



Stability of some essential B-spectra of pencil operators and application

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Abstract: In this paper, we give some results on the essential B-spectra of a linear operator pencil, which are used to determine the essential B-spectra of an integro-differential operator with abstract boundary conditions in the Banach space $L_p([-a, a] \times [-1, 1])$, $p \geq 1$ and $a > 0$.

Key words: Operator pencil, finite-rank and power finite-rank perturbations, essential B-spectra, transport operator.

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1. INTRODUCTION

Let X be a Banach space. We will denote by $\mathcal{C}(X)$ (resp. $L(X)$) the set of all closed linear (resp. the algebra of all bounded) linear operators from X into X . For $T \in \mathcal{C}(X)$, we write $D(T) \subset X$ for the domain, $N(T) \subset X$ for the null space and $R(T) \subset X$ for the range of T . We denote by $\alpha(T)$ the dimension of $N(T)$ and $\beta(T)$ the codimension of $R(T)$ in X . For $T \in \mathcal{C}(X)$ and $M \in L(X)$, we define the resolvent of the linear operator pencil $\lambda M - T$, where $\lambda \in \mathbb{C}$, or the M-resolvent of T by

$$\rho_M(T) := \{\lambda \in \mathbb{C} : \lambda M - T \text{ has a bounded inverse}\},$$

and its spectrum by

$$\sigma(M, T) = \mathbb{C} \setminus \rho_M(T).$$

For $T \in \mathcal{C}(X)$, we define the set

$$\Delta(T) = \{n \in \mathbb{N} : \forall m \in \mathbb{N}, m \geq n \Rightarrow R(T^n) \cap N(T) \subset R(T^m) \cap N(T)\}.$$

The degree of stable iteration of T is defined as $\text{dis}(T) = \inf \Delta(T)$, where $\text{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.



We define the set of upper semi-Fredholm operators by

$$\Phi_+(X) = \{T \in \mathcal{C}(X) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } X\},$$

and the set of lower semi-Fredholm operators by

$$\Phi_-(X) = \{T \in \mathcal{C}(X) : \beta(T) < \infty \text{ and } R(T) \text{ is closed in } X\}.$$

$\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ will denote the set of Fredholm operators from X into X . The index of a Fredholm operator T is defined by $i(T) = \alpha(T) - \beta(T)$.

According to [17], an operator $T \in \mathcal{C}(X)$ is called quasi-Fredholm of degree $d \in \mathbb{N}$ if the following three conditions are satisfied:

- (i) $\text{dis}(T) = d$;
- (ii) $R(T^d) \cap N(T)$ is a closed and complemented subspace of X ;
- (iii) $R(T) + N(T^d)$ is a closed and complemented subspace of X .

This set of operators will be denoted by $QF(d)$.

Following [10, Definition 2.4], an operator $T \in \mathcal{C}(X)$ is called upper semi B-Fredholm (resp. lower semi B-Fredholm) if there exists an integer $d \in \mathbb{N}$ such that $T \in QF(d)$ and such that $N(T) \cap R(T^d)$ is of finite dimension (resp. $R(T) + N(T^d)$ is of finite codimension). These sets are denoted respectively by $\Phi_B^+(X)$ and $\Phi_B^-(X)$. We denote by $\Phi_B(X) := \Phi_B^+(X) \cap \Phi_B^-(X)$, the set of B-Fredholm operators from X into X . In this case, the index of T is defined as the integer: $\text{ind}(T) = \dim(N(T) \cap R(T^d)) - \text{codim}(R(T) + N(T^d))$. An operator $T \in \mathcal{C}(X)$ is called B-Weyl if it is a B-Fredholm operator of index zero. We will denote this set by $BW(X)$.

For $T \in \mathcal{C}(X)$, we define the ascent $a(T)$ of T by

$$a(T) = \inf \{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\},$$

and the descent $d(T)$ of T by

$$d(T) = \inf \{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}.$$

We define respectively the set of Drazin invertible operators, left Drazin invertible operators and right Drazin invertible operators as follows:

$$\begin{aligned} DR(X) &:= \{T \in \mathcal{C}(X) : a(T) \text{ and } d(T) \text{ are both finite}\}, \\ LD(X) &:= \{T \in \mathcal{C}(X) : a(T) \text{ is finite and } R(T^{a(T)+1}) \text{ is closed}\}, \\ RD(X) &:= \{T \in \mathcal{C}(X) : d(T) \text{ is finite and } R(T^{d(T)}) \text{ is closed}\}. \end{aligned}$$

An operator $T \in \mathcal{C}(X)$ is of Kato type if there exist an integer $d \in \mathbb{N}$ and a pair of two closed subspaces (N_1, N_2) of X such that:

- (i) $X = N_1 \oplus N_2$;
- (ii) $T(N_1) \subset N_1$ and $T_{/N_1}$ is semi-regular;
- (iii) $T(N_2) \subset N_2$ and $(T_{/N_2})^d = 0$, i.e., $T_{/N_2}$ is nilpotent.

Note that, in the case of Hilbert spaces, Labrousse in [17, Theorem 3.2.2] has shown that, the set of Quasi-Fredholm operators coincides with the Kato type operators for the class of closed operators. According to [17, p. 206, Remark], this equivalence is also true in the case of Banach spaces.

For $T \in \mathcal{C}(X)$ and $M \in L(X)$, we define the B-Fredholm spectrum, Drazin spectrum, the upper semi B-Fredholm spectrum, the lower semi B-Fredholm spectrum, the left Drazin spectrum, the right Drazin spectrum, the B-Weyl spectrum, the closed-range spectrum and the Kato spectrum of the linear operator pencil $\lambda M - T$, where $\lambda \in \mathbb{C}$, or the pair (M, T) as follows:

$$\begin{aligned}
\sigma_{BF}(M, T) &= \{ \lambda \in \mathbb{C} : \lambda M - T \notin \Phi_B(X) \}, \\
\sigma_D(M, T) &= \{ \lambda \in \mathbb{C} : \lambda M - T \notin DR(X) \}, \\
\sigma_{BF+}(M, T) &= \{ \lambda \in \mathbb{C} : \lambda M - T \notin \Phi_B^+(X) \}, \\
\sigma_{BF-}(M, T) &= \{ \lambda \in \mathbb{C} : \lambda M - T \notin \Phi_B^-(X) \}, \\
\sigma_{LD}(M, T) &= \{ \lambda \in \mathbb{C} : \lambda M - T \notin LD(X) \}, \\
\sigma_{RD}(M, T) &= \{ \lambda \in \mathbb{C} : \lambda M - T \notin RD(X) \}, \\
\sigma_{BW}(M, T) &= \{ \lambda \in \mathbb{C} : \lambda M - T \notin BW(X) \}, \\
\sigma_{ec}(M, T) &= \{ \lambda \in \mathbb{C} : R(\lambda M - T) \text{ is not closed} \}, \\
\sigma_{ek}(M, T) &= \{ \lambda \in \mathbb{C} : \lambda M - T \text{ is not Kato} \}.
\end{aligned}$$

The point spectrum, the residual spectrum and the continuous spectrum of the pair (M, T) , when $T \in \mathcal{C}(X)$ and $M \in L(X)$, are defined respectively by:

$$\begin{aligned}
\sigma_p(M, T) &= \{ \lambda \in \mathbb{C} : \lambda M - T \text{ is not injective} \}, \\
\sigma_r(M, T) &= \{ \lambda \in \mathbb{C} : N(\lambda M - T) = \{0\} \text{ and } \overline{R(\lambda M - T)} \subsetneq X \}, \\
\sigma_c(M, T) &= \{ \lambda \in \mathbb{C} : N(\lambda M - T) = \{0\}, \overline{R(\lambda M - T)} = X \\
&\quad \text{and } R(\lambda M - T) \neq X \},
\end{aligned}$$

where, $\overline{R(\lambda M - T)}$ is the closure of $R(\lambda M - T)$. The collection $\{\sigma_p(M, T), \sigma_r(M, T), \sigma_c(M, T)\}$ forms a partition of the spectrum $\sigma(M, T)$, which means that they are pairwise disjoint and $\sigma(M, T) = \sigma_p(M, T) \cup \sigma_r(M, T) \cup \sigma_c(M, T)$.

For $T \in \mathcal{C}(X)$ and $M \in L(X)$, the upper semi B-Fredholm, the lower semi B-Fredholm and the B-Fredholm resolvent of the linear operator pencil $\lambda M - T$, where $\lambda \in \mathbb{C}$, are defined respectively by

$$\begin{aligned}\rho_{BF^+}(M, T) &= \mathbb{C} \setminus \sigma_{BF^+}(M, T), \\ \rho_{BF^-}(M, T) &= \mathbb{C} \setminus \sigma_{BF^-}(M, T), \\ \rho_{BF}(M, T) &= \rho_{BF^+}(M, T) \cap \rho_{BF^-}(M, T).\end{aligned}$$

The present paper is a generalization of the results obtained by A. Jeribi et al. in [15] and some of stability results obtained by M. Berkani et al. in [6] for the usual essential B-spectra. It generalizes also the works obtained by A. Jeribi in [13, 14] about the invariance of the S-essential spectra under weakly compact or strictly singular perturbations, which are not applied in the B-Fredholm theory. So that, we can use, by adding some hypothesis, the perturbations of the B-Fredholm spectra under finite-rank and power finite-rank commuting operators. More precisely, let $T_1, T_2 \in \mathcal{C}(X)$ be two commuting closed linear operators such that the bounded linear operator M commutes in the resolvent sense with T_1 and T_2 (see Definition 2.2) and satisfying $(\lambda M - T_1)^{-1} - (\lambda M - T_2)^{-1} \in \mathcal{F}(X)$ (resp. $(\lambda M - T_1)^{-1} - (\lambda M - T_2)^{-1} \in \mathcal{F}_p(X)$ or nilpotent) for some $\lambda \in \rho_M(T_1) \cap \rho_M(T_2)$. Then, we prove that $\sigma_*(M, T_1) = \sigma_*(M, T_2)$, where

$$\begin{aligned}\sigma_*(M, \cdot) \in \{ &\sigma_{BF}(M, \cdot), \sigma_{BF^+}(M, \cdot), \sigma_{BF^-}(M, \cdot), \\ &\sigma_{BW}(M, \cdot), \sigma_{LD}(M, \cdot), \sigma_{RD}(M, \cdot), \sigma_D(M, \cdot)\}.\end{aligned}$$

These perturbation results are needed to extend the results obtained in [15], on the S-essential spectra of closed densely defined linear operators to essential B-spectra of operator pencil $\lambda M - T$, when $T \in \mathcal{C}(X)$, $M \in L(X)$ and $\lambda \in \mathbb{C}$. Moreover, under the additional hypothesis $M(C(T)) = C(T)$ (see Definition 2.1), we give the relationship between the closed-range spectrum and the Kato spectrum of the linear operator pencil $\lambda M - T$, when $T \in \mathcal{C}(X)$, $M \in L(X)$ and $\lambda \in \mathbb{C}$ (see Proposition 2.4), which generalizes a result obtained in [9, Proposition 3.2] in the case of bounded operators and [8, Corollary 3] for closed densely defined linear operators for the usual spectrum. We establish also, the equality between the closed-range spectrum and some essential B-spectra of operator pencil acting on a Banach space (Theorem 2.5). The obtained results, are finally used to describe the essential B-spectra of the operator pencil of the following integro-differential operator with abstract boundary conditions in the Banach space $X_p := L_p([-a, a] \times [-1, 1], dx dy)$, $a > 0$, $1 \leq p < \infty$,

$$A_H = T_H + K,$$

where T_H , K and M are defined by

$$\begin{cases} T_H : D(T_H) \subseteq X_p \longrightarrow X_p \\ \psi \longmapsto -\xi \frac{\partial \psi}{\partial x}(x, \xi) - \sigma(\xi)\psi(x, \xi), \\ D(T_H) = \{\psi \in \mathcal{W}_p : \psi^i = H\psi^o\}, \end{cases}$$

where $\mathcal{W}_p := \{\psi \in X_p : \xi \frac{\partial \psi}{\partial x} \in X_p\}$,

$$\begin{cases} K : X_p \longrightarrow X_p \\ u \longmapsto \int_{-1}^1 k(x, \xi, \nu)u(x, \nu)d\nu, \end{cases}$$

and

$$\begin{cases} M : X_p \longrightarrow X_p \\ \varphi \longmapsto M(\varphi)(x, \xi) = \eta(\xi)\varphi(x, \xi), \end{cases}$$

where $\sigma(\cdot)$ and $\eta(\cdot)$ are in $L^\infty(-1, 1)$, $k(\cdot, \cdot, \cdot)$ is a measurable function, and H is the boundary operator connecting the outgoing and the incoming fluxes.

The outline of this work is organized in the following way: in Section 2, we give some stability results of some essential B-spectra of linear operator pencil. The main results of this section are Theorem 2.3 and Theorem 2.5. In Section 3, we apply the results developed in Section 2 to characterize the B-essential spectra of a transport operator with abstract boundary conditions on L_p -spaces, $1 \leq p < \infty$.

2. STABILITY OF SOME ESSENTIAL B-SPECTRA OF PENCIL OPERATORS

We are interested, in this section, in some of stability results of the essential B-spectra of an operator pencil $\lambda M - T$, where $M \in L(X)$, $T \in \mathcal{C}(X)$ and $\lambda \in \mathbb{C}$.

Since, the Kato decomposition theorem, remains true in the case of Banach spaces as shown in [17, p. 206], we can directly use the following proposition inspired from [10], when necessary, in the case of Banach spaces without proof.

PROPOSITION 2.1. *Let $T \in \mathcal{C}(X)$. If T is a semi B-Fredholm operator, then there exist two closed subspaces X_0 and X_1 of X such that*

- (i) $X = X_0 \oplus X_1$,
- (ii) $T(X_0) \subset X_0$ and $T_0 = T|_{X_0}$ is a semi-Fredholm operator,
- (iii) $T(X_1) \subset X_1$ and $T_1 = T|_{X_1}$ is a nilpotent operator.

First we recall the following subspace, introduced by P. Saphar in [20], and it was defined by P. Aiena in [1] in purely algebraic terms.

DEFINITION 2.1. The algebraic core $C(T)$ of a linear operator T is defined to be the greatest subspace $N \subset D(T)$ for which $T(N) = N$. For more details for the algebraic core $C(T)$, we can refer to [1].

THEOREM 2.1. *Let $T \in \mathcal{C}(X)$ and $M \in L(X)$ such that $M(C(T)) = C(T)$ and $\rho_M(T) \neq \emptyset$. If T is a semi B-Fredholm operator, then there exists $\varepsilon > 0$ such that $T - \mu M$ is a semi-Fredholm operator, for each $\mu \in D(0, \varepsilon) \setminus \{0\}$. Moreover, we have $\alpha(T - \mu M)$ and $\beta(T - \mu M)$ are constants on $D(0, \varepsilon) \setminus \{0\}$.*

Proof. If $M = I$, then we obtain the result established in [7].

If $M \neq I$, then the fact that T is a semi B-Fredholm operator, this implies from Proposition 2.1, the existence of two T -invariant closed subspaces X_0 and X_1 such that

- $X = X_0 \oplus X_1$,
- $T(X_0) \subset X_0$ and $T_0 = T|_{X_0}$ is a semi-Fredholm operator,
- $T(X_1) \subset X_1$ and $T_1 = T|_{X_1}$ is nilpotent.

Since $M(C(T)) = C(T)$ and $T(C(T)) = C(T)$, then we can conclude, by using the definition of $C(T)$, that X_0 and X_1 are invariants under the operator M . So, we can consider $M_0 = M|_{X_0}$ and $M_1 = M|_{X_1}$ such that $M = M_0 \oplus M_1$.

Case 1: If $X_0 = \{0\}$, then we get $M_0 = 0$, $T_0 = 0$, $M = M_1$ and $T = T_1$ is nilpotent. Since, the operator T_1 is nilpotent then we have $T_1 - \mu I_1$ is invertible, for $\mu \neq 0$, where $I_1 = I|_{X_1}$. We have

$$T_1 - \mu M_1 = (T_1 - \mu I_1)[I_1 - \mu(T_1 - \mu I_1)^{-1}(M_1 - I_1)]$$

is invertible for μ such that

$$0 < |\mu| < \frac{1}{\|(T_1 - \mu I_1)^{-1}(M_1 - I_1)\|} = \gamma.$$

Hence, $T - \mu M = T_1 - \mu M_1$ is a semi-Fredholm operator, for μ such that $0 < |\mu| < \gamma$. Moreover, we have $\alpha(T - \mu M) = \alpha(T_1 - \mu M_1) = 0$ and $\beta(T - \mu M) = \beta(T_1 - \mu M_1) = 0$ on $D(0, \gamma) \setminus \{0\}$.

Case 2: Suppose $X_0 \neq \{0\}$.

• If $M_0 = 0$, then by using Case 1, we obtain that $T - \mu M = T_0 \oplus T_1 - \mu M_1$ is a semi-Fredholm operator for μ such that $0 < |\mu| < \gamma$.

• If $M_0 \neq 0$, then $T - \mu M = T_0 - \mu M_0 \oplus T_1 - \mu M_1$. It follows from [19, Theorem 7.9], that $T_0 - \mu M_0$ is a semi-Fredholm for μ such that

$\mu \in D(0, \frac{\varepsilon'}{\|M_0\|}) \setminus \{0\}$, for some $\varepsilon' > 0$. Set $\varepsilon = \min(\frac{\varepsilon'}{\|M_0\|}, \gamma)$. Therefore, the operator $T - \mu M$ is semi-Fredholm for μ such that $\mu \in D(0, \varepsilon) \setminus \{0\}$. Again, from [19, Theorem 7.9], we get $\alpha(T - \mu M) = \alpha(T_0 - \mu M_0)$ and $\beta(T - \mu M) = \beta(T_0 - \mu M_0)$ are constants on $D(0, \varepsilon) \setminus \{0\}$. ■

Using Theorem 2.1, we can deduce the following result which is a generalization of [15, Proposition 2.1]

COROLLARY 2.1. *Let $T \in \mathcal{C}(X)$ and $M \in L(X)$ such that $M(C(T)) = C(T)$ and $\rho_M(T) \neq \emptyset$. Then,*

- (i) $\rho_{BF^+}(M, T)$, $\rho_{BF^-}(M, T)$ and $\rho_{BF}(M, T)$ are open subsets of \mathbb{C} ;
- (ii) $\text{ind}(\lambda M - T)$ is constant on any component of $\rho_{BF^+}(M, T)$, $\rho_{BF^-}(M, T)$ and $\rho_{BF}(M, T)$.

Proof. (i) Let $\lambda_0 \in \rho_{BF^+}(M, T)$, then from Theorem 2.1 there exists an $\varepsilon > 0$ such that $T - \mu M$ is an upper semi-Fredholm operator, for each $\mu \in D(\lambda_0, \varepsilon) \setminus \{0\}$. This implies that $\rho_{BF^+}(M, T)$ is an open subset of \mathbb{C} . The same proof is used to show that, $\rho_{BF^-}(M, T)$ and $\rho_{BF}(M, T)$ are open subsets of \mathbb{C} .

(ii) Let Ω be a component of $\rho_{BF^+}(M, T)$ (resp. $\rho_{BF^-}(M, T)$), $\lambda_0 \in \Omega$ be a fixed point and $\lambda_1 \in \Omega$ be an arbitrary point that are connected by a polygonal line Γ contained in Ω . It follows from the assertion (i) of this corollary that, for each $\mu \in \Gamma$, there exists an open disc $D(\mu, \varepsilon)$, such that $\text{ind}(\mu M - T) = \text{ind}(\lambda M - T)$, for each $\lambda \in D(\mu, \varepsilon)$. By the Heine-Borel theorem, there exist a finite number of open discs that cover Γ . This allows us to deduce that $\text{ind}(\lambda_0 M - T) = \text{ind}(\lambda_1 M - T)$. ■

Now, we recall the following definition considered in [12] for bounded linear operators and it remains also true in the general case of closed linear operators:

DEFINITION 2.2. Let $M \in L(X)$ and $T \in \mathcal{C}(X)$ such that $\rho_M(T) \neq \emptyset$. We say that M and T commute in the sense of resolvent if for all $\lambda \in \rho_M(T)$,

$$M(T - \lambda M)^{-1} = (T - \lambda M)^{-1}M.$$

Remarks 2.1. (a) If M and T commute in the sense of the resolvent, the assumption $M(C(T)) \subset C(T)$ is verified. Indeed, let $x \in C(T)$. Then, from [1, Theorem 1.8], there exists a sequence $(u_n) \subset D(T)$ such that $x = u_0$ and $Tu_{n+1} = u_n$, for every $n \in \mathbb{Z}_+$. Set $y_n = Mu_n$. The commutativity

of the resolvent of the operators M and T , permits us to deduce that $M = (\lambda M - T)^{-1}M(\lambda M - T)$ on $D(T)$ and $TM = MT$ on $D(T)$, which entails that $Mu_n \in D(T)$. Thus, we get $y_0 = Mu_0 = Mx$ and $Ty_{n+1} = TMu_{n+1} = MTu_{n+1} = Mu_n = y_n$, for every $n \in \mathbb{Z}_+$. This, implies that $Mx \in M(C(T))$ and finally we obtain that $M(C(T)) \subset C(T)$.

(b) If M and T commute in the sense of the resolvent and M is invertible, then we get $M^{-1}(C(T)) \subset C(T)$ and finally we can conclude that $C(T) \subset M(C(T))$.

PROPOSITION 2.2. *Let $M \in L(X)$ be an invertible operator and $T \in \mathcal{C}(X)$ such that $0 \in \rho_M(T)$. If $MT^{-1} = T^{-1}M$, then*

- (i) for $\lambda \neq 0$ and $n \geq 1$: $(T - \lambda M)^n (T^{-1})^n = (-\lambda)^n M^n (T^{-1} - \lambda^{-1}M^{-1})^n$;
- (ii) for all $n \geq 1$: $R((T - \lambda M)^n) = R((T^{-1} - \lambda^{-1}M^{-1})^n)$.

Proof. (i) For $n = 1$, the equality is obvious. Let $n \geq 1$ and assume that $(T - \lambda M)^n (T^{-1})^n = (-\lambda)^n M^n (T^{-1} - \lambda^{-1}M^{-1})^n$, then

$$\begin{aligned} (T - \lambda M)^{n+1} (T^{-1})^{n+1} &= (T - \lambda M) [(T - \lambda M)^n (T^{-1})^n] T^{-1} \\ &= (T - \lambda M) [(-\lambda)^n M^n (T^{-1} - \lambda^{-1}M^{-1})^n] T^{-1} \\ &= (T - \lambda M) T^{-1} [(-\lambda)^n M^n (T^{-1} - \lambda^{-1}M^{-1})^n] \\ &= (-\lambda)^{n+1} M^{n+1} (T^{-1} - \lambda^{-1}M^{-1})^{n+1}. \end{aligned}$$

(ii) It follows from (i), that

$$R[M^n (T^{-1} - \lambda^{-1}M^{-1})^n] = R[(T^{-1} - \lambda^{-1}M^{-1})^n] \subseteq R[(T - \lambda M)^n].$$

Conversely, if $y \in R((T - \lambda M)^n)$, then there exists $x \in D(T^n)$ such that $y = (T - \lambda M)^n x$. The fact that $(T^{-1})^n(X) = D(T^n)$, this enable us the existence of $t \in X$ such that $x = (T^{-1})^n(t)$. Hence, $y = (T - \lambda M)^n (T^{-1})^n(t)$ and finally $y = (-\lambda)^n M^n (T^{-1} - \lambda^{-1}M^{-1})^n(t) \in R[(T^{-1} - \lambda^{-1}M^{-1})^n]$. ■

PROPOSITION 2.3. *Let $M \in L(X)$ be an invertible operator and $T \in \mathcal{C}(X)$ such that $0 \in \rho_M(T)$. If $MT^{-1} = T^{-1}M$, then*

- (i) for $\lambda \neq 0$ and $n \geq 1$: $(T^{-1})^n (T - \lambda M)^n = (-\lambda)^n M^n (T^{-1} - \lambda^{-1}M^{-1})^n$ and $T^n (T^{-1} - \lambda^{-1}M^{-1})^n = (-\lambda^{-1})^n (M^{-1})^n (T - \lambda M)^n$ on $D(T^n)$;
- (ii) for all $n \geq 1$: $N((T - \lambda M)^n) = N((T^{-1} - \lambda^{-1}M^{-1})^n)$.

Proof. (i) Since T is invertible and $D(T - \lambda M) = D(T)$, then $R((T^{-1})^n) = D(T^n)$ and $R(T^n) = X$. Therefore, the operators $(T^{-1})^n(T - \lambda M)^n$ and $T^n(T^{-1} - \lambda^{-1}M^{-1})^n$ are well defined. Then, we verify directly that $(T^{-1})^n(T - \lambda M)^n = (-\lambda)^n M^n (T^{-1} - \lambda^{-1}M^{-1})^n$ and $T^n(T^{-1} - \lambda^{-1}M^{-1})^n = (-\lambda^{-1})^n (M^{-1})^n (T - \lambda M)^n$ on $D(T^n)$.

(ii) It is a direct consequence of (i). ■

Remark 2.2. If $M = I$, we recover the results obtained in [10].

Now, we state the main result of this section.

THEOREM 2.2. *Let $M \in L(X)$ be an invertible operator and $T \in \mathcal{C}(X)$ such that $0 \in \rho_M(T)$. If $MT^{-1} = T^{-1}M$, then*

$$\sigma_*(M, T) = \{\lambda^{-1} : \lambda \in \sigma_*(M^{-1}, T^{-1}) \setminus \{0\}\},$$

where

$$\sigma_*(M, T) \in \{\sigma_{BF^+}(M, T), \sigma_{BF^-}(M, T), \sigma_{BF}(M, T), \sigma_D(M, T), \sigma_{BW}(M, T), \sigma_{LD}(M, T), \sigma_{RD}(M, T)\}.$$

Proof. Let $\lambda \neq 0$. By using Proposition 2.2 and Proposition 2.3, we get that $\lambda M^{-1} - T^{-1}$ is a B-Fredholm operator if and only if $\lambda^{-1}M - T$ is a B-Fredholm one. The same arguments are used to prove the other B-spectra. ■

In order to give, the stability result of some essential B-spectra of operator pencil by means of some class of commuting perturbations which generalizes some results established in [6], we shall define the following class of operators: We say that a linear operator is of finite-rank if its range is of finite dimension. If there exists an integer $p \in \mathbb{N}^*$ such that $\dim R(T^p) < \infty$, then it is called a power finite-rank operator. We will denote by $\mathcal{F}(X)$ (resp. $\mathcal{F}_p(X)$) the set of all finite-rank linear bounded (resp. power finite-rank) operators.

In many applications (see Section 3) the perturbed operator is not of finite rank but we have some information about the difference of the resolvent, so the usual result.

THEOREM 2.3. *Let $M \in L(X)$ be an invertible operator and $T_1, T_2 \in \mathcal{C}(X)$ such that M commutes with T_1 and T_2 in the sense of resolvent. If for some $\lambda \in \rho_M(T_1) \cap \rho_M(T_2)$, the operator $(T_1 - \lambda M)^{-1} - (T_2 - \lambda M)^{-1} \in \mathcal{F}(X)$,*

then

$$\sigma_*(M, T_1) = \sigma_*(M, T_2),$$

where, $\sigma_*(M, \cdot) \in \{\sigma_{BF}(M, \cdot), \sigma_{BF^+}(M, \cdot), \sigma_{BF^-}(M, \cdot), \sigma_{BW}(M, \cdot)\}$.

Proof. Without loss of generality, we can assume that $\lambda = 0$, then $T_1^{-1} - T_2^{-1} \in \mathcal{F}(X)$. Let $\mu \neq 0$. The use of Theorem 2.2 shows that, $\mu M - T_1$ is a B-Fredholm operator if and only if $\mu^{-1}M^{-1} - T_1^{-1}$ is a B-Fredholm one. Since $T_1^{-1} - T_2^{-1} \in \mathcal{F}(X)$, then by using [3, Corollary 3.10], we get $\mu^{-1}M^{-1} - T_2^{-1}$ is a B-Fredholm operator. Again by Theorem 2.2, this is equivalent to $\mu M - T_2$ is also a B-Fredholm operator, which finish the proof of the B-Fredholm spectrum equality. For the upper semi B-Fredholm spectrum, the lower semi B-Fredholm spectrum and the B-Weyl spectrum, we use the same technique as above and [11, Proposition 2.7]. ■

DEFINITION 2.3. ([18]) Let X be a Banach space, $A : D(A) \subset X \rightarrow X$ and $T : D(T) \subset X \rightarrow X$ two linear operators. We say that A commutes with T , and we denote $AT = TA$, if

- (i) $D(A) \subset D(T)$;
- (ii) $Tx \in D(A)$ whenever $x \in D(A)$;
- (iii) $AT = TA$ on $\{x \in D(A) : Ax \in D(T)\}$.

Under the additional hypothesis of commutativity of operators, we get a stronger version of Theorem 2.3:

THEOREM 2.4. Let $M \in L(X)$ be an invertible operator and $T_1, T_2 \in \mathcal{C}(X)$ such that M commutes with T_1 and T_2 in the resolvent sense and $T_1T_2 = T_2T_1$. If for some $\lambda \in \rho_M(T_1) \cap \rho_M(T_2)$, the operator $(T_1 - \lambda M)^{-1} - (T_2 - \lambda M)^{-1} \in \mathcal{F}_p(X)$, then

$$\sigma_*(M, T_1) = \sigma_*(M, T_2),$$

where

$$\sigma_*(M, T) \in \{\sigma_{BF}(M, \cdot), \sigma_{BF^+}(M, \cdot), \sigma_{BF^-}(M, \cdot), \sigma_{BW}(M, \cdot), \sigma_{LD}(M, \cdot), \sigma_{RD}(M, \cdot), \sigma_D(M, \cdot)\}.$$

Proof. Without loss of generality, we can assume that $\lambda = 0$, then $T_1^{-1} - T_2^{-1} \in \mathcal{F}_p(X)$. Let $\mu \neq 0$. It follows from Theorem 2.2, that $\mu M - T_1$ is

a B-Fredholm operator if and only if $\mu^{-1}M^{-1} - T_1^{-1}$ is a B-Fredholm one. Since, $T_1^{-1} - T_2^{-1} \in \mathcal{F}_p(X)$, then from [16], we obtain that $\mu^{-1}M^{-1} - T_2^{-1}$ is also a B-Fredholm operator, which is equivalent to $\mu M - T_1$ is a B-Fredholm operator by Theorem 2.2. This, shows that $\sigma_{BF}(M, T_1) = \sigma_{BF}(M, T_2)$. For the other equalities, we use the same technique as above. ■

COROLLARY 2.2. *Let $M \in L(X)$ be an invertible operator and $T_1, T_2 \in \mathcal{C}(X)$ such that M commutes with T_1 and T_2 in the resolvent sense and $T_1T_2 = T_2T_1$. If for some $\lambda \in \rho_M(T_1) \cap \rho_M(T_2)$, the operator $(T_1 - \lambda M)^{-1} - (T_2 - \lambda M)^{-1}$ is nilpotent, then*

$$\sigma_*(M, T_1) = \sigma_*(M, T_2),$$

where

$$\sigma_*(.) \in \{\sigma_{BF}(M, T), \sigma_{BF^+}(M, T), \sigma_{BF^-}(M, T), \\ \sigma_{BW}(M, T), \sigma_{LD}(M, T), \sigma_{RD}(M, T), \sigma_D(M, T)\}.$$

COROLLARY 2.3. *Let $T \in \mathcal{C}(X)$, $M \in L(X)$ be an invertible operator and $Q \in L(X)$ a nilpotent operator such that $\rho_M(T) \neq \emptyset$. If $TQ = QT$ on $D(T)$, $MQ = QM$ and M, T commute in the resolvent sense, then*

$$\sigma_*(M, T) = \sigma_*(M, T + Q),$$

where

$$\sigma_*(.) \in \{\sigma_{BF}(M, T), \sigma_{BF^+}(M, T), \sigma_{BF^-}(M, T), \\ \sigma_{BW}(M, T), \sigma_{LD}(M, T), \sigma_{RD}(M, T), \sigma_D(M, T)\}.$$

Proof. Since, $TQ = QT$ and Q is a nilpotent operator, then $(\lambda M - T)^{-1}Q$ is nilpotent, for all $\lambda \in \rho_M(T)$. Thus, its spectral radius is equal to zero, which implies that $\lambda \in \rho_M(T + Q)$ and that

$$\begin{aligned} (\lambda M - T - Q)^{-1} &= (\lambda M - T)^{-1}(M - (\lambda I - T)^{-1}Q)^{-1} \\ &= (\lambda M - T)^{-1} \sum_{k=0}^n ((\lambda M - T)^{-1}Q)^k \\ &= (\lambda M - T)^{-1} + (\lambda M - T)^{-1}Q \sum_{k=1}^{n-1} ((\lambda M - T)^{-1})^k Q^{k-1} \end{aligned}$$

with n is the nilpotent-index of $(\lambda M - T)^{-1}Q$. Hence, $(\lambda M - T - Q)^{-1} - (\lambda M - T)^{-1}$ is nilpotent. So, we deduce from Corollary 2.2, that $\sigma_*(M, T + Q) = \sigma_*(M, T)$. ■

The following proposition is proved in [12] for bounded linear operators and it holds also true in the general case of closed densely-defined linear operators.

PROPOSITION 2.4. *Let $T \in \mathcal{C}(X)$ be densely-defined linear operator and $M \in L(X)$ such that $M(C(T)) = C(T)$. If $\lambda \in \sigma_{ec}(M, T)$ is non-isolated, then $\lambda \in \sigma_{ek}(M, T)$.*

Remark 2.3. Proposition 2.4 is also true if we replace $\sigma_{ek}(M, T)$ by $\sigma_{qf}(M, T)$, where $\sigma_{qf}(M, T) = \{\lambda \in \mathbb{C} : \lambda M - T \text{ is not a Quazi-Fredholm operator}\}$.

The following theorem, shows the equality between the closed-range spectrum and some essential B-spectra of operator pencil acting on the Banach space.

THEOREM 2.5. *Let $T \in \mathcal{C}(X)$ be densely-defined linear operator and $M \in L(X)$ such that $M(C(T)) = C(T)$. If $\sigma_{ec}(M, T) = \sigma(M, T)$ and every $\lambda \in \sigma_{ec}(M, T)$ is non-isolated. Then*

$$\begin{aligned} \sigma(M, T) &= \sigma_{BF}(M, T) = \sigma_{BW}(M, T) \\ &= \sigma_{BF^+}(M, T) = \sigma_{BF^-}(M, T) = \sigma_D(M, T). \end{aligned}$$

Proof. Since $\sigma_{BF}(M, T) \subset \sigma(M, T)$, it suffices to show that $\sigma(M, T) \subset \sigma_{BF}(M, T)$. Let $\lambda \in \sigma(M, T)$, then from Proposition 2.4, we have $\lambda \in \sigma_{ek}(M, T)$. Since, a B-Fredholm operator is a Quazi-Fredholm one, this shows that $\sigma_{ek}(M, T) \subset \sigma_{BF}(M, T)$. The same arguments are used for the upper semi B-Fredholm, the lower semi B-Fredholm and the B-Weyl spectrum. The fact that, a Drazin invertible operator is a B-Fredholm one, then by using the same arguments as above we can prove that, $\sigma(M, T) = \sigma_D(M, T)$. Finally, we conclude that $\sigma(M, T) = \sigma_{ec}(M, T) = \sigma_{BF}(M, T) = \sigma_{BW}(M, T) = \sigma_{BF^+}(M, T) = \sigma_{BF^-}(M, T) = \sigma_D(M, T)$. ■

3. APPLICATION

In this section, we will use the previous results to treat the essential B-spectra of a transport operator with abstract boundary conditions. Let

$$X_p := L_p((-a, a) \times (-1, 1), dx d\xi), \quad a > 0, \quad 1 \leq p < \infty.$$

We consider the following integro-differential operator with abstract boundary conditions:

$$A_H = T_H + K,$$

where T_H is defined by

$$\left\{ \begin{array}{l} T_H : D(T_H) \subseteq X_p \longrightarrow X_p \\ \psi \longmapsto T_H \psi(x, \xi) = -\xi \frac{\partial \psi}{\partial x}(x, \xi) - \sigma(\xi) \psi(x, \xi), \\ D(T_H) = \{ \psi \in \mathcal{W}_p : \psi^i = H \psi^o \}, \end{array} \right.$$

where $\mathcal{W}_p := \{ \varphi \in X_p : \xi \frac{\partial \varphi}{\partial x} \in X_p \}$ and $\sigma(\cdot) \in L^\infty(-1, 1)$; ψ^o, ψ^i are, respectively, the outgoing and the incoming fluxes related by the boundary operator H (“o” for the outgoing and “i” for the incoming), and given by

$$\left\{ \begin{array}{ll} \psi^i(\xi) = \psi(-a, \xi), & \xi \in (0, 1), \\ \psi^i(\xi) = \psi(a, \xi), & \xi \in (-1, 0), \\ \psi^o(\xi) = \psi(-a, \xi), & \xi \in (-1, 0), \\ \psi^o(\xi) = \psi(a, \xi), & \xi \in (0, 1). \end{array} \right.$$

The bounded collision operator K is defined by

$$\left\{ \begin{array}{l} K : X_p \longrightarrow X_p \\ u \longmapsto K(u)(x, \xi) = \int_{-1}^1 k(x, \xi, \nu) u(x, \nu) d\nu, \end{array} \right.$$

where the kernel $k : (-a, a) \times (-1, 1) \times (-1, 1) \longrightarrow \mathbb{R}$ is assumed to be measurable.

The following boundary spaces denoted by X_p^o and X_p^i are defined as follows:

$$X_p^o := L_p[\{-a\} \times (-1, 0); |\xi| d\xi] \times L_p[\{a\} \times (0, 1); |\xi| d\xi] := X_{1,p}^o \times X_{2,p}^o$$

equipped with the norm

$$\begin{aligned} \|u^o, X_p^o\| &:= \left(\|u_1^o, X_{1,p}^o\|^p + \|u_2^o, X_{2,p}^o\|^p \right)^{\frac{1}{p}} \\ &= \left[\int_{-1}^0 |u(-a, v)|^p |v| dv + \int_0^1 |u(a, v)|^p |v| dv \right]^{\frac{1}{p}}, \end{aligned}$$

and

$$X_p^i := L_p[\{-a\} \times (0, 1); |\xi|d\xi] \times L_p[\{a\} \times (-1, 0); |\xi|d\xi] := X_{1,p}^i \times X_{2,p}^i$$

equipped with the norm

$$\begin{aligned} \|u^i, X_p^i\| &:= \left(\|u_1^i, X_{1,p}^i\|^p + \|u_2^i, X_{2,p}^i\|^p \right)^{\frac{1}{p}} \\ &= \left[\int_0^1 |u(-a, v)|^p |v| dv + \int_{-1}^0 |u(a, v)|^p |v| dv \right]^{\frac{1}{p}}. \end{aligned}$$

In this part, we will determine the essential B-spectra of the pair (M, A_H) , where M is the operator defined by

$$\begin{cases} M : X_p & \longrightarrow X_p \\ \varphi & \longmapsto M(\varphi)(x, \xi) = \eta(\xi)\varphi(x, \xi), \end{cases}$$

where $\eta(\cdot) \in L^\infty(-1, 1)$.

To clarify our subsequent analysis, we define the following bounded operators introduced in [15]:

$$\begin{cases} M_\lambda : X_p^i \longrightarrow X_p^o, & M_\lambda u := (M_\lambda^+ u, M_\lambda^- u), & \text{with} \\ (M_\lambda^+ u)(-a, \xi) := u(-a, \xi) e^{\frac{-2a}{|\xi|}(\lambda\eta(\xi)+\sigma(\xi))}, & 0 < \xi < 1, \\ (M_\lambda^- u)(a, \xi) := u(a, \xi) e^{\frac{-2a}{|\xi|}(\lambda\eta(\xi)+\sigma(\xi))}, & -1 < \xi < 0, \end{cases}$$

$$\begin{cases} B_\lambda : X_p^i \longrightarrow X_p, & B_\lambda u := \chi_{(-1,0)}(\xi)B_\lambda^- u + \chi_{(0,1)}(\xi)B_\lambda^+ u, & \text{with} \\ (B_\lambda^- u)(x, \xi) := u(a, \xi) e^{-\frac{(\lambda\eta(\xi)+\sigma(\xi))}{|\xi|}|a-x|}, & -1 < \xi < 0, \\ (B_\lambda^+ u)(x, \xi) := u(-a, \xi) e^{-\frac{(\lambda\eta(\xi)+\sigma(\xi))}{|\xi|}|a+x|}, & 0 < \xi < 1, \end{cases}$$

$$\begin{cases} G_\lambda : X_p \longrightarrow X_p^o, & G_\lambda \varphi := (G_\lambda^+ \varphi, G_\lambda^- \varphi), & \text{with} \\ G_\lambda^- \varphi(a, \xi) := \frac{1}{|\xi|} \int_{-a}^a e^{-\frac{(\lambda\eta(\xi)+\sigma(\xi))}{|\xi|}|a+x|} \varphi(x, \xi) dx, & -1 < \xi < 0, \\ G_\lambda^+ \varphi(-a, \xi) := \frac{1}{|\xi|} \int_{-a}^a e^{-\frac{(\lambda\eta(\xi)+\sigma(\xi))}{|\xi|}|a-x|} \varphi(x, \xi) dx, & 0 < \xi < 1, \end{cases}$$

and finally

$$\left\{ \begin{array}{l} C_\lambda : X_p \longrightarrow X_p, \quad C_\lambda \varphi = \chi_{(-1,0)} C_\lambda^- \varphi + \chi_{(0,1)} C_\lambda^+ \varphi, \quad \text{with} \\ C_\lambda^- \varphi(x, \xi) := \frac{1}{|\xi|} \int_x^a e^{-\frac{(\lambda\eta(\xi)+\sigma(\xi))}{|\xi|}|x-x'|} \varphi(x', \xi) dx', \quad -1 < \xi < 0, \\ C_\lambda^+ \varphi(x, \xi) := \frac{1}{|\xi|} \int_{-a}^x e^{-\frac{(\lambda\eta(\xi)+\sigma(\xi))}{|\xi|}|x-x'|} \varphi(x', \xi) dx', \quad 0 < \xi < 1, \end{array} \right.$$

where $\chi_{(0,1)}(\cdot)$ and $\chi_{(-1,0)}(\cdot)$ denote, respectively, the characteristic functions of the intervals $(0, 1)$ and $(-1, 0)$.

Note that, from [15, Proposition 3.1], the operators M_λ , B_λ , G_λ and C_λ are bounded respectively by $\exp(-2a\mu^* \operatorname{Re} \lambda)$, $(p\mu^* \operatorname{Re} \lambda)^{-1/p}$, $(\mu^* \operatorname{Re} \lambda)^{-1/q}$ and $(\mu^* \operatorname{Re} \lambda)^{-1}$ where q is the conjugate of p .

In what follows, we will show that the M-spectrum of T_0 (i.e., T_H with $H = 0$) is the continuous spectrum $\sigma_c(M, T_0)$ of the pair (M, T_0) .

- LEMMA 3.1. (i) *The point spectrum of the pair (M, T_0) is empty.*
 (ii) *The residual spectrum of the pair (M, T_0) is empty.*

Proof. (i) We consider for $\lambda \in \mathbb{C}$ such that $\operatorname{Re}(\lambda) \leq 0$, the eigenvalue problem $(\lambda M - T_0)\psi = 0$, where the unknown ψ must be in $D(T_0)$. His solution is formally given by

$$\psi(x, \xi) = K(\xi) e^{-\frac{1}{|\xi|}[\lambda\eta(\xi)+\sigma(\xi)]x}.$$

Moreover, since $\psi \in D(T_0)$, then we get $\psi^i = 0$. So we obtain $K(\xi) = 0$ on $(-1, 1)$. Consequently, $\psi = 0$.

(ii) To prove that the residual spectrum $\sigma_r(M, T_0)$ is also empty, we shall determine the point spectrum of the adjoint operator pencil densely defined $\lambda M - T_0$, where $\lambda \in \mathbb{C}$.

The adjoint operators T_0^* and M^* are, respectively, given by:

$$\left\{ \begin{array}{l} T_0^* : D(T_0^*) \subseteq X_q \longrightarrow X_q \\ \psi \longmapsto T_0^* \psi(x, \xi) = \xi \frac{\partial \psi}{\partial x}(x, \xi) - \sigma(\xi) \psi(x, \xi), \\ D(T_0^*) = \{\psi \in \mathcal{W}_q : \psi^o = 0\}, \end{array} \right.$$

where q is the conjugate of p ($\frac{1}{p} + \frac{1}{q} = 1$),

$$\left\{ \begin{array}{l} M^* : X_q \longrightarrow X_q \\ \varphi \longrightarrow M^*(\varphi)(x, v) = \eta(v) \varphi(x, v). \end{array} \right.$$

Let us consider the eigenvalue problem

$$(\lambda M^* - T_0^*)\psi = 0. \quad (3.1)$$

In view of the boundary conditions, a straightforward estimation shows that, the problem (3.1) admits only the trivial solution. Then, we obtain $\sigma_p(M^*, T_0^*) = \emptyset$. Since, $\sigma_r(M, T_0) \subseteq \sigma_p(M^*, T_0^*)$, then we can easily obtain the desired result. ■

Now, by using the previous results, we can deduce the following theorem:

THEOREM 3.1. *Let $M \in L(X)$. Then,*

$$\sigma(M, T_0) = \sigma_c(M, T_0) = \sigma_{ec}(M, T_0) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\}.$$

Proof. Since, $\sigma(M, T_0) = \sigma_p(M, T_0) \cup \sigma_r(M, T_0) \cup \sigma_c(M, T_0)$, then by using Lemma 3.1, we deduce $\sigma(M, T_0) = \sigma_c(M, T_0)$. On the other hand, it follows from [15, Theorem 3.1], that $\sigma(M, T_0) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\}$. Combining these two results, we can obtain the assertion of the present theorem. ■

Now, we are able to express the B-spectra of the pair (M, T_H) :

THEOREM 3.2. *If the boundary operator H is of finite rank and commutes with M , and every point λ in $\sigma_{ec}(M, T_0)$ is non-isolated, then*

$$\sigma_*(M, T_H) = \sigma_*(M, T_0) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\},$$

where $\sigma_*(M, \cdot) \in \{\sigma_{BF}(M, \cdot), \sigma_{BF^+}(M, \cdot), \sigma_{BF^-}(M, \cdot), \sigma_{BW}(M, \cdot)\}$.

Proof. According to [15], we have

$$(\lambda M - T_H)^{-1} = H \sum_{n \geq 0} B_\lambda (M_\lambda H)^n G_\lambda + C_\lambda, \quad (3.2)$$

where $C_\lambda = (\lambda M - T_0)^{-1}$. Since $(\lambda M - T_H)^{-1} - (\lambda M - T_0)^{-1} \in \mathcal{F}(X)$, this implies by Theorem 2.3 that $\sigma_*(M, T_H) = \sigma_*(M, T_0)$. The fact that M and T_H commute in the sense of the resolvent and M is invertible this allows us, by the use of Remarks 2.1, Theorem 2.5 and Theorem 3.1 to conclude that

$$\sigma_*(M, T_0) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\},$$

where $\sigma_*(M, \cdot) \in \{\sigma_{BF}(M, \cdot), \sigma_{BF^+}(M, \cdot), \sigma_{BF^-}(M, \cdot), \sigma_{BW}(M, \cdot)\}$ ■

THEOREM 3.3. *Suppose that the boundary operator H and the collision operator K are of finite rank. If $MH = HM$ and every point λ in $\sigma_{ec}(M, T_0)$ is non-isolated, then we get*

$$\sigma_*(M, A_H) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\},$$

where $\sigma_*(M, \cdot) \in \{\sigma_{BF}(M, \cdot), \sigma_{BF^+}(M, \cdot), \sigma_{BF^-}(M, \cdot), \sigma_{BW}(M, \cdot)\}$.

Proof. Since, the collision operator K is finite rank, then it follows from Theorem 2.3 and Theorem 3.2, that $\sigma_*(M, A_H) = \sigma_*(M, T_H + K) = \sigma_*(M, T_H) = \sigma_*(M, T_0)$, and finally we obtain that

$$\sigma_*(M, A_H) = \{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) \leq 0\},$$

where $\sigma_*(M, \cdot) \in \{\sigma_{BF}(M, \cdot), \sigma_{BF^+}(M, \cdot), \sigma_{BF^-}(M, \cdot), \sigma_{BW}(M, \cdot)\}$. ■

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