and Y are sequentially Right, since the sequentialy Right property is inherited by quotients [26, Proposition 3.8]. We will show that $\ell_1 \not\hookrightarrow X$ or $\ell_1 \not\hookrightarrow Y$. Suppose that $\ell_1 \hookrightarrow X$ and $\ell_1 \hookrightarrow Y$. Hence $L_1 \hookrightarrow X^*$ ([32, Theorem 3.4], [8, p. 212]). Also, the Rademacher functions span ℓ_2 inside of L_1, and thus $\ell_2 \hookrightarrow X^*$. Similarly $\ell_2 \hookrightarrow Y^*$. Then $c_0 \hookrightarrow K(X, Y^*)$ ([15, Theorem 3], [21, Corollary 21]). Thus $\ell_1 \overset{c}{\hookrightarrow} X \otimes_\pi Y$ ([3, Theorem 4], [8, Theorem 10, p. 48]), a contradiction with Corollary 5 (iii). ∎

Observation 2. If $\ell_1 \hookrightarrow X$ and $\ell_1 \hookrightarrow Y$, then $\ell_2 \hookrightarrow X^*$ and $\ell_2 \hookrightarrow Y^*$, and $c_0 \hookrightarrow K(X, Y^*)$ ([15, Theorem 3], [21, Corollary 21]). More generally, if $\ell_1 \hookrightarrow X$ and $\ell_p \hookrightarrow Y^*$, $p \geq 2$, then $c_0 \hookrightarrow K(X, Y^*)$ ([15], [21]). Thus $\ell_1 \overset{c}{\hookrightarrow} X \otimes_\pi Y$ ([3, Theorem 4], [8, Theorem 10, p. 48]). Hence $X \otimes_\pi Y$ is not weak sequentially Right (and does not have the wL-limited property), by Corollary 5 (iii).

COROLLARY 29. *Suppose that $L(X, Y^*) = K(X, Y^*)$.*

1. *If $X \otimes_\pi Y$ is weak sequentially Right, then X and Y are weak sequentially Right and at least one of them does not contain ℓ_1.*
2. *If $X \otimes_\pi Y$ has the wL-limited property, then X and Y have the wL-limited property and at least one of them does not contain ℓ_1.*

Proof. We only prove 1. The other proof is similar. If $X \otimes_\pi Y$ is weak sequentially Right, then X and Y are weak sequentially Right, since the weak sequentially Right property is inherited by quotients (by Corollary 3). Apply Observation 2. ∎

COROLLARY 30. ([7, Theorem 22]) *Suppose that X and Y have the DPP. The following statements are equivalent:*

(i) *X and Y are sequentially Right and at least one of them does not contain ℓ_1.*

(ii) *$X \otimes_\pi Y$ is sequentially Right.*

Proof. (i)⇒(ii) Suppose that X and Y have the DPP. Without loss of generality suppose that $\ell_1 \not\hookrightarrow X$. Then X^* has the Schur property [9]. Apply Corollary 27.

(ii)⇒(i) by Observation 2. ∎

By Corollary 30, the space $C(K_1) \otimes_\pi C(K_2)$ is sequentially Right if and only if either K_1 or K_2 is dispersed.

Next we present some results about the necessity of the condition $L(X, Y^*) = K(X, Y^*)$. It is implicit in [6] that a Banach space X has all bilinear forms weakly sequentially continuous if and only if every operator $S : X \to X^*$ transforms weakly null sequences into L-sets. Emmanuelle shows in [13] that a Banach space X does not contain ℓ_1 if and only if every L-set in X^* is relatively compact. Then, it is easy to see that if X and Y are not containing ℓ_1, then $L(X, Y^*) = K(X, Y^*)$ if and only if every operator $T : X \to Y^*$ transforms weakly null sequences into L-sets (for more details see [6]).

A Banach space X has the *approximation property* if for each norm compact subset M of X and $\epsilon > 0$, there is a finite rank operator $T : X \to X$ such that $\|Tx - x\| < \epsilon$ for all $x \in M$. If in addition T can be found with $\|T\| \leq 1$, then X is said to have the *metric approximation property*. $C(K)$ spaces, c_0, ℓ_p, $1 \leq p < \infty$, $L_p(\mu)$ (μ any measure), $1 \leq p < \infty$, and their duals have the metric approximation property [10, p. 238].

A separable Banach space X has an *unconditional compact expansion of the identity (u.c.e.i)* if there is a sequence (A_n) of compact operators from X to X such that $\sum A_n(x)$ converges unconditionally to x for all $x \in X$ [17]. In this case, (A_n) is called an (u.c.e.i.) of X.

A sequence (X_n) of closed subspaces of a Banach space X is called an *unconditional Schauder decomposition* of X if every $x \in X$ has a unique representation of the form $x = \sum x_n$, with $x_n \in X_n$, for every n, and the series converges unconditionally [30, p. 48].

The space X has (Rademacher) *cotype* q for some $2 \leq q \leq \infty$ if there is a constant C such that for for every n and every $x_1, x_2, \ldots, x_n$ in X,

$$\left(\sum_{i=1}^{n} \|x_i\|^q \right)^{1/q} \leq C \left(\int_0^1 \|r_i(t) x_i\|^q dt \right)^{1/q},$$

where (r_n) are the Radamacher functions. A Hilbert space has cotype 2 [8, p. 118]. $\mathcal{L}_p$-spaces have cotype 2, if $1 \leq p \leq 2$ [8, p. 118].

THEOREM 31. *Assume one of the following holds:*

(i) *If $T : X \to Y^*$ is an operator which is not compact, then there is a sequence (T_n) in $K(X, Y^*)$ such that for each $x \in X$, the series $\sum T_n(x)$ converges unconditionally to $T(x)$.*
(ii) *X is an $\mathcal{L}_\infty$-space and Y^* is a subspace of an $\mathcal{L}_1$-space.*
(iii) *$X = C(K)$, K a compact Hausdorff space, and Y^* is a space with cotype 2.*
(iv) *Either X or Y^* has an (u.c.e.i.).*
(v) *X has the DPP and $\ell_1 \hookrightarrow Y$.*
(vi) *X and Y have the DPP.*

If $X \otimes_\pi Y$ is weak sequentially Right, then $L(X, Y^*) = K(X, Y^*)$.

Proof. Suppose that $X \otimes_\pi Y$ is weak sequentially Right. Then X and Y are weak sequentially Right.

(i) Let $T : X \to Y^*$ be a noncompact operator. Let (T_n) be a sequence as in the hypothesis. By the Uniform Boundedness Principle, $\{\sum_{n \in A} T_n : A \subseteq \mathbb{N}, A \text{ finite}\}$ is bounded in $K(X, Y^*)$. Then $\sum T_n$ is wuc and not unconditionally convergent (since T is noncompact). Hence $c_0 \hookrightarrow K(X, Y^*)$ ([3, Theorem 5]), $\ell_1 \overset{c}{\hookrightarrow} X \otimes_\pi Y$ ([3, Theorem 4]), and we have a contradiction with Corollary 5 (iii).

Suppose (ii) or (iii) holds. It is known that any operator $T : X \to Y^*$ is 2-absolutely summing ([8, p. 189]), hence it factorizes through a Hilbert space. If $L(X, Y^*) \neq K(X, Y^*)$, then $c_0 \hookrightarrow K(X, Y^*)$ (by [16, Remark 3]), a contradiction.

(iv) If $L(X, Y^*) \neq K(X, Y^*)$, then $c_0 \hookrightarrow K(X, Y^*)$ (by [27, Theorem 6]), a contradiction.

(v) Suppose that X has the DPP and $\ell_1 \hookrightarrow Y$. By Observation 1, $\ell_1 \not\hookrightarrow X$. Then X^* has the Schur property ([9, Theorem 3]). Let $T : Y \to X^*$ be an operator. Then T is pseudo weakly compact (since X^* has the Schur property), and thus weakly precompact (by Corollary 5 (i)). Then $L(Y, X^*) = K(Y, X^*)$. Hence $L(X, Y^*) = K(X, Y^*)$, by Observation 1.

(vi) Suppose that X and Y have the DPP. Then $L(X, Y^*) = K(X, Y^*)$, either by (v) if $\ell_1 \hookrightarrow Y$, or since Y^* has the Schur property ([9, Theorem 3]) if $\ell_1 \not\hookrightarrow Y$ (by an argument similar to the one in (v)). ∎

Assumption (i) of the previous theorem is satisfied, for instance, if X^* (or Y^*) has an (u.c.e.i.).

Examples. By Theorem 31, the space $\ell_p \otimes \ell_q$, where $1 < p \leq q' < \infty$ and q and q' are conjugate, is not weak sequentially Right, since the natural inclusion map $i : \ell_p \to \ell_{q'}$ is not compact.

The space $C(K) \otimes_\pi \ell_p$, with K not dispersed and $1 < p \leq 2$, is not weak sequentially Right (by Observation 2, since $\ell_1 \hookrightarrow C(K)$ and $\ell_2 \hookrightarrow \ell_p^*$).

For $1 < p_1, p_2 < \infty$, $L_{p_1}[0,1] \otimes_\pi L_{p_2}[0,1]$ is not weak sequentially Right by Corollary 5 (iii), since $\ell_1 \overset{c}{\hookrightarrow} L_{p_1}[0,1] \otimes_\pi L_{p_2}[0,1]$ ([38, Corollary 2.26]).

Theorem 32. (i) *Suppose Y^* is complemented in a Banach space Z which has an unconditional Schauder decomposition (Z_n), and $L(X, Z_n) = K(X, Z_n)$ for all n. If $X \otimes_\pi Y$ is weak sequentially Right, then $L(X, Y^*) = K(X, Y^*)$.*

(ii) *Suppose either X^* or Y^* has the metric approximation property. If $X \otimes_\pi Y$ is sequentially Right, then $W(X, Y^*) = K(X, Y^*)$.*

Proof. (i) Let $T : X \to Y^*$ be a noncompact operator, $P_n : Z \to Z_n$, $P_n(\sum z_i) = z_n$, and let P be the projection of Z onto Y^*. Define $T_n : X \to Y^*$ by $T_n(x) = PP_nT(x)$, $x \in X$, $n \in \mathbb{N}$. Note that P_nT is compact since $L(X, Z_n) = K(X, Z_n)$. Then T_n is compact for each n. For each $z \in Z$, $\sum P_n(z)$ converges unconditionally to z; thus $\sum T_n(x)$ converges unconditionally to $T(x)$ for each $x \in X$. Then $\sum T_n$ is wuc and not unconditionally converging. Hence $c_0 \hookrightarrow K(X, Y^*)$ ([3, Theorem 5]), and we obtain a contradiction.

(ii) Since $X \otimes_\pi Y$ is sequentially Right, $(X \otimes_\pi Y)^* \simeq L(X, Y^*)$ is weakly sequentially complete ([26, Corollary 3.26]). Under assumption (ii), [29, Corollary 2.4] implies $W(X, Y^*) = K(X, Y^*)$.

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