EXTRACTA MATHEMATICAE Vol. **33**, Núm. 1, 67 – 108 (2018)

# Partial Differential Equations and Strictly Plurisubharmonic Functions in Several Variables

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Presented by Manuel Maestre  Received September 22, 2014

*Abstract*: Using algebraic methods, we prove that there exists a fundamental relation between partial differential equations and strictly plurisubharmonic functions over domains of $\mathbb{C}^n$ $(n \geq 1)$.

*Key words*: Analytic convex and plurisubharmonic functions, harmonic function, maximal plurisubharmonic, differential equation, analysis, inequalities.

AMS *Subject Class.* (2010): Primary 32A10, 32A60, 32A70, 32U05; Secondary 32W50.

## 1. Introduction

We investigate in this paper the relation between strictly plurisubharmonic functions and partial differential equations in domains of $\mathbb{C}^n$, $(n \geq 1)$. Various related results are obtained in this context. Several papers developed by Lelong [14, 15, 16, 17, 18], Sadullaev [20], Oka [19], Bremermann [4, 5],Siciak [21], Abidi [1], Cegrell [6] and others studied plurisubharmonic functions and related topics are of particular importance in this context. For example we can state that strictly plurisubharmonic functions and analytic subsets are related in domains of $\mathbb{C}^n$ as follows. Let $A = \{z \in \mathbb{C} : f(z) = 0\}$ and $B = \{z \in \mathbb{C} : g(z) = 0\}$ two analytic subsets of $\mathbb{C}$, where $f, g : \mathbb{C} \to \mathbb{C}$ be 2 analytic functions, $fg \neq 0$. Put $f_1$ and $g_1$ some analytic primitives of $f$ and $g$ respectively over $\mathbb{C}$. Then $A \cap B = \emptyset$ if and only if the function $u$,

$$u(z,w) = |w - f_1(z)|^2 + |w - g_1(z)|^2 \ , \qquad (z,w) \in \mathbb{C}^2 \, ,$$

is strictly psh in $\mathbb{C}^2$. Some good references for the study of convex functions are [11, 13, 3]. For the study of analytic functions we cite the references [12, 10, 13]. For the study of the extension problem of analytic and plurisubharmonic functions we cite the references [7, 9, 6, 18, 8, 22, 23].

As usual, $\mathbb{N} := \{1, 2, \dots\}$, $\mathbb{R}$ and $\mathbb{C}$ are the sets of all natural, real and complex numbers, respectively. Let $U$ be a domain of $\mathbb{R}^d$, $(d \geq 2)$; $m_d$ is the Lebesgue measure on $\mathbb{R}^d$. Let $f : U \to \mathbb{C}$ be a function; $|f|$ is the modulus of $f$, $\operatorname{Re}(f)$ and $\operatorname{Im}(f)$ are respectively the real and imaginary parts of $f$. Let $g : D \to \mathbb{C}$ be an analytic function, $D$ is a domain of $\mathbb{C}$. We denote by $g^{(0)} = g$, $g^{(1)} = g'$ is the holomorphic derivative of $g$ over $D$. $g^{(2)} = g''$, $g^{(3)} = g'''$. In general $g^{(m)} = \frac{\partial^m g}{\partial z^m}$ is the derivative of $g$ of order $m$ for all $m \in \mathbb{N}$. Let $z \in \mathbb{C}^n$, $z = (z_1, \dots, z_n)$, $n \geq 2$. For $j \in \{1, \dots, n\}$, we write $z = (z_j, Z_j) = (z_1, \dots, z_{j-1}, z_j, z_{j+1}, \dots, z_n)$ where $Z_j = (z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_n) \in \mathbb{C}^{n-1}$. $C^k(U) = \{\varphi : U \to \mathbb{C} : \varphi \text{ is of class } C^k \text{ in } U\}$, $k \in \mathbb{N} \cup \{\infty\}$. Let $\varphi : U \to \mathbb{C}$ be a function of class $C^2$. $\Delta(\varphi)$ is the Laplacian of $\varphi$. Let $D$ be a domain of $\mathbb{C}^n$, $(n \geq 1)$; $\operatorname{psh}(D)$ and $\operatorname{prh}(D)$ are respectively the class of plurisubharmonic and pluriharmonic functions on $D$. For all $a \in \mathbb{C}$, $|a|$ is the modulus of $a$; $\operatorname{Re}(a)$ and $\operatorname{Im}(a)$ are the real and the imaginary parts of $a$ respectively.

## 2. Main results

We begin this study by the next result.

THEOREM 2.1. *Let $h_1, \dots, h_N : \mathbb{C} \to \mathbb{R}$ be $N$ harmonic functions, $N \geq 1$. For the function $u(z, w) = |w - h_1(z)|^2 + \dots + |w - h_N(z)|^2$, $(z, w) \in \mathbb{C}^2$, the following conditions are equivalent:*

(a) *$u$ is strictly psh in $\mathbb{C}^2$;*

(b) $\left\{z \in \mathbb{C} : \frac{\partial h_1}{\partial z}(z) = 0\right\} \cap \dots \cap \left\{z \in \mathbb{C} : \frac{\partial h_N}{\partial z}(z) = 0\right\} = \emptyset.$

*Proof.* (a) $\Rightarrow$ (b) Because $|w - h_1|^2 + \dots + |w - h_N|^2 = N|w|^2 + (h_1^2 + \dots + h_N^2) - w(h_1 + \dots + h_N) - \overline{w}(h_1 + \dots + h_N)$, $u$ is a function of class $C^\infty$ in $\mathbb{C}^2$. Now let $(z, w) \in \mathbb{C}^2$,

$$\frac{\partial^2 u}{\partial z \partial \overline{z}}(z, w) = 2\left[\left|\frac{\partial h_1}{\partial z}(z)\right|^2 + \dots + \left|\frac{\partial h_N}{\partial z}(z)\right|^2\right],$$

$$\frac{\partial^2 u}{\partial w \partial \overline{w}}(z, w) = N,$$

$$\frac{\partial^2 u}{\partial z \partial \overline{w}}(z, w) = -\left(\frac{\partial h_1}{\partial z}(z) + \dots + \frac{\partial h_N}{\partial z}(z)\right).$$

The Levi hermitian form associated to $u$ is now

$$\begin{aligned}L(u)(z,w)(\alpha,\beta) &= \frac{\partial^2 u}{\partial z\partial\overline{z}}(z,w)\alpha\overline{\alpha} + \frac{\partial^2 u}{\partial w\partial\overline{w}}(z,w)\beta\overline{\beta} \\ &\qquad + 2\,\mathrm{Re}[\frac{\partial^2 u}{\partial z\partial\overline{w}}(z,w)\alpha\overline{\beta}] \\ &= 2\left[\left|\frac{\partial h_1}{\partial z}(z)\right|^2 + \cdots + \left|\frac{\partial h_N}{\partial z}(z)\right|^2\right]\alpha\overline{\alpha} + N\beta\overline{\beta} \\ &\qquad + 2\,\mathrm{Re}\left[-\left(\frac{\partial h_1}{\partial z}(z) + \cdots + \frac{\partial h_N}{\partial z}(z)\right)\alpha\overline{\beta}\right] > 0\,,\end{aligned}$$

for all $(z,w)\in\mathbb{C}^2$ and $(\alpha,\beta)\in\mathbb{C}^2\backslash\{(0,0)\}$. Thus

$$\left|\frac{\partial h_1}{\partial z}(z) + \cdots + \frac{\partial h_N}{\partial z}(z)\right|^2 < 2N\left[\left|\frac{\partial h_1}{\partial z}(z)\right|^2 + \cdots + \left|\frac{\partial h_N}{\partial z}(z)\right|^2\right]$$

for each $z\in\mathbb{C}$.

Now we use the following lemma.

LEMMA 2.2. *Let $a_1,\ldots,a_N\in\mathbb{C}$ and $N\geq 1$. We have*

(i) $N(|a_1|^2+\cdots+|a_N|^2)\geq|a_1+\cdots+a_N|^2$ *;*

(ii) $M(|a_1|^2+\cdots+|a_N|^2)>|a_1+\cdots+a_N|^2$ *if $M>N$ and there exists $j_0$ such that $a_{j_0}\neq 0$.*

*Proof.*

$$\begin{aligned}\left(\sum_{j=1}^N a_j\right)\left(\sum_{k=1}^N \overline{a_k}\right) &= \sum_{j,k=1}^N a_j\overline{a_k} \leq \sum_{j,k=1}^N |a_j||a_k| \leq \sum_{j,k=1}^N\left(\frac{|a_j|^2}{2}+\frac{|a_j|^2}{2}\right) \\ &= 2\sum_{j,k=1}^N \frac{|a_j|^2}{2} = \sum_{k=1}^N\sum_{j=1}^N |a_j|^2 = N\sum_{j=1}^N |a_j|^2.\end{aligned}$$

∎

Now we complete the proof of the theorem. Put

$$\begin{aligned}A &= \left[\left|\frac{\partial h_1}{\partial z}\right|^2 + \cdots + \left|\frac{\partial h_N}{\partial z}\right|^2\right] \\ &\quad + \left[(2N-1)\left(\left|\frac{\partial h_1}{\partial z}\right|^2 + \cdots + \left|\frac{\partial h_N}{\partial z}\right|^2\right) - \left|\frac{\partial h_1}{\partial z} + \cdots + \frac{\partial h_N}{\partial z}\right|^2\right];\end{aligned}$$

$A > 0$ over $\mathbb{C}$. $A = B + C$, where $B \geq 0$, $C \geq 0$. Then $A = 0$ if and only if $B = C = 0$. Thus if $z \in \mathbb{C}$ such that $B(z) = \left|\frac{\partial h_1}{\partial z}(z)\right|^2 + \cdots + \left|\frac{\partial h_N}{\partial z}(z)\right|^2 = 0$, then $\frac{\partial h_1}{\partial z}(z) = \cdots = \frac{\partial h_N}{\partial z}(z) = 0$. Therefore $C(z) = (2N-1)\Big(\left|\frac{\partial h_1}{\partial z}(z)\right|^2 + \cdots + \left|\frac{\partial h_N}{\partial z}(z)\right|^2\Big) - \left|\frac{\partial h_1}{\partial z}(z) + \cdots + \frac{\partial h_N}{\partial z}(z)\right|^2 = 0$.

The converse is also true. We conclude that $A(z) > 0$ if and only if $B(z) > 0$, for all $z \in \mathbb{C}$. Then $\left|\frac{\partial h_1}{\partial z}\right|^2 + \cdots + \left|\frac{\partial h_N}{\partial z}\right|^2 > 0$ over $\mathbb{C}$ if and only if $u$ is strictly psh in $\mathbb{C}^2$. But $\left|\frac{\partial h_1}{\partial z}\right|^2 + \cdots + \left|\frac{\partial h_N}{\partial z}\right|^2 > 0$ in $\mathbb{C}$ if and only if

$$\left\{z \in \mathbb{C} \,:\, \frac{\partial h_1}{\partial z}(z) = 0\right\} \cap \cdots \cap \left\{z \in \mathbb{C} \,:\, \frac{\partial h_N}{\partial z}(z) = 0\right\} = \emptyset\,.$$

By this proof we deduce also (b) $\Rightarrow$ (a). ∎

For analytic functions, we have now.

Theorem 2.3. *Let $g_1, \ldots, g_N : D \to \mathbb{C}$ be $N$ analytic functions, $N \in \mathbb{N}\backslash\{1\}$, $D$ is a domain of $\mathbb{C}$. Put $u(z,w) = |w - g_1(z)|^2 + \cdots + |w - g_N(z)|^2$, $(z,w) \in D \times \mathbb{C}$. Then $u$ is strictly psh in $D \times \mathbb{C}$ if and only if*

$$\sum_{j,k=1}^{N} g_j' \overline{g_k'} \delta_{jk} > 0 \qquad \text{over } D\,,$$

*where $\delta_{jk} = (N-1$ if $j = k$ and $-1$ if $j \neq k)$, $j,k \in \{1,\ldots,N\}$.*

*Proof.* The function $u = N|w|^2 + \big(|g_1|^2 + \cdots + |g_N|^2\big) - w(\overline{g_1} + \cdots + \overline{g_N}) - \overline{w}(g_1 + \cdots + g_N)$ is of class $C^\infty$ over $D \times \mathbb{C}$. Let $(z,w) \in \mathbb{C}^2$,

$$\begin{aligned}
\frac{\partial^2 u}{\partial z \partial \overline{z}}(z,w) &= |g_1'(z)|^2 + \cdots + |g_N'(z)|^2\,,\\
\frac{\partial^2 u}{\partial w \partial \overline{w}}(z,w) &= N\,,\\
\frac{\partial^2 u}{\partial z \partial \overline{w}}(z,w) &= -\big(g_1'(z) + \cdots + g_N'(z)\big)\,.
\end{aligned}$$

Assume that $u$ is strictly psh in $D \times \mathbb{C}$. The Levi hermitian form of $u$ is now $L(u)(z,w)(\alpha,\beta) = \big[|g_1'(z)|^2 + \cdots + |g_N'(z)|^2\big]\alpha\overline{\alpha} + N\beta\overline{\beta} + 2\,\mathrm{Re}\big[-(g_1'(z) +$

$\cdots + g_N'(z))\alpha\overline{\beta}\big] > 0$ for all $(\alpha,\beta) \in \mathbb{C}^2\backslash\{(0,0)\}$. Thus $|g_1' + \cdots + g_N'|^2 < N[|g_1'|^2 + \cdots + |g_N'|^2]$ over $D$. Then

$$\sum_{j,k=1 (j\neq k)}^{N} g_j'\overline{g_k'}\delta_{jk} + (N-1)\big[|g_1'|^2 + \cdots + |g_N'|^2\big] = \sum_{j,k=1}^{N} g_j'\overline{g_k'}\delta_{jk} > 0 \quad \text{on } D\,.$$

Now assume that $\sum_{j,k=1}^{N} g_j'\overline{g_k'}\delta_{jk} > 0$ on $D$. By the above proof, $|g_1' + \cdots + g_N'|^2 < N\big[|g_1'|^2 + \cdots + |g_N'|^2\big]$. It follows that $L(u)(z,w)(\alpha,\beta) > 0$ for each $(\alpha,\beta) \in \mathbb{C}^2\backslash\{0\}$. ∎

The theorem below gives a fundamental part of this paper and the study of the relation between partial differential equations and strictly plurisubharmonic functions over domains of $\mathbb{C}^n$, $(n \geq 1)$.

THEOREM 2.4. *Let $g : D \to \mathbb{C}$ be a function, $D$ is a domain of $\mathbb{C}$. Put $v(z,w) = |w - g(z)|^2$, for $(z,w) \in D \times \mathbb{C}$. The following assertions are equivalent:*

(a) *$v$ is strictly psh in $D \times \mathbb{C}$;*

(b) *$g$ is harmonic in $D$ and $\big\{z \in D : \frac{\partial g}{\partial \overline{z}}(z) = 0\big\} = \emptyset$.*

*Proof.* (a) $\Rightarrow$ (b) $v$ is strictly psh in $D \times \mathbb{C}$, then $v$ is psh in $D \times \mathbb{C}$. Therefore $g$ is harmonic in $D$ by Abidi [1]. It follows that $v$ is a function of class $C^\infty$ in $D\times\mathbb{C}$. Let $(z,w) \in D\times\mathbb{C}$. Write $v(z,w) = |w|^2 + |g(z)|^2 - \overline{w}g(z) - w\overline{g}(z)$. We have

$$\frac{\partial^2 v}{\partial z \partial \overline{z}}(z,w)\alpha\overline{\alpha} + \frac{\partial^2 v}{\partial w \partial \overline{w}}(z,w)\beta\overline{\beta} + 2\,\mathrm{Re}\left[\frac{\partial^2 v}{\partial z \partial \overline{w}}(z,w)\alpha\overline{\beta}\right] > 0$$

for all $(\alpha,\beta) \in \mathbb{C}^2\backslash\{0\}$, and

$$\frac{\partial^2 v}{\partial z \partial \overline{z}} = \left|\frac{\partial g}{\partial z}\right|^2 + \left|\frac{\partial g}{\partial \overline{z}}\right|^2, \qquad \frac{\partial^2 v}{\partial w \partial \overline{w}} = 1\,, \qquad \frac{\partial^2 v}{\partial z \partial \overline{w}} = -\frac{\partial g}{\partial z}\,.$$

Therefore $\big| - \frac{\partial g}{\partial z}\big|^2 < \big|\frac{\partial g}{\partial z}\big|^2 + \big|\frac{\partial g}{\partial \overline{z}}\big|^2$, and consequently $\big|\frac{\partial g}{\partial \overline{z}}\big|^2 > 0$ over $D$.

(b) $\Rightarrow$ (a) Since $g$ is harmonic in $D$, then $v$ is a function of class $C^\infty$ in $D \times \mathbb{C}$. We have

$$\frac{\partial^2 v}{\partial z \partial \overline{z}} = \left|\frac{\partial g}{\partial z}\right|^2 + \left|\frac{\partial g}{\partial \overline{z}}\right|^2, \qquad \frac{\partial^2 v}{\partial w \partial \overline{w}} = 1\,, \qquad \frac{\partial^2 v}{\partial z \partial \overline{w}} = -\frac{\partial g}{\partial z}\,.$$

The Levi hermitian form of $v$ is

$$L(v)(z,w)(\alpha,\beta) = \frac{\partial^2 v}{\partial z\partial\overline{z}}(z,w)\alpha\overline{\alpha} + \frac{\partial^2 v}{\partial w\partial\overline{w}}(z,w)\beta\overline{\beta} + 2\,\mathrm{Re}\left[\frac{\partial^2 v}{\partial z\partial\overline{w}}(z,w)\alpha\overline{\beta}\right]$$
$$= \left[\left|\frac{\partial g}{\partial z}(z)\right|^2 + \left|\frac{\partial g}{\partial\overline{z}}(z)\right|^2\right]\alpha\overline{\alpha} + \beta\overline{\beta} + 2\,\mathrm{Re}\left[-\frac{\partial g}{\partial z}(z)\alpha\overline{\beta}\right],$$

for $(z,w) \in D \times \mathbb{C}$, $(\alpha,\beta) \in \mathbb{C}^2$. Since $\left|\frac{\partial g}{\partial\overline{z}}(z)\right|^2 > 0$, for every $z \in D$, then

$$\left|-\frac{\partial g}{\partial z}(z)\right|^2 < \left|\frac{\partial g}{\partial z}(z)\right|^2 + \left|\frac{\partial g}{\partial\overline{z}}(z)\right|^2$$

for each $z \in D$. Therefore,

$$L(v)(z,w)(\alpha,\beta) > 0\,, \qquad \forall\ (z,w) \in D \times \mathbb{C}\,,\ \forall\ (\alpha,\beta) \in \mathbb{C}^2\backslash\{0\}.$$

Consequently, $v$ is strictly psh in $D \times \mathbb{C}$. ∎

Observe that if $k = k_1 + \overline{k_2}$, where $k_1, k_2 : D \to \mathbb{C}$ be 2 analytic functions in the domain $D \subset \mathbb{C}$, and $u(z,w) = |w - k(z)|^2$, $(z,w) \in D \times \mathbb{C}$, then the strict plurisubharmonicity of $u$ is independent of the function $k_1$. On the other hand if we replace the strict inequality $<$ by the large inequality $\leq$, then the above theorem is false.

*Remark* 2.5. Let $k : D \to \mathbb{C}$ be an analytic function, $D$ is a domain of $\mathbb{C}$. Put $u(z,w) = |w - k(z)|^2$, $v(z,w) = |w - \overline{k}(z)|^2$, where $(z,w) \in D \times \mathbb{C}$. Then $u$, $\log(u)$ and $\log(v)$ are not strictly psh functions on any not empty domain of $D \times \mathbb{C}$; $v$ is strictly psh in $D \times \mathbb{C}$ if and only if $\left|\frac{\partial\overline{k}}{\partial\overline{z}}\right| = \left|\frac{\partial k}{\partial z}\right| > 0$ in $D$.

Example. Let $k(z) = \exp(z)$, $z \in \mathbb{C}$, and $v_1(z,w) = |w - \exp(z)|^2$, $v_2(z,w) = |w - \exp(\overline{z})|^2$, for $(z,w) \in \mathbb{C}^2$; $v_1$ is not strictly psh on any open of $\mathbb{C}^2$, but $v_2$ is strictly psh in all $\mathbb{C}^2$. Note that $\log(v_2)$ is not strictly psh on any domain of $\big\{(z,w) : |w - \exp(\overline{z})|^2 > 0\big\}$.

On the other hand, $g_1(z) = z$ and $g_2(z) = 1 - z$ $(z \in \mathbb{C})$ are analytic functions over $\mathbb{C}$. Set $v(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2$, $(z,w) \in \mathbb{C}^2$. Let $(\alpha,\beta) \in \mathbb{C}^2$. The Levi hermitian form of $v$ is

$$L(v)(z,w)(\alpha,\beta) = \alpha\overline{\alpha} + \beta\overline{\beta} + 2\,\mathrm{Re}\big[-\alpha\overline{\beta}\big] + \alpha\overline{\alpha} + \beta\overline{\beta} + 2\,\mathrm{Re}\big[\alpha\overline{\beta}\big]$$
$$= 2(\alpha\overline{\alpha} + \beta\overline{\beta}) > 0\,, \quad \forall\ (z,w) \in \mathbb{C} \times \mathbb{C}\,,\ \forall\ (\alpha,\beta) \in \mathbb{C}^2\backslash\{0\}.$$

Then $v$ is strictly psh in $\mathbb{C}^2$. Observe that in this case if we put $u_1(z,w) = |w - g_1(z)|^2$, $u_2(z,w) = |w - g_2(z)|^2$, then $u_1$ and $u_2$ are plurisubharmonic over $\mathbb{C}^2$ but not strictly psh functions on any domain of $\mathbb{C}^2$. But $v = (u_1 + u_2)$ is strictly psh in $\mathbb{C}^2$.

In fact we have the following result.

Claim 2.6. Let $g_1, g_2 : D \to \mathbb{C}$ be 2 analytic functions, $D$ is a domain of $\mathbb{C}$ and $v(z,w) = |w - g_1(z)|^2 + |w - g_2(z)|^2$, where $(z,w) \in D \times \mathbb{C}$. Then $v$ is strictly psh in $D \times \mathbb{C}$ if the function $\operatorname{Re}\left[g_1' \overline{g_2'}\right] < 0$ over $D$.

If $D = \mathbb{C}$, then $v$ is strictly psh in $\mathbb{C}^2$ if for example $(g_1' \overline{g_2'})$ is equal a constant $c$ over $\mathbb{C}$ and $\operatorname{Re}(c) < 0$.

According to the paper Abidi [1], we can prove the following extension.

Claim 2.7. Let $a, b \in \mathbb{C}$. Put $v(z,w) = |(w - \overline{z})^2 - (a+b)(w - \overline{z}) + ab|$, where $(z,w) \in \mathbb{C}^2$. Then $v$ is strictly psh on $\mathbb{C}^2$ if and only if $a = b$.

In general we can state the following result: For all $g : \mathbb{C} \to \mathbb{C}$ be analytic, if we put

$$u(z,w) = \left|(w - \overline{g}(z))^2 - (a+b)(w - \overline{g}(z)) + ab\right|,$$

where $(z,w) \in \mathbb{C}^2$, then $u$ is strictly psh on $\mathbb{C}^2$ if and only if ($a = b$ and $\left|\frac{\partial g}{\partial z}(z)\right| > 0$ for all $z \in \mathbb{C}$).

Theorem 2.8. *Let $D$ be a domain of $\mathbb{C}$ and $g : D \to \mathbb{C}$ be an analytic function. The following statements are equivalent:*

($a_1$) $|w - \overline{g}|^2$ *is strictly psh in $D \times \mathbb{C}$;*

($a_2$) $|w - g|^2 + |w - \overline{g}|^2$ *is strictly psh in $D \times \mathbb{C}$;*

($a_3$) $|\frac{\partial g}{\partial z}| > 0$ *in $D$;*

($a_4$) $|w - cg - \overline{g}|^2$ *is strictly psh in $D \times \mathbb{C}$, where $c \in \mathbb{C} \backslash \{0\}$;*

($a_5$) $|w_1 - g|^2 + |w_2 - \overline{g}|^2$ *is strictly psh in $D \times \mathbb{C} \times \mathbb{C}$;*

($a_6$) *for all $n \in \mathbb{N}$, $\left(|w_1 - g|^2 + \cdots + |w_n - g|^2 + |w_{n+1} - \overline{g}|^2\right)$ is strictly psh in $D \times \mathbb{C}^{n+1}$.*

Proposition 2.9. *Let $g : D \to \mathbb{C}$ be analytic, $D$ is a domain of $\mathbb{C}$. $g = h + ik$, $h = \operatorname{Re}(g)$, $k = \operatorname{Im}(g)$. Let $a, b \in \mathbb{C}$, ($a \neq 0$ or $b \neq 0$). Put $u(z,w) = |w - \overline{g}(z)|^2$, $v(z,w) = |w - ah(z)|^2 + |w - bk(z)|^2$, $u_1(z,w) = |w - h(z)|^2$, $u_2(z,w) = |w - k(z)|^2$, where $(z,w) \in D \times \mathbb{C}$. We have the equivalents:*

(a) $u$ *is strictly psh in* $D \times \mathbb{C}$*;*

(b) $u_1$ *is strictly psh in* $D \times \mathbb{C}$*;*

(c) $u_2$ *is strictly psh in* $D \times \mathbb{C}$*;*

(d) $v$ *is strictly psh in* $D \times \mathbb{C}$*.*

Observe that in general we can not compare the structure strictly psh of the functions $v_1$ and $v_2$ where $v_1(z,w) = |w - g(z)|^2$, $v_2(z,w) = |w - \overline{g}(z)|^2$, $g : \mathbb{C} \to \mathbb{C}$ be analytic and $(z,w) \in \mathbb{C}^2$. But if we add another function constructed according to the expression of $g$ we have the following extension.

Claim 2.10. Let $g : \mathbb{C}^n \to \mathbb{C}$ be analytic $g = h + ik$, $h = \mathrm{Re}(g)$, $n \in \mathbb{N}$. Denote by $\varphi(z,w) = |w-g(z)|^2$, $\varphi_1(z,w) = |w-h(z)|^2+|w-g(z)|^2$, $\varphi_2(z,w) = |w - h(z)|^2 + |w - \overline{g}(z)|^2$, $\varphi_3(z,w) = |w - \overline{g}(z)|^2$, where $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. We have the equivalents:

(a) $\varphi_1$ is strictly psh in $\mathbb{C}^n \times \mathbb{C}$;

(b) $\varphi_2$ is strictly psh in $\mathbb{C}^n \times \mathbb{C}$;

(c) $n = 1$ and $\varphi_3$ is strictly psh in $\mathbb{C}^2$.

Note that $\varphi$ is not strictly psh on all not empty domain of $\mathbb{C}^2$.

At this stage of the development, observe that if $f : \mathbb{C}^n \to \mathbb{R}$ is pluriharmonic ($n \geq 1$), and $F(z,w) = |w - f(z)|^2$, where $(z,w) \in \mathbb{C}^n \times \mathbb{C}$, then $F$ is not strictly psh on any not empty domain of $\mathbb{C}^n \times \mathbb{C}$ if and only if
($a_1$) $n = 1$ and $f$ is constant in $\mathbb{C}$, or
($a_2$) $n \geq 2$ and $f$ is an arbitrary prh function over $\mathbb{C}^n$.
The function $f$ have real valued is of great importance in this subject.

Some fundamental remarks concerning strictly psh functions. At the beginning of this statements we observe the following assertions: Let $h : D \to \mathbb{C}$ be a function, $D$ is a convex domain of $\mathbb{C}$. If $|w - h|^2$ is psh (resp. convex) in $D \times \mathbb{C}$, then $\left|w - \overline{h}\right|^2$ is psh (resp. convex) in $D \times \mathbb{C}$ and conversely.

But we can obtain $|w - h|^2$ is strictly psh (resp. strictly psh and convex) in $D \times \mathbb{C}$ and $\left|w - \overline{h}\right|^2$ is not strictly psh (resp. not strictly psh and convex) on any domain subset of $D \times \mathbb{C}$. This is one of the great differences between the classes of functions psh, convex, of the first part and the classes of strictly psh, (strictly psh and convex) functions for the second part. Consequently, if we replace the large inequality $\leq$ by the strict inequality $<$ the above result is not true.

Now let $g_1, \ldots, g_N : \mathbb{C}^n \to \mathbb{C}$ be $N$ analytic functions, where $n, N \geq 1$. Put $u_1(z,w) = |w - g_1(z)|^2 + \cdots + |w - g_N(z)|^2$, $v_1(z,w) = \left|w - \overline{g_1}(z)\right|^2 + \cdots + \left|w - \overline{g_N}(z)\right|^2$, $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. If $u_1$ is strictly psh in $\mathbb{C}^n \times \mathbb{C}$, then $\left\{\frac{\partial}{\partial z_1}(g_1, \ldots, g_N), \ldots, \frac{\partial}{\partial z_n}(g_1, \ldots, g_N)\right\}$ is linearly independent over $\mathbb{C}^N$ and $n < N$ (by using the hermitian Levi form of the function $u_1$). If $u_1$ is strictly psh in $\mathbb{C}^n \times \mathbb{C}$, then $v_1$ is strictly psh in $\mathbb{C}^n \times \mathbb{C}$. But not conversely.

Example. The functions $k_1(z) = z$, $k_2(z) = z^2$ $(z \in \mathbb{C})$ are analytic over $\mathbb{C}$. Let $v_1(z,w) = \left|w - \overline{k_1}(z)\right|^2 + \left|w - \overline{k_2}(z)\right|^2$, where $(z,w) \in \mathbb{C}^2$; $v_1$ is strictly psh on $\mathbb{C}^2$. Put $u_1(z,w) = |w-z|^2 + |w-z^2|^2$, where $(z,w) \in \mathbb{C}^2$. Let $\alpha, \beta \in \mathbb{C}$. The Levi hermitian form of $u_1$ is $L(u_1)(z,w)(\alpha,\beta) = |\beta - \alpha|^2 + |\beta - 2z\alpha|^2$. If $z = \frac{1}{2}$, then we have

$$L(u_1)\left(\frac{1}{2}, w\right)(\alpha,\alpha) = 0 \qquad \text{for each } \alpha \in \mathbb{C}\backslash\{0\}.$$

Therefore $u_1$ is not strictly psh in $\mathbb{C}^2$. Put $u_2(z,w) = |w_1 - g_1(z)|^2 + \cdots + |w_N - g_N(z)|^2$, $z \in \mathbb{C}^n$, $w = (w_1, \ldots, w_N) \in \mathbb{C}^N$; $u_2$ is not strictly psh in any not empty domain of $\mathbb{C}^n \times \mathbb{C}^N$.

Now put $v_2(z,w) = \left|w_1 - \overline{g_1}(z)\right|^2 + \cdots + \left|w_N - \overline{g_N}(z)\right|^2$. If for all fixed $z$ in $\mathbb{C}^n$, the system

$$\begin{cases} \frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \cdots + \frac{\partial g_1}{\partial z_n}(z)\alpha_n = 0 \\ \qquad\vdots \\ \frac{\partial g_N}{\partial z_1}(z)\alpha_1 + \cdots + \frac{\partial g_N}{\partial z_n}(z)\alpha_n = 0 \end{cases}$$

$(\alpha_1, \ldots, \alpha_n \in \mathbb{C})$ has only the solution $(\alpha_1, \ldots, \alpha_n) = (0, \ldots, 0)$, then $v_2$ is strictly psh in $\mathbb{C}^n \times \mathbb{C}^N$. Therefore $u_2$ and $v_2$ do not have the same structure in the theory of the strictly plurisubharmonic functions.

Put $u_3(z,w) = \left|w - \overline{\varphi_1}(z)\right|^2 + \left|w - \varphi_2(z)\right|^2$, where $(z,w) \in \mathbb{C}^2$, $\varphi_1, \varphi_2 : \mathbb{C} \to \mathbb{C}$ are analytic functions, $u_4(z,w) = \left|w - \varphi_1(z)\right|^2 + \left|w - \overline{\varphi_2}(z)\right|^2$. Then $u_3$ is strictly psh in $\mathbb{C}^2$ if and only if $u_4$ is strictly psh in $\mathbb{C}^2$.

Question 2.11. An original problem of the theory of functions in several complex variables is now the following. Let $f_0, \ldots, f_{k-1} : \mathbb{C}^n \to \mathbb{C}$ be $k$

analytic functions, $(n, k \geq 1)$. Set

$$u(z,w) = \left|w^k + f_{k-1}(z)w^{k-1} + \cdots + f_1(z)w + f_0(z)\right|,$$
$$v(z,w) = \left|w^k + \overline{f_{k-1}}(z)w^{k-1} + \cdots + \overline{f_1}(z)w + \overline{f_0}(z)\right|,$$

where $(z,w) \in \mathbb{C}^n \times \mathbb{C}$. $u$ is convex in $\mathbb{C}^n \times \mathbb{C}$ if and only if $v$ is convex in $\mathbb{C}^n \times \mathbb{C}$. Now note that $u$ is psh in $\mathbb{C}^n \times \mathbb{C}$, but $v$ is not in general (example take $v_1(z,w) = \left|w^2 + \bar{z}w\right|$ is not psh in $\mathbb{C}^2$). Find the condition described by the functions $f_0, \ldots, f_{k-1}$ such that $v$ is psh in $\mathbb{C}^n \times \mathbb{C}$. (Observe that we can consider in this study the question of a power series).

*Remark* 2.12. The above proposition is not true if $g : D \to \mathbb{C}$ is harmonic. For example, if $g : \mathbb{C} \to \mathbb{R}$, $g(z) = x_1$, $z = (x_1 + ix_2) \in \mathbb{C}$, where $x_1, x_2 \in \mathbb{R}$, then $\left|w - \overline{g}\right|^2$ is strictly psh in $\mathbb{C}^2$. But $\mathrm{Im}(g) = 0$ and $|w - 0|^2 = |w|^2$ is not strictly psh on any domain of $\mathbb{C}^2$.

THEOREM 2.13. *Let* $g_1, \ldots, g_N : \mathbb{C} \to \mathbb{C}$, $u(z,w) = |w_1 - g_1(z)|^2 + \cdots + |w_N - g_N(z)|^2$, *where* $(z,w) = (z, w_1, \ldots, w_N) \in \mathbb{C} \times \mathbb{C}^N, N \in \mathbb{N}$. *$u$ is strictly psh in $\mathbb{C} \times \mathbb{C}^N$ if and only if $g_1, \ldots, g_N$ are harmonic functions in $\mathbb{C}$ and* $\left|\frac{\partial g_1}{\partial \bar{z}}\right|^2 + \cdots + \left|\frac{\partial g_N}{\partial \bar{z}}\right|^2 > 0$ *on* $\mathbb{C}$.

*Proof.* Assume that $u$ is strictly psh on $\mathbb{C} \times \mathbb{C}^N$. Note that $u$ is a function of class $C^\infty$ on $\mathbb{C} \times \mathbb{C}^N$. Let $(z,w) = (z, w_1, \ldots, w_N) \in \mathbb{C} \times \mathbb{C}^N$. Fix $w_2, \ldots, w_N \in \mathbb{C}$. Then the function $u(.,., w_2, \ldots, w_N)$ is strictly psh on $\mathbb{C}^2$. By Abidi [1], $g_1$ is harmonic on $\mathbb{C}$. Consequently, $g_1, \ldots, g_N$ are harmonic functions on $\mathbb{C}$.

Put $g_j = f_j + \overline{k}_j$, where $f_j, k_j : \mathbb{C} \to \mathbb{C}$ be two analytic functions and $j \in \{1, \ldots, N\}$. Let $(\alpha, \beta) = (\alpha, \beta_1, \ldots, \beta_N) \in \mathbb{C} \times \mathbb{C}^N \backslash \{(0,0)\}$. The Levi hermitian form of $u$ is now

$$L(u)(z,w)(\alpha,\beta) = \left|\beta_1 - \frac{\partial f_1}{\partial z}(z)\alpha\right|^2 + \left|\frac{\partial k_1}{\partial z}(z)\alpha\right|^2 + \cdots + \left|\beta_N - \frac{\partial f_N}{\partial z}(z)\alpha\right|^2 + \left|\frac{\partial k_N}{\partial z}(z)\alpha\right|^2.$$

Assume that $\alpha \neq 0$. Put $\beta_1 = \frac{\partial f_1}{\partial z}(z)\alpha, \ldots, \beta_N = \frac{\partial f_N}{\partial z}(z)\alpha$. Then

$$L(u)(z,w)(\alpha,\beta) = \left(\left|\frac{\partial k_1}{\partial z}(z)\alpha\right|^2 + \cdots + \left|\frac{\partial k_N}{\partial z}(z)\alpha\right|^2\right)$$

for each $z \in \mathbb{C}$. Thus

$$\left|\frac{\partial g_1}{\partial \bar{z}}(z)\right|^2 + \cdots + \left|\frac{\partial g_N}{\partial \bar{z}}(z)\right|^2 > 0\,.$$

The converse is trivial.

Observe that the notion $u$ is strictly psh in $\mathbb{C} \times \mathbb{C}^N$ on the above theorem is independent of $f_1, \ldots, f_N$, where $g_j = f_j + \overline{k_j}$, $f_j, k_j : \mathbb{C} \to \mathbb{C}$ are analytic functions $(1 \leq j \leq N)$. ∎

PROPOSITION 2.14. *For every $g : D \to \mathbb{C}$ analytic, $D$ is a domain of $\mathbb{C}^n$, $(n \geq 2)$, $u = |g|^2$ is not strictly psh on any domain $D_1 \subset D$. Indeed $e^{|g|^2}$, $|g|^2 e^{|g|^2}$, $|g|^2 e^{|g|^2} e^{e^{|g|^2}}$ are not strictly psh functions in any domain $D_2 \subset D$. For example let $v = |g_1|^2 + \cdots + |g_n|^2$, where $g_1, \ldots, g_n : \mathbb{C}^n \to \mathbb{C}$ are analytic functions. Then $v$ is strictly psh in $\mathbb{C}^n$ if and only if the determinant $\det\left(\frac{\partial g_j}{\partial z_k}(z)\right)_{j,k} \neq 0$, for all $z \in \mathbb{C}^n$.*

Note that we have the assertion. Let $g_1, \ldots, g_N : D \to \mathbb{C}$ be $N$ analytic functions, $D$ is a domain of $\mathbb{C}^n$, $n \geq 2$, $N \geq 1$. If $N < n$, then $u = |g_1|^2 + \cdots + |g_N|^2$ is not strictly psh on any domain $D_1 \subset D$. In fact $u$ is a function of class $C^\infty$ in $D$. The Levi hermitian form of $u$ is

$$\begin{aligned} L(u)(z)(\alpha) &= \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k} \\ &= \left|\sum_{j=1}^{n} \frac{\partial g_1}{\partial z_j}(z)\alpha_j\right|^2 + \cdots + \left|\sum_{j=1}^{n} \frac{\partial g_N}{\partial z_j}(z)\alpha_j\right|^2 \end{aligned}$$

for each $z = (z_1, \ldots, z_n) \in D$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$.

Suppose that $u$ is strictly psh in $D$. Then for all $z \in D$, for all $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$, $L(u)(z)(\alpha_1, \ldots, \alpha_n) = 0$ if and only if

$$\begin{cases} \frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \cdots + \frac{\partial g_1}{\partial z_n}(z)\alpha_n = 0 \\ \qquad\qquad\vdots \\ \frac{\partial g_N}{\partial z_1}(z)\alpha_1 + \cdots + \frac{\partial g_N}{\partial z_n}(z)\alpha_n = 0. \end{cases}$$

Then

$$\alpha_1\left(\frac{\partial g_1}{\partial z_1}(z), \ldots, \frac{\partial g_N}{\partial z_1}(z)\right) + \cdots + \alpha_n\left(\frac{\partial g_1}{\partial z_n}(z), \ldots, \frac{\partial g_N}{\partial z_n}(z)\right) = 0$$

implies that $\alpha_1 = \cdots = \alpha_n = 0$. Therefore the subset of vectors

$$\left\{ \left( \frac{\partial g_1}{\partial z_1}(z), \dots, \frac{\partial g_N}{\partial z_1}(z) \right), \dots, \left( \frac{\partial g_1}{\partial z_n}(z), \dots, \frac{\partial g_N}{\partial z_n}(z) \right) \right\}$$

is a free family of $n$ vectors of $\mathbb{C}^N$, and $N < n$. This is a contradiction. Consequently, $u$ is not strictly psh on any domain $D_1 \subset D$.

But we have the following result: For all $n \in \mathbb{N}$, there exists $u_1, \dots, u_n : \mathbb{C}^{2n} \to \mathbb{C}$ be $n$ pluriharmonic functions such that $v = \big(|u_1|^2 + \cdots + |u_n|^2\big)$ is strictly psh in $\mathbb{C}^{2n}$.

Example. Put $u_j(z) = z_j + \overline{z_{n+j}}$, $1 \le j \le n$, where $z = (z_1, \dots, z_{2n}) \in \mathbb{C}^{2n}$. $u_j$ is in fact prh in $\mathbb{C}^{2n}$; $|u_1(z)|^2 = |z_1 + \overline{z_{n+1}}|^2 = |z_1|^2 + |z_{n+1}|^2 + z_1 z_{n+1} + \overline{z_1 z_{n+1}}$. Note that the function $K_1(z) = z_1 z_{n+1} + \overline{z_1 z_{n+1}}$, $K_1$ is pluriharmonic in $\mathbb{C}^{2n}$ and therefore the Levi hermitian form of $K_1$ is equal 0 over $\mathbb{C}^{2n} \times \mathbb{C}^{2n}$. Then $L\big(|u_1|^2\big)(z)(\alpha_1, \dots, \alpha_{2n}) = |\alpha_1|^2 + |\alpha_{n+1}|^2$. Then

$$L(v)(z)(\alpha_1, \dots, \alpha_{2n}) = \sum_{j=1}^{2n} |\alpha_j|^2 > 0 \qquad \text{if } (\alpha_1, \dots, \alpha_{2n}) \in \mathbb{C}^{2n} \backslash \{0\}.$$

Then $v$ is strictly psh in $\mathbb{C}^{2n}$, but $n < 2n$.

In fact for all $n \ge 1$, there exists a function $u : \mathbb{C}^n \to \mathbb{R}$ pluriharmonic such that $|u|^2$ is not strictly psh in $\mathbb{C}^n$, $u$ is not constant. Observe that we have if $h : \mathbb{C}^3 \to \mathbb{C}$ is pluriharmonic, then $|h|^2$ is not strictly psh in $\mathbb{C}^3$. Exactly we have for all $h_1, \dots, h_s : \mathbb{C}^n \to \mathbb{C}$ prh, if $s < \frac{n}{2}$, then $(|h_1|^2 + \cdots + |h_s|^2)$ is not strictly psh in $\mathbb{C}^n$. Now if one of the function have real valued, one of the above result is not true. For example, if $u : \mathbb{C}^2 \to \mathbb{R}$ is a pluriharmonic function, then $u^2$ is not strictly psh on $\mathbb{C}^2$.

Theorem 2.15. *Let $u_1, \dots, u_n : \mathbb{C}^{2n} \to \mathbb{R}$ be $n$ pluriharmonic functions, $n \in \mathbb{N}$. Set $u = u_1^2 + \cdots + u_n^2$. Then $u$ is not strictly psh on any domain of $\mathbb{C}^{2n}$.*

*Proof.* The functions $u_1^2, \dots, u_n^2$ and $u$ are of class $C^\infty$ in $\mathbb{C}^{2n}$. Denote by

$$L(u)(z)(\alpha_1, \dots, \alpha_{2n}) = \sum_{j,k=1}^{2n} \frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}(z) \alpha_j \overline{\alpha_k}$$

for all $z = (z_1, \dots, z_{2n}) \in \mathbb{C}^{2n}$ and for all $\alpha = (\alpha_1, \dots, \alpha_{2n}) \in \mathbb{C}^{2n}$. We have

$$L(u)(z)(\alpha_1, \dots, \alpha_{2n}) = L(u_1^2)(z)(\alpha_1, \dots, \alpha_{2n}) + \cdots + L(u_n^2)(z)(\alpha_1, \dots, \alpha_{2n})$$

and

$$L(u_1^2)(z)(\alpha_1,\ldots,\alpha_{2n}) = \sum_{j,k=1}^{2n} \frac{\partial^2(u_1^2)}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k} = 2\sum_{j,k=1}^{2n} \frac{\partial u_1}{\partial z_j}(z)\frac{\partial u_1}{\partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k}$$

$$= 2\left(\sum_{j=1}^{2n} \frac{\partial u_1}{\partial z_j}(z)\alpha_j\right)\left(\sum_{k=1}^{2n} \overline{\frac{\partial u_1}{\partial z_k}(z)\alpha_k}\right)$$

$$= 2\left|\sum_{j=1}^{2n} \frac{\partial u_1}{\partial z_j}(z)\alpha_j\right|^2.$$

Consequently,

$$L(u)(z)(\alpha_1,\ldots,\alpha_{2n}) = 2\left|\sum_{j=1}^{2n} \frac{\partial u_1}{\partial z_j}(z)\alpha_j\right|^2 + \cdots + 2\left|\sum_{j=1}^{2n} \frac{\partial u_n}{\partial z_j}(z)\alpha_j\right|^2.$$

Fix $z = (z_1,\ldots,z_{2n}) \in \mathbb{C}^{2n}$. $L(u)(z)(\alpha_1,\ldots,\alpha_{2n}) = 0$ if and only if

$$\sum_{j=1}^{2n} \frac{\partial u_1}{\partial z_j}(z)\alpha_j = 0\,, \quad \ldots\,, \quad \sum_{j=1}^{2n} \frac{\partial u_n}{\partial z_j}(z)\alpha_j = 0.$$

Then

$$\begin{cases} \frac{\partial u_1}{\partial z_1}(z)\alpha_1 + \cdots + \frac{\partial u_1}{\partial z_{2n}}(z)\alpha_{2n} = 0 \\ \vdots \\ \frac{\partial u_n}{\partial z_1}(z)\alpha_1 + \cdots + \frac{\partial u_n}{\partial z_{2n}}(z)\alpha_{2n} = 0. \end{cases}$$

Thus

$$\alpha_1\big(\tfrac{\partial u_1}{\partial z_1}(z),\ldots,\tfrac{\partial u_n}{\partial z_1}(z)\big) + \cdots + \alpha_{2n}\big(\tfrac{\partial u_1}{\partial z_{2n}}(z),\ldots,\tfrac{\partial u_n}{\partial z_{2n}}(z)\big) = (0,\ldots,0) \in \mathbb{C}^n,$$

where $\alpha_1,\ldots,\alpha_{2n} \in \mathbb{C}$. We have $2n$ vectors of $\mathbb{C}^n$ (considered a vector space). Therefore the subset of the above $2n$ vectors is not a linearly independent family in the $\mathbb{C}$-vector space $\mathbb{C}^n$ of dimension $n$. Then there exists $(\alpha_1,\ldots,\alpha_{2n}) \in \mathbb{C}^{2n}\backslash\{0\}$ such that $L(u)(z)(\alpha_1,\ldots,\alpha_{2n}) = 0$. Consequently, $u$ is not strictly psh on any not empty domain of $\mathbb{C}^{2n}$. ∎

Definition 2.16. (Klimek [12]) Let $u : D \to \mathbb{R}$ be a psh function, where $D$ is an open of $\mathbb{C}^n$, $n \geq 1$. $u$ is maximal psh on $D$ if for all relatively compact open $G$ subset of $D$ and for each upper semi continuous function $v$ on $\overline{G}$ such that $v$ is psh on $G$ and $v \leq u$ on $\partial G$, we have $v \leq u$ on $G$.

*Remark* 2.17. (a) Let $n \in \mathbb{N}$, $n \geq 2$. Given $u_1, \ldots, u_{n-1} : D \to \mathbb{R}$ be $n-1$ pluriharmonic functions, where $D$ is a domain of $\mathbb{C}^n$. Then $u = \left(u_1^2 + \cdots + u_{n-1}^2\right)$ is not strictly psh on any domain $D_1 \subset D$.

(b) Let $n \in \mathbb{N}$ and $D$ a domain of $\mathbb{C}^n$. Consider $h_1, \ldots, h_n : D \to \mathbb{R}$ be $n$ pluriharmonic functions and put $u = h_1^2 + \cdots + h_n^2$. $u$ is psh on $D$. Then $u$ is strictly psh on $D$ if and only if $\det\left(\frac{\partial h_j}{\partial z_k}(z)\right)_{1 \leq j,\, k \leq n} \neq 0$ for all $z \in D$.

(c) Let $g : D \to \mathbb{C}$ be analytic, $D$ is a domain of $\mathbb{C}^n$, $(n \geq 2)$. $u = |g|^2$ is maximal plurisubharmonic (in the sense of Klimek [12] or Sadullaev [20]). But if $k : \mathbb{C} \to \mathbb{C}$ is analytic not constant, then $|k|^2 = v$ is not maximal subharmonic because $v$ is not harmonic. This is one of the great differences between the theory of functions of one complex variable and the same theory in several complex variables. In one complex variable, the sum of 2 maximal subharmonic functions is maximal subharmonic.

If now $g_1, g_2 : \mathbb{C}^2 \to \mathbb{C}$ be 2 analytic functions such that $|g_1|^2 + |g_2|^2 = \varphi$ is strictly psh in $\mathbb{C}^2$, then $|g_1|^2 + |g_2|^2$ is psh but not maximal plurisubharmonic on any open of $\mathbb{C}^2$. In this case $|g_1|^2$ and $|g_2|^2$ are maximal plurisubharmonic functions on $\mathbb{C}^2$. But the sum $|g_1|^2 + |g_2|^2 = \varphi$ is not maximal psh on any not empty open of $\mathbb{C}^2$. But we have the following result. Let $g_1, \ldots, g_N : D \to \mathbb{C}$ be $N$ analytic functions $(N \geq 1)$. Then if $N < n$, $u = |g_1|^2 + \cdots + |g_N|^2$ is maximal plurisubharmonic on $D$.

Proposition 2.18. *There exists a function $u : \mathbb{C}^2 \to \mathbb{R}$, $u$ real analytic on $\mathbb{C}^2$, $u$ is maximal plurisubharmonic on $\mathbb{C}^2$, but $e^u$ is plurisubharmonic on $\mathbb{C}^2$ and not maximal plurisubharmonic on any not empty domain of $\mathbb{C}^2$.*

*Moreover, for all $v : D \to \mathbb{R}$ prh, ($D$ is a domain of $\mathbb{C}^n$, $n \geq 2$) the function $e^v$ is maximal plurisubharmonic on $D$.*

*Proof.* Let $u(z_1, z_2) = x_1^2 + x_2$, where $z_1 = (x_1 + ix_3)$, $z_2 = (x_2 + ix_4) \in \mathbb{C}$ $(x_1, x_2, x_3, x_4 \in \mathbb{R})$. $u$ is plurisubharmonic in $\mathbb{C}^2$ and real analytic. We have the determinant

$$\det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z_k}}(z)\right)_{j,k} = 0$$

for each $z \in \mathbb{C}^2$. By Klimek [12], $u$ is maximal plurisubharmonic in $\mathbb{C}^2$.

Now

$$\det\left(\frac{\partial^2 (e^u)}{\partial z_j \partial \overline{z_k}}(z)\right)_{j,k} = \frac{1}{8} \neq 0$$

for every $z \in \mathbb{C}^2$. By Klimek [12, Proposition 3.1.6], $e^u$ is not maximal psh on any domain of $\mathbb{C}^2$. ∎

LEMMA 2.19. *Let $u : D \to \mathbb{R}$ be plurisubharmonic, where $D$ is a domain of $\mathbb{C}^n$, $n \geq 1$. If $e^u$ is maximal psh on $D$, then $u$ is maximal psh on $D$.*

*Proof.* Let $G$ be a relatively compact open subset of $D$ and $v : \overline{G} \to [-\infty, +\infty[$ be an upper semi continuous function such that $v$ is psh on $G$ and $v \leq u$ on $\partial G$. Then $e^v \leq e^u$ on $\partial G$ and consequently, $e^v \leq e^u$ on $G$. It follows that $v \leq u$ on $G$. ∎

*Remark* 2.20. For all $n \geq 1$, for all domain $D$ of $\mathbb{C}^n$, there exists $u : D \to \mathbb{R}$ be $C^\infty$ psh such that $e^u$ is strictly psh on $D$ but $u$ is not strictly psh on any domain $D_1 \subset D$.

In general we have the following lemma.

LEMMA 2.21. *Let $A, B$ two hermitian matrix of type $(n, n)$ with coefficients in $\mathbb{C}$. Suppose that $A$ and $B$ are positive semi-definite.*

(a) *If $A$ is positive definite then $A + B$ is positive definite on $\mathbb{C}^n$.*
(b) *If the determinant $\det(A) \neq 0$ then $A$ is positive definite on $\mathbb{C}^n$.*
(c) *If $A + B$ is positive definite, we can not conclude that $A$ or $B$ is positive definite on $\mathbb{C}^n$ if $n \geq 2$.*

EXAMPLE. Let $D$ be a domain of $\mathbb{C}^2$. Let $F = \{(g_1, g_2) \ / \ g_1, g_2 : D \to \mathbb{C}$ be analytic functions such that $(|g_1|^2 + |g_2|^2)$ is strictly psh in $\mathbb{C}^2\}$. Let $(g_1, g_2) \in F$. Fix $z = (z_1, z_2) \in D$. Put

$$A = \left(\frac{\partial^2 |g_1|^2}{\partial z_j \partial \overline{z_k}}(z)\right)_{j,\,k}, \qquad B = \left(\frac{\partial^2 |g_2|^2}{\partial z_j \partial \overline{z_k}}(z)\right)_{j,\,k}.$$

$A$ and $B$ are hermitian matrix positive semi definite on $\mathbb{C}^2$. Then $A + B$ is an hermitian matrix positive definite, but $A$ and $B$ are not positive definite over $\mathbb{C}^2$.

Now we can prove the following result.

THEOREM 2.22. *Let $u : D \to \mathbb{R}$ be a function of class $C^2$, $D$ is a domain of $\mathbb{C}^n$, $n \geq 1$. Suppose that $u$ is psh on $D$. Then $e^u$ is maximal psh on $D$ if and only if $e^{e^u}$ is maximal psh on $D$. Therefore if $e^u$ is maximal psh on $D$, then $F_s(u)$ is maximal psh on $D$, for each $s \in \mathbb{N}$, where $F_s = \exp \circ \exp \circ \cdots \circ \exp$ ($s$ times).*

This theorem have good and several applications in problems and exercises.

*Proof.* If $e^u$ is maximal psh on $D$. Since $e^u$ is psh and of class $C^2$ in $D$, then $\det\left(\frac{\partial^2(e^u)}{\partial z_j\partial\overline{z_k}}(z)\right)_{j,k}=0$ for all $z\in D$. Fix $z\in D$. Thus the matrix

$$A=\left(\frac{\partial^2(u)}{\partial z_j\partial\overline{z_k}}(z)+\frac{\partial u}{\partial z_j}(z)\frac{\partial u}{\partial\overline{z_k}}(z)\right)_{j,k}$$

is not an injection. Hence there exists $\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{C}^n\backslash\{0\}$ such that $A\alpha=0$. If $<.,.>$ is the hermitian habitual product on $\mathbb{C}^n$, then $<\alpha,A\alpha>=0$,

$$\sum_{j,k=1}^n\frac{\partial^2(u)}{\partial z_j\partial\overline{z_k}}(z)\alpha_j\overline{\alpha_k}+\sum_{j,k=1}^n\frac{\partial u}{\partial z_j}(z)\frac{\partial u}{\partial\overline{z_k}}(z)\alpha_j\overline{\alpha_k}=0\,.$$

As a consequence

$$\sum_{j,k=1}^n\frac{\partial^2(u)}{\partial z_j\partial\overline{z_k}}(z)\alpha_j\overline{\alpha_k}+\left|\sum_{j=1}^n\frac{\partial u}{\partial z_j}(z)\alpha_j\right|^2=0\,.$$

Since $u$ is psh and of class $C^2$ in $D$, thus

$$\sum_{j,k=1}^n\frac{\partial^2(u)}{\partial z_j\partial\overline{z_k}}(z)\alpha_j\overline{\alpha_k}\geq 0.$$

Now since $\left|\sum_{j=1}^n\frac{\partial u}{\partial z_j}(z)\alpha_j\right|^2\geq 0$, it follows that $\sum_{j,k=1}^n\frac{\partial^2(u)}{\partial z_j\partial\overline{z_k}}(z)\alpha_j\overline{\alpha_k}=0$ and $\left|\sum_{j=1}^n\frac{\partial u}{\partial z_j}(z)\alpha_j\right|^2=\sum_{j,k=1}^n\frac{\partial u}{\partial z_j}(z)\frac{\partial u}{\partial\overline{z_k}}(z)\alpha_j\overline{\alpha_k}=0$, and thus

$$\left(1+e^{u(z)}\right)\sum_{j,k=1}^n\frac{\partial u}{\partial z_j}(z)\frac{\partial u}{\partial\overline{z_k}}(z)\alpha_j\overline{\alpha_k}=0\,.$$

Consequently,

$$\sum_{j,k=1}^n\frac{\partial^2(u)}{\partial z_j\partial\overline{z_k}}(z)\alpha_j\overline{\alpha_k}+\left(1+e^{u(z)}\right)\sum_{j,k=1}^n\frac{\partial u}{\partial z_j}(z)\frac{\partial u}{\partial\overline{z_k}}(z)\alpha_j\overline{\alpha_k}=$$

$$\sum_{j,k=1}^n\left(\frac{\partial^2(u)}{\partial z_j\partial\overline{z_k}}(z)+\left(1+e^{u(z)}\right)\frac{\partial u}{\partial z_j}(z)\frac{\partial u}{\partial\overline{z_k}}(z)\right)\alpha_j\overline{\alpha_k}=0\,.$$

Now the matrix

$$B = \left( \frac{\partial^2(u)}{\partial z_j \partial \overline{z_k}}(z) + (1 + e^{u(z)}) \frac{\partial u}{\partial z_j}(z) \frac{\partial u}{\partial \overline{z_k}}(z) \right)_{j,\,k}$$

is an hermitian matrix positive semi definite because $e^{e^u}$ is psh on $D$. If $\det(B) \neq 0$, then $B$ is positive definite on $\mathbb{C}^n$. But there exists $\alpha \in \mathbb{C}^n \backslash \{0\}$ such that $< \alpha, B\alpha >= 0$. Then $B$ is not definite positive in $\mathbb{C}^n$. Consequently, $\det(B) = 0$ and we have $e^{e^u}$ is maximal psh on $D$.

The converse is trivial. ∎

Example. Let $h : D \to \mathbb{R}$ be prh, where $D$ is a domain of $\mathbb{C}^n$, $n \geq 2$. We denote by $F_s = \exp \circ \exp \circ \cdots \circ \exp$ ($s$ times), for $s \in \mathbb{N}$ (and $F_0$ is the identity operator). Then $F_s(h)$ is maximal plurisubharmonic in $D$. Now let $s, t \in \mathbb{N}$. Thus the function $F_s(h) - F_t(h)$ is maximal plurisubharmonic in $D$ in the case $s \geq t$. (We prove that $\det \left( \frac{\partial^2(F_s(h) - F_t(h))}{\partial z_j \partial \overline{z_k}} \right)_{1 \leq j,k \leq n} = 0$. By Klimek [12, Corollary 3.1.8] we conclude the required property).

The following two theorems have several applications in the theory of functions.

Theorem 2.23. *Let $f : D \to \mathbb{R}$ be a function, $D$ is a domain of $\mathbb{C}^n$, $n \geq 1$. Put $u(z, w) = |w - f(z)|^2$, where $(z, w) \in D \times \mathbb{C}$. The following two conditions are equivalent:*

(a) *$u$ is strictly psh in $D \times \mathbb{C}$;*

(b) *$n = 1$, $f$ is harmonic in $D$ and $\frac{\partial f}{\partial z}(z) \neq 0$ for each $z \in D$.*

*Proof.* (a) ⇒ (b) Since $u$ is strictly psh in $D \times \mathbb{C}$, then $u$ is psh in $D \times \mathbb{C}$ and consequently, $f$ is pluriharmonic in $D$. Therefore $u$ is a function of class $C^\infty$ in $D$.

Suppose that $n \geq 2$. Let $z^0 = (z_1^0, \ldots, z_n^0) \in D$. Consider now $R > 0$ such that $P(z^0, R) = D(z_1^0, R) \times D(z_2^0, R) \times \cdots \times D(z_n^0, R) \subset D$. We consider the function $f(.,., z_3^0, \ldots, z_n^0)$ defined and prh in $D(z_1^0, R) \times D(z_2^0, R) = A$. Let $f_1 = f(.,., z_3^0, \ldots, z_n^0)$ and

$$\begin{aligned} u_1(z_1, z_2, w) &= u(z_1, z_2, z_3^0, \ldots, z_n^0, w) \\ &= \left| w - f(z_1, z_2, z_3^0, \ldots, z_n^0) \right|^2 = |w - f_1(z_1, z_2)|^2, \end{aligned}$$

where $(z_1, z_2, w) \in D(z_1^0, R) \times D(z_2^0, R) \times \mathbb{C}$. Note that $f_1$ is prh in $A$ and $u_1$ is strictly psh in $A \times \mathbb{C}$,

$$u_1(z_1, z_2, w) = |w|^2 + |f_1(z_1, z_2)|^2 - \overline{w} f_1(z_1, z_2) - w f_1(z_1, z_2) .$$

Fix $w_0 = 0 \in \mathbb{C}$. The Levi hermitian form of $u_1(.,.,0)$ is

$$\begin{aligned} L(u_1)(z_1, z_2, 0)(\alpha_1, \alpha_2) = & \frac{\partial^2 u_1}{\partial z_1 \partial \overline{z_1}}(z_1, z_2, 0)\alpha_1\overline{\alpha_1} + \frac{\partial^2 u_1}{\partial z_2 \partial \overline{z_2}}(z_1, z_2, 0)\alpha_2\overline{\alpha_2} \\ & + 2\operatorname{Re}\left[\frac{\partial^2 u_1}{\partial z_1 \partial \overline{z_2}}(z_1, z_2, 0)\alpha_1\overline{\alpha_2}\right] > 0 , \end{aligned}$$

for all $(z_1, z_2) \in A$ and for all $(\alpha_1, \alpha_2) \in \mathbb{C}^2 \backslash \{(0,0)\}$. Moreover

$$2\left|\frac{\partial f_1}{\partial z_1}\right|^2 \alpha_1\overline{\alpha_1} + 2\left|\frac{\partial f_1}{\partial z_2}\right|^2 \alpha_2\overline{\alpha_2} + 2\operatorname{Re}\left[2\frac{\partial f_1}{\partial z_1}\frac{\partial f_1}{\partial \overline{z_2}}\alpha_1\overline{\alpha_2}\right] > 0 ,$$

on $A$ for all $(\alpha_1, \alpha_2) \in \mathbb{C}^2 \backslash \{0\}$. Then $\left|2\frac{\partial f_1}{\partial z_1}\frac{\partial f_1}{\partial \overline{z_2}}\right|^2 < 4\left|\frac{\partial f_1}{\partial z_1}\right|^2\left|\frac{\partial f_1}{\partial z_2}\right|^2$ over $A$. But we have $\left|\frac{\partial f_1}{\partial z_1}\frac{\partial f_1}{\partial \overline{z_2}}\right| = \left|\frac{\partial f_1}{\partial z_1}\right|\left|\frac{\partial f_1}{\partial \overline{z_2}}\right| = \left|\frac{\partial f_1}{\partial z_1}\right|\left|\frac{\partial f_1}{\partial z_2}\right| < \left|\frac{\partial f_1}{\partial z_1}\right|\left|\frac{\partial f_1}{\partial z_2}\right|$ in $A$ (because $f_1$ has real valued). A contradiction. Then $n = 1$.

Now the Levi hermitian form of $u$ is

$$\begin{aligned} L(u)(z, w)(\alpha, \beta) &= \frac{\partial^2 u}{\partial z \partial \overline{z}}(z, w)\alpha\overline{\alpha} + \frac{\partial^2 u}{\partial w \partial \overline{w}}(z, w)\beta\overline{\beta} + 2\operatorname{Re}\left[\frac{\partial^2 u}{\partial z \partial \overline{w}}(z, w)\alpha\overline{\beta}\right] \\ &= 2\left|\frac{\partial f}{\partial z}(z)\right|^2 \alpha\overline{\alpha} + \beta\overline{\beta} + 2\operatorname{Re}\left[-\frac{\partial f}{\partial z}(z)\alpha\overline{\beta}\right] > 0 , \end{aligned}$$

for all $(z, w) \in D \times \mathbb{C}$ and for all $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{0\}$. Then

$$\left|\frac{\partial f}{\partial z}(z)\right|^2 < 2\left|\frac{\partial f}{\partial z}(z)\right|^2$$

for each $z \in D$. Thus $\frac{\partial f}{\partial z}(z) \neq 0$ for every $z \in D$. Consequently, $n = 1$, $f$ is harmonic in $D$ and $\left\{z \in D : \frac{\partial f}{\partial z}(z) = 0\right\} = \emptyset$.

(b) $\Rightarrow$ (a) The Levi hermitian form of $u$ is

$$L(u)(z, w)(\alpha, \beta) = 2\left|\frac{\partial f}{\partial z}(z)\right|^2 \alpha\overline{\alpha} + \beta\overline{\beta} + 2\operatorname{Re}\left[-\frac{\partial f}{\partial z}(z)\alpha\overline{\beta}\right] > 0$$

for each $(z,w) \in D \times \mathbb{C}$ and $(\alpha,\beta) \in \mathbb{C}^2 \backslash \{0\}$. We have

$$L(u)(z,w)(\alpha,\beta) > 0 \qquad \forall\,(z,w) \in D \times \mathbb{C}\,,\ \forall\,(\alpha,\beta) \in \mathbb{C}^2 \backslash \{0\}$$

if and only if

$$\left|\frac{\partial f}{\partial z}(z)\right|^2 < 2\left|\frac{\partial f}{\partial z}(z)\right|^2.$$

But this is equivalent to $\frac{\partial f}{\partial z}(z) \neq 0$ for all $z \in D$. ∎

Now the case where the function is complex valued, we prove the following extension.

THEOREM 2.24. *Let $g : D \to \mathbb{C}$ be a function, $D$ is a domain of $\mathbb{C}^n$, $n \geq 1$. Put $v(z,w) = |w - g(z)|^2$, where $(z,w) \in D \times \mathbb{C}$. The following two conditions are equivalent:*

(a) *$v$ is strictly psh in $D \times \mathbb{C}$;*

(b) *$n = 1$, $g$ is harmonic in $D$ and $\left\{z \in D \,:\, \frac{\partial g}{\partial \bar{z}}(z) = 0\right\} = \emptyset$.*

*Proof.* (a) $\Rightarrow$ (b) Since $v$ is strictly plurisubharmonic in $D \times \mathbb{C}$, then $v$ is plurisubharmonic in $D \times \mathbb{C}$. Consequently, $g$ is pluriharmonic in $D$. Let $z^0 = (z_1^0, \dots, z_n^0) \in D$, $R > 0$ such that $D(z_1^0, R) \times \cdots \times D(z_n^0, R) = A \subset D$. Put $g = g_1 + \overline{g_2}$ in the convex domain $A$, where $g_1, g_2 : A \to \mathbb{C}$ be two analytic functions. Now we use the following fundamental decomposition

$$\begin{aligned} v(z,w) &= |w - g_1(z) - \overline{g_2}(z)|^2 \\ &= |w - g_1(z)|^2 + |g_2(z)|^2 - (w - g_1(z))g_2(z) - (\overline{w - g_1(z)})\overline{g_2}(z) \end{aligned}$$

for each $(z,w) \in A \times \mathbb{C}$. Suppose that $n \geq 2$.

CASE 1: $n = 2$. We have $v_1(z,w) = |w - g_1(z)|^2$, $v_2(z,w) = |g_2(z)|^2$, $v_3(z,w) = -(w - g_1(z))g_2(z) - (\overline{w - g_1(z)})\overline{g_2}(z)$, where $(z,w) \in A \times \mathbb{C}$; $v_1, v_2$ and $v_3$ are $C^\infty$ functions in the domain $A \times \mathbb{C}$, and $v_3$ is pluriharmonic in $A \times \mathbb{C}$. Then the Levi hermitian form of $v_3$ is

$$L(v_3)(z_1, z_2, w)(\alpha_1, \alpha_2, \beta) = 0$$

for all $(z_1, z_2, w) \in A \times \mathbb{C}$ and for all $(\alpha_1, \alpha_2, \beta) \in \mathbb{C}^3$. The Levi hermitian form of $v_2$ is

$$\begin{aligned} L(v_2)(z_1, z_2, w)(\alpha_1, \alpha_2, \beta) &= \left|\frac{\partial g_2}{\partial z_1}(z)\right|^2 \alpha_1\overline{\alpha_1} + \left|\frac{\partial g_2}{\partial z_2}(z)\right|^2 \alpha_2\overline{\alpha_2} \\ &\quad + 2\,\mathrm{Re}\left[\frac{\partial g_2}{\partial z_1}(z)\frac{\partial \overline{g_2}}{\partial \overline{z_2}}(z)\alpha_1\overline{\alpha_2}\right] \\ &= \left|\frac{\partial g_2}{\partial z_1}(z)\alpha_1 + \frac{\partial g_2}{\partial z_2}(z)\alpha_2\right|^2 \end{aligned}$$

for each $(z, w) = (z_1, z_2, w) \in A \times \mathbb{C}$ and $(\alpha_1, \alpha_2, \beta) \in \mathbb{C}^3$. The Levi hermitian form of $v_1$ is

$$\begin{aligned} L(v_1)(z_1, z_2, w)(\alpha_1, \alpha_2, \beta) &= \left|\frac{\partial g_1}{\partial z_1}(z)\right|^2 \alpha_1\overline{\alpha_1} + \left|\frac{\partial g_1}{\partial z_2}(z)\right|^2 \alpha_2\overline{\alpha_2} + \beta\overline{\beta} \\ &\quad + 2\,\mathrm{Re}\left[\frac{\partial g_1}{\partial z_1}(z)\frac{\partial \overline{g_1}}{\partial \overline{z_2}}(z)\alpha_1\overline{\alpha_2}\right] \\ &\quad + 2\,\mathrm{Re}\left[-\frac{\partial g_1}{\partial z_1}(z)\alpha_1\overline{\beta}\right] + 2\,\mathrm{Re}\left[-\frac{\partial g_1}{\partial z_2}(z)\alpha_2\overline{\beta}\right] \\ &= \left|\frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2\right|^2 + |\beta|^2 \\ &\quad + 2\,\mathrm{Re}\left[-\left(\frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2\right)\overline{\beta}\right] \\ &= \left|\beta - \left[\frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2\right]\right|^2, \end{aligned}$$

where $(z, w) = (z_1, z_2, w) \in A \times \mathbb{C}$.

Now we have

$$L(v)(z, w)(\alpha_1, \alpha_2, \beta) = \left|\beta - \left[\frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2\right]\right|^2 + \left|\frac{\partial g_2}{\partial z_1}(z)\alpha_1 + \frac{\partial g_2}{\partial z_2}(z)\alpha_2\right|^2$$

where $(z, w) = (z_1, z_2, w) \in A \times \mathbb{C}$ and $(\alpha_1, \alpha_2, \beta) \in \mathbb{C}^3$.

Let $z \in A$. Choose $(\alpha_1, \alpha_2) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ such that $\frac{\partial g_2}{\partial z_1}(z)\alpha_1 + \frac{\partial g_2}{\partial z_2}(z)\alpha_2 = 0$. Now let $\beta = \frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2$. We have $(\alpha_1, \alpha_2, \beta) \in \mathbb{C}^3 \setminus \{(0, 0, 0)\}$ and $L(v)(z, w)(\alpha_1, \alpha_2, \beta) = 0$. This proves for example that $v$ is not strictly psh on any open of $D \times \mathbb{C}$. A contradiction.

Case 2: $n \geq 3$. We deduce by in fact the formula

$$L(v)(z,w)(\alpha_1,\ldots,\alpha_n,\beta) = \left|\beta - \sum_{j=1}^{n} \frac{\partial g_1}{\partial z_j}(z)\alpha_j\right|^2 + \left|\sum_{j=1}^{n} \frac{\partial g_2}{\partial z_j}(z)\alpha_j\right|^2,$$

where $(z,w) = (z_1,\ldots,z_n,w) \in A \times \mathbb{C}$ and $(\alpha_1,\ldots,\alpha_n,\beta) \in \mathbb{C}^{n+1}$.

Let $z \in A$. Now it is possible to choose $(\alpha_1,\ldots,\alpha_n) \in \mathbb{C}^n \backslash \{0\}$ such that

$$\sum_{j=1}^{n} \frac{\partial g_2}{\partial z_j}(z)\alpha_j = 0 \qquad (\text{because } n \geq 3)\,.$$

Let $\beta = \sum_{j=1}^{n} \frac{\partial g_1}{\partial z_j}(z)\alpha_j$. We have $(\alpha_1,\ldots,\alpha_n,\beta) \in \mathbb{C}^{n+1} \backslash \{0\}$ and

$$L(v)(z,w)(\alpha_1,\ldots,\alpha_n,\beta) = 0\,.$$

Therefore in fact $v$ is not strictly psh on any domain of $A \times \mathbb{C}$. A contradiction.

Consequently, $n = 1$. By the above theorem, $g$ is harmonic in $D$ and $\left|\frac{\partial g}{\partial \overline{z}}\right| > 0$ in $D$.

(b) $\Rightarrow$ (a) By the above theorem we deduce this assertion in fact. ∎

Theorem 2.25. *Let $g_1,\ldots,g_N : \mathbb{C}^2 \to \mathbb{C}$, $v(z,w) = |w_1 - g_1(z)|^2 + \cdots + |w_N - g_N(z)|^2$, where $z \in \mathbb{C}^2$, $w = (w_1,\ldots,w_N) \in \mathbb{C}^N$ and $N \in \mathbb{N}$. The following conditions are equivalent:*

(a$_1$) *$v$ is strictly psh in $\mathbb{C}^2 \times \mathbb{C}^N$;*

(a$_2$) *$g_j$ is pluriharmonic in $\mathbb{C}^2$, $g_j = f_j + \overline{k_j}$, where $f_j, k_j : \mathbb{C}^2 \to \mathbb{C}$ are analytic functions, for all $1 \leq j \leq N$ ($N \geq 2$). The functions $k_1,\ldots,k_N$ satisfies an algebraic condition, that is for each $z \in \mathbb{C}^2$, the set $\{s_1,\ldots,s_N\}$ is a generating family of the $\mathbb{C}$-vector space $\mathbb{C}^2$, $s_1 = \left(\frac{\partial k_1}{\partial z_1}(z), \frac{\partial k_1}{\partial z_2}(z)\right),\ldots, s_N = \left(\frac{\partial k_N}{\partial z_1}(z), \frac{\partial k_N}{\partial z_2}(z)\right)$;*

(a$_3$) *$g_j$ is pluriharmonic in $\mathbb{C}^2$, $g_j = f_j + \overline{k_j}$, where $f_j, k_j : \mathbb{C}^2 \to \mathbb{C}$ are analytic functions for all $j \in \{1,\ldots,N\}$, $N \geq 2$ and for all $z \in \mathbb{C}^2$, there exist $R > 0$ and there exists $s, t \in \{1,\ldots,N\}$ ($s \neq t$) such that $v_1$ is strictly psh in $B(z,R) \times \mathbb{C}^2$, where $v_1(z,t) = |w_s - g_s(z)|^2 + |w_t - g_t(z)|^2$, for $(z,w) \in B(z,R) \times \mathbb{C}^2$ and $w = (w_s,w_t)$;*

(a$_4$) *$g_j$ is pluriharmonic on $\mathbb{C}^2$, $g_j = f_j + \overline{k_j}$, where $f_j, k_j : \mathbb{C}^2 \to \mathbb{C}$ are analytic functions for all $1 \le j \le N$. $k_1, \dots, k_N$ satisfies $\Big\{\Big(\frac{\partial k_1}{\partial z_1}(z), \dots, \frac{\partial k_N}{\partial z_1}(z)\Big), \Big(\frac{\partial k_1}{\partial z_2}(z), \dots, \frac{\partial k_N}{\partial z_2}(z)\Big)\Big\}$ is a free family in the $\mathbb{C}$-vector space $\mathbb{C}^N$, for all fixed $z \in \mathbb{C}^2$.*

*Proof.* (a$_1$) $\Rightarrow$ (a$_2$) Firstly we prove that $g_1, \dots, g_N$ are continuous functions over $\mathbb{C}^2$. Let $z^0 \in \mathbb{C}^2$. Put $\zeta_1 = g_1(z^0), \dots, \zeta_N = g_N(z^0) \in \mathbb{C}$; $v(z^0, \zeta_1, \dots, \zeta_N) = 0$. Let $\epsilon > 0$. Since $v$ is upper semi-continuous in the point $(z^0, \zeta_1, \dots, \zeta_N)$ then there exists $\delta > 0$ such that $\|z - z^0\| + |w_1 - \zeta_1| + \cdots + |w_N - \zeta_N| < \delta$ implies that $|w_1 - g_1(z)|^2 + \cdots + |w_N - g_N(z)|^2 \le \epsilon^2$. Let $j \in \{1, \dots, N\}$. If we put $w_1 = \zeta_1, \dots, w_{j-1} = \zeta_{j-1}, w_{j+1} = \zeta_{j+1}, \dots, w_N = \zeta_N$, then we have $\|z - z^0\| + |w_j - \zeta_j| < \delta$ implies that $|w_j - g_j(z)|^2 < \epsilon^2$. Let $w_j = \zeta_j = g_j(z^0)$. Then $\|z - z^0\| < \delta$ implies that $|g_j(z) - g_j(z^0)| < \epsilon$. Then $g_j$ is continuous in the point $z^0 \in \mathbb{C}^2$. Consequently, $g_1, \dots, g_N$ are continuous functions on $\mathbb{C}^2$.

We have $v$ is strictly psh in $\mathbb{C}^2 \times \mathbb{C}^N$, therefore $v$ is psh in $\mathbb{C}^2 \times \mathbb{C}^N$. Therefore the function of two variables $v(.,.,0, \dots, 0)$ is psh in $\mathbb{C}^2 \times \mathbb{C}$, where

$$v(z, w_1, 0, \dots, 0) = v_1(z, w_1) = |w_1 - g_1(z)|^2 + |g_2(z)|^2 + \cdots + |g_N(z)|^2.$$

Let $\varphi : \mathbb{C}^2 \to \mathbb{R}_+$, $\varphi$ is of class $C^\infty$ and have a compact support in $\mathbb{C}^2$. Let

$$\Delta = 4\bigg(\frac{\partial^2}{\partial z_1 \partial \overline{z_1}} + \frac{\partial^2}{\partial z_2 \partial \overline{z_2}}\bigg)$$

the Laplace operator on $\mathbb{C}^2$. Then we have

$$\int |w_1 - g_1(z)|^2 \Delta\varphi(z)\, \mathrm{d}m_4(z) + \sum_{j=2}^{N} \int |g_j(z)|^2 \Delta\varphi(z)\, \mathrm{d}m_4(z) \ge 0$$

for each $w_1 \in \mathbb{C}$. Let $w_1 \in \mathbb{R}$. Then we have

$$-w_1 \int [g_1(z) + \overline{g_1}(z)] \Delta\varphi(z)\, \mathrm{d}m_4(z) + \sum_{j=1}^{N} \int |g_j(z)|^2 \Delta\varphi(z)\, \mathrm{d}m_4(z) \ge 0$$

for all $w_1 \in \mathbb{R}$. If $\int [g_1(z) + \overline{g_1}(z)] \Delta\varphi(z)\, \mathrm{d}m_4(z) > 0$, then we obtain a contradiction by letting $w_1$ to $+\infty$. If $\int [g_1(z) + \overline{g_1}(z)] \Delta\varphi(z)\, \mathrm{d}m_4(z) < 0$, then we have a contradiction by letting $w_1$ go to $(-\infty)$. Consequently,

$$\int [g_1(z) + \overline{g_1}(z)] \Delta\varphi(z)\, \mathrm{d}m_4(z) = 0.$$

Since $g_1 + \overline{g_1}$ is a continuous function in $\mathbb{C}^2$, then $g_1 + \overline{g_1}$ is harmonic in $\mathbb{C}^2$.

Let $w_1 \in i\mathbb{R}$. Then $\overline{w_1} = -w_1$. In this case we prove that $(g_1 - \overline{g_1})$ is harmonic in $\mathbb{C}^2$. Now since $g_1 = \frac{1}{2}\left[(g_1 + \overline{g_1}) + (g_1 - \overline{g_1})\right]$, then $g_1$ is harmonic in $\mathbb{C}^2$. Let $T_1 : \mathbb{C}^2 \to \mathbb{C}^2$ be a $\mathbb{C}$-linear bijective transformation. Consider now $T(z, w_1) = (T_1(z), w_1)$, where $z \in \mathbb{C}^2$ and $w_1 \in \mathbb{C}$. Note that $T : \mathbb{C}^3 \to \mathbb{C}^3$ is a $\mathbb{C}$-linear bijective transformation. $v_1$ is psh in $\mathbb{C}^2 \times \mathbb{C}$, then $v_1 \circ T$ is psh in $\mathbb{C}^2 \times \mathbb{C}$,

$$v_1 \circ T(z, w_1) = |w_1 - g_1 \circ T_1(z)|^2 + |g_2 \circ T_1(z)|^2 + \cdots + |g_N \circ T_1(z)|^2 ,$$

$(z, w_1) \in \mathbb{C}^2 \times \mathbb{C}$. By the above development we have $g_1 \circ T_1$ is harmonic in $\mathbb{C}^2$. Consequently, $g_1$ is a pluriharmonic function on $\mathbb{C}^2$. Therefore $g_1, \ldots, g_N$ are pluriharmonic functions on $\mathbb{C}^2$; $g_j = f_j + \overline{k_j}$, $f_j, k_j : \mathbb{C}^2 \to \mathbb{C}$ are analytic functions , $1 \leq j \leq N$.

Consider now $a_1(z, w) = |w_1 - g_1(z)|^2$, where $(z, w_1) \in \mathbb{C}^2 \times \mathbb{C}$; $a_1(z, w_1) = |w_1 - f_1(z) - \overline{k_1}(z)|^2$. We consider now the following decomposition $a_1(z, w_1) = |w_1 - f_1(z)|^2 + |k_1(z)|^2 - k_1(z)(w_1 - f_1(z)) - \overline{k_1(z)(w_1 - f_1(z))}$. $a_1$ is a function of class $C^\infty$ in $\mathbb{C}^2 \times \mathbb{C}$.

Let $H_1(z, w_1) = k_1(z)(w_1 - f_1(z)) + \overline{k_1(z)(w_1 - f_1(z))}$, where $(z, w_1) \in \mathbb{C}^2 \times \mathbb{C}$; $H_1$ is pluriharmonic in $\mathbb{C}^2 \times \mathbb{C}$. Therefore the Levi hermitian form of $H_1$ is $L(H_1)(z, w_1)(\alpha_1, \alpha_2, \beta_1) = 0$, for all $(z, w_1) \in \mathbb{C}^2 \times \mathbb{C}$ and for all $(\alpha_1, \alpha_2, \beta_1) \in \mathbb{C}^3$.

Then the Levi hermitian form of $a_1$ is

$$L(a_1)(z, w_1)(\alpha_1, \alpha_2, \beta_1) = L(b_1)(z, w_1)(\alpha_1, \alpha_2, \beta_1) + L(c_1)(z, w_1)(\alpha_1, \alpha_2, \beta_1) ,$$

where $b_1(z, w_1) = |w_1 - f_1(z)|^2$, $c_1(z, w_1) = |k_1(z)|^2$ and $(z, w_1) \in \mathbb{C}^2 \times \mathbb{C}$. $b_1$ and $c_1$ are in particular functions of class $C^\infty$ in $\mathbb{C}^2 \times \mathbb{C}$. We have

$$\begin{aligned} L(b_1)(z, w_1)(\alpha_1, \alpha_2, \beta_1) &= \frac{\partial^2 b_1}{\partial z_1 \partial \overline{z_1}}(z, w_1)\alpha_1\overline{\alpha_1} + \frac{\partial^2 b_1}{\partial z_2 \partial \overline{z_2}}(z, w_1)\alpha_2\overline{\alpha_2} \\ &\quad + 2\,\mathrm{Re}\left[\frac{\partial^2 b_1}{\partial z_1 \partial \overline{z_2}}(z, w_1)\alpha_1\overline{\alpha_2}\right] + \frac{\partial^2 b_1}{\partial w_1 \partial \overline{w_1}}(z, w_1)\beta_1\overline{\beta_1} \\ &\quad + 2\,\mathrm{Re}\left[\frac{\partial^2 b_1}{\partial z_1 \partial \overline{w_1}}(z, w_1)\alpha_1\overline{\beta_1} + \frac{\partial^2 b_1}{\partial z_2 \partial \overline{w_1}}(z, w_1)\alpha_2\overline{\beta_1}\right] \\ &= \left|\frac{\partial f_1}{\partial z_1}(z)\right|^2\alpha_1\overline{\alpha_1} + \left|\frac{\partial f_1}{\partial z_2}(z)\right|^2\alpha_2\overline{\alpha_2} + 2\,\mathrm{Re}\left[\frac{\partial f_1}{\partial z_1}(z)\frac{\partial f_1}{\partial \overline{z_2}}(z)\alpha_1\overline{\alpha_2}\right] + \beta_1\overline{\beta_1} \\ &\quad + 2\,\mathrm{Re}\left[-\frac{\partial f_1}{\partial z_1}(z)\alpha_1\overline{\beta_1} - \frac{\partial f_1}{\partial z_2}(z)\alpha_2\overline{\beta_1}\right] \end{aligned}$$

$$= \left|\frac{\partial f_1}{\partial z_1}(z)\alpha_1 + \frac{\partial f_1}{\partial z_2}(z)\alpha_2\right|^2 + |\beta_1|^2 - 2\,\mathrm{Re}\left[\left(\frac{\partial f_1}{\partial z_1}(z)\alpha_1 + \frac{\partial f_1}{\partial z_2}(z)\alpha_2\right)\overline{\beta_1}\right]$$
$$= \left|\beta_1 - \left[\frac{\partial f_1}{\partial z_1}(z)\alpha_1 + \frac{\partial f_1}{\partial z_2}(z)\alpha_2\right]\right|^2.$$

Since $c_1(z,w_1) = |k_1(z)|^2$, then

$$\begin{aligned} L(c_1)(z,w_1)(\alpha_1,\alpha_2,\beta_1) &= \frac{\partial^2 c_1}{\partial z_1 \partial \overline{z_1}}(z,w_1)\alpha_1\overline{\alpha_1} + \frac{\partial^2 c_1}{\partial z_2 \partial \overline{z_2}}(z,w_1)\alpha_2\overline{\alpha_2} \\ &\quad + 2\,\mathrm{Re}\left[\frac{\partial^2 c_1}{\partial z_1 \partial \overline{z_2}}(z,w_1)\alpha_1\overline{\alpha_2}\right] \\ &= \left|\frac{\partial k_1}{\partial z_1}(z)\right|^2 \alpha_1\overline{\alpha_1} + \left|\frac{\partial k_1}{\partial z_2}(z)\right|^2 \alpha_2\overline{\alpha_2} + 2\,\mathrm{Re}\left[\frac{\partial k_1}{\partial z_1}(z)\frac{\partial \overline{k_1}}{\partial \overline{z_2}}(z)\alpha_1\overline{\alpha_2}\right] \\ &= \left|\frac{\partial k_1}{\partial z_1}(z)\alpha_1 + \frac{\partial k_1}{\partial z_2}(z)\alpha_2\right|^2. \end{aligned}$$

Consequently, $L(a_1)(z,w_1)(\alpha_1,\alpha_2,\beta_1) = \left|\beta_1 - \left[\frac{\partial f_1}{\partial z_1}(z)\alpha_1 + \frac{\partial f_1}{\partial z_2}(z)\alpha_2\right]\right|^2 + \left|\frac{\partial k_1}{\partial z_1}(z)\alpha_1 + \frac{\partial k_1}{\partial z_2}(z)\alpha_2\right|^2$ for each $(z,w_1) \in \mathbb{C}^2 \times \mathbb{C}$ and $(\alpha_1,\alpha_2,\beta_1) \in \mathbb{C}^3$.

Since $v$ is a function of class $C^\infty$ in $\mathbb{C}^2 \times \mathbb{C}^N$, then we have for each $(z,w_1,\dots,w_N) \in \mathbb{C}^2 \times \mathbb{C}^N$, $z = (z_1,z_2) \in \mathbb{C}^2$ and all $(\alpha_1,\alpha_2,\beta_1,\dots,\beta_N) \in \mathbb{C}^{N+2}$, the Levi hermitian form of $v$ is

$$\begin{aligned} &L(v)(z,w_1,\dots,w_N)(\alpha_1,\alpha_2,\beta_1,\dots,\beta_N) \\ &\quad = \left|\beta_1 - \left[\frac{\partial f_1}{\partial z_1}(z)\alpha_1 + \frac{\partial f_1}{\partial z_2}(z)\alpha_2\right]\right|^2 + \left|\frac{\partial k_1}{\partial z_1}(z)\alpha_1 + \frac{\partial k_1}{\partial z_2}(z)\alpha_2\right|^2 \\ &\qquad + \cdots + \left|\beta_N - \left[\frac{\partial f_N}{\partial z_1}(z)\alpha_1 + \frac{\partial f_N}{\partial z_2}(z)\alpha_2\right]\right|^2 + \left|\frac{\partial k_N}{\partial z_1}(z)\alpha_1 + \frac{\partial k_N}{\partial z_2}(z)\alpha_2\right|^2. \end{aligned}$$

Fix $z \in \mathbb{C}^2$. If $L(v)(z,w_1,\dots,w_N)(\alpha_1,\alpha_2,\beta_1,\dots,\beta_N) = 0$, then

$$\begin{cases} \frac{\partial k_1}{\partial z_1}(z)\alpha_1 + \frac{\partial k_1}{\partial z_2}(z)\alpha_2 = 0 \\ \qquad\vdots \\ \frac{\partial k_N}{\partial z_1}(z)\alpha_1 + \frac{\partial k_N}{\partial z_2}(z)\alpha_2 = 0. \end{cases}$$

Therefore if $\alpha_1, \alpha_2 \in \mathbb{C}$ such that

$$\alpha_1\left(\frac{\partial k_1}{\partial z_1}(z),\dots,\frac{\partial k_N}{\partial z_1}(z)\right) + \alpha_2\left(\frac{\partial k_1}{\partial z_2}(z),\dots,\frac{\partial k_N}{\partial z_2}(z)\right) = (0,\dots,0) \in \mathbb{C}^N,$$

then $\alpha_1 = \alpha_2 = 0$. Thus $N \geq 2$ and there exists $s, t \in \{1, \ldots, N\}$ $(s \neq t)$ such that $\left\{\left(\frac{\partial k_s}{\partial z_1}(z), \frac{\partial k_s}{\partial z_2}(z)\right), \left(\frac{\partial k_t}{\partial z_1}(z), \frac{\partial k_t}{\partial z_2}(z)\right)\right\}$ is a basis of the $\mathbb{C}$-vector space $\mathbb{C}^2$. Then $\left\{\left(\frac{\partial k_1}{\partial z_1}(z), \frac{\partial k_1}{\partial z_2}(z)\right), \ldots, \left(\frac{\partial k_N}{\partial z_1}(z), \frac{\partial k_N}{\partial z_2}(z)\right)\right\}$ is a generating family of the $\mathbb{C}$-vector space $\mathbb{C}^2$. Observe that locally $(s, t)$ is independent of $z \in \mathbb{C}^2$, but not globally if $N \geq 3$.

$(\mathrm{a}_2) \Rightarrow (\mathrm{a}_1)$ Let $z \in \mathbb{C}^2$. Since $\left\{\left(\frac{\partial k_1}{\partial z_1}(z), \frac{\partial k_1}{\partial z_2}(z)\right), \ldots, \left(\frac{\partial k_N}{\partial z_1}(z), \frac{\partial k_N}{\partial z_2}(z)\right)\right\}$ is a generating family of the $\mathbb{C}$-vector space $\mathbb{C}^2$, then $N \geq 2$ and we can exhibit a family of 2 vectors which is a basis of $\mathbb{C}^2$. Without loss of generality we suppose that $\left\{\left(\frac{\partial k_1}{\partial z_1}(z), \frac{\partial k_1}{\partial z_2}(z)\right), \left(\frac{\partial k_2}{\partial z_1}(z), \frac{\partial k_2}{\partial z_2}(z)\right)\right\}$ is a basis of $\mathbb{C}^2$. Therefore the matrix $(\lambda_{\mu\nu})_{1 \leq \mu, \nu \leq 2}$ have a determinant $\det(\lambda_{\mu\nu})_{1 \leq \mu, \nu \leq 2} = \varphi(z) \neq 0$, where $\lambda_{\mu\nu} = \frac{\partial k_\mu}{\partial z_\nu}(z)$. Since the function $\varphi$ is analytic in $\mathbb{C}^2$, then $|\varphi| > 0$ on a neighborhood $B(z, r)$ of the point $z$ $(r > 0)$. Then for all $\xi \in B(z, r)$ and $(\alpha_1, \alpha_2) \in \mathbb{C}^2$, we have

$$\begin{cases} \frac{\partial k_1}{\partial z_1}(\xi)\alpha_1 + \frac{\partial k_1}{\partial z_2}(\xi)\alpha_2 = 0 \\ \frac{\partial k_2}{\partial z_1}(\xi)\alpha_1 + \frac{\partial k_2}{\partial z_2}(\xi)\alpha_2 = 0 \end{cases}$$

if and only if $\alpha_1 = \alpha_2 = 0$. Thus if $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \mathbb{C}^4$, $\xi \in B(z, r)$, $\left|\beta_1 - \left[\frac{\partial f_1}{\partial z_1}(\xi)\alpha_1 + \frac{\partial f_1}{\partial z_2}(\xi)\alpha_2\right]\right|^2 + \left|\frac{\partial k_1}{\partial z_1}(\xi)\alpha_1 + \frac{\partial k_1}{\partial z_2}(\xi)\alpha_2\right|^2 + \left|\beta_2 - \left[\frac{\partial f_2}{\partial z_1}(\xi)\alpha_1 + \frac{\partial f_2}{\partial z_2}(\xi)\alpha_2\right]\right|^2 + \left|\frac{\partial k_2}{\partial z_1}(\xi)\alpha_1 + \frac{\partial k_2}{\partial z_2}(\xi)\alpha_2\right|^2 = 0$, then

$$\begin{cases} \frac{\partial k_1}{\partial z_1}(\xi)\alpha_1 + \frac{\partial k_1}{\partial z_2}(\xi)\alpha_2 = 0 \\ \frac{\partial k_2}{\partial z_1}(\xi)\alpha_1 + \frac{\partial k_2}{\partial z_2}(\xi)\alpha_2 = 0. \end{cases}$$

It follows that $\alpha_1 = \alpha_2 = 0$. Thus $\beta_1 = \beta_2 = 0$. Consequently, $\varphi_1(\xi, w_1, w_2) = |w_1 - g_1(\xi)|^2 + |w_2 - g_2(\xi)|^2$ is strictly psh in $B(z, r) \times \mathbb{C} \times \mathbb{C}$. In fact we can prove that $\varphi_1$ is strictly psh in $(\mathbb{C}^2 \backslash A) \times \mathbb{C}^2$, where $A$ is an analytic subset of $\mathbb{C}^2$.

Now the above proof implies that the assertions $(a_1)$, $(a_3)$ and $(a_4)$ are equivalent. ∎

Corollary 2.26. *Let $g_1, g_2 : \mathbb{C}^2 \to \mathbb{C}$ be two analytic functions. Put $u(z, w_1, w_2) = |w_1 - g_1(z)|^2 + |w_2 - g_2(z)|^2$, where $(z, w_1, w_2) \in \mathbb{C}^2 \times \mathbb{C} \times \mathbb{C}$. Let $A \subset \mathbb{C}^2$, $A$ closed and bounded in $\mathbb{C}^2$. Suppose that $u$ is strictly psh in $\mathbb{C}^2 \times (\mathbb{C}^2 \backslash A)$. Then $u$ is strictly psh in $\mathbb{C}^2 \times \mathbb{C}^2$.*

*Proof.* Note that $u$ is a function of class $C^\infty$ on $\mathbb{C}^2 \times \mathbb{C}^2$. Assume that $u$ is not strictly psh at the point $(z_0, w_0) \in \mathbb{C}^2 \times \mathbb{C}^2$. Then there exists $\big((\alpha_1, \alpha_2), (\beta_1, \beta_2)\big) \in \mathbb{C}^2 \times \mathbb{C}^2 \backslash \{(0,0)\}$ such that the Levi hermitian form of $u$ verify

$$L(u)(z_0,w_0)\big((\alpha_1,\alpha_2),(\beta_1,\beta_2)\big) = \left|\beta_1 - \sum_{j=1}^{2} \frac{\partial g_1}{\partial z_j}(z_0)\alpha_j\right|^2 + \left|\beta_2 - \sum_{j=1}^{2} \frac{\partial g_2}{\partial z_j}(z_0)\alpha_j\right|^2 = 0\,.$$

Let $b_0 \in \mathbb{C}^2 \backslash A$. Since $u$ is strictly psh on $\mathbb{C}^2 \times (\mathbb{C}^2 \backslash A)$, then $u$ is strictly psh at the point $(z_0, b_0)$. But we have

$$L(u)(z_0,b_0)\big((\alpha_1,\alpha_2),(\beta_1,\beta_2)\big) = \left|\beta_1 - \sum_{j=1}^{2} \frac{\partial g_1}{\partial z_j}(z_0)\alpha_j\right|^2 + \left|\beta_2 - \sum_{j=1}^{2} \frac{\partial g_2}{\partial z_j}(z_0)\alpha_j\right|^2 = 0\,.$$

and $\big((\alpha_1, \alpha_2), (\beta_1, \beta_2)\big) \in \mathbb{C}^2 \times \mathbb{C}^2 \backslash \{(0,0)\}$. A contradiction. Consequently, $u$ is strictly psh on $\mathbb{C}^2 \times \mathbb{C}^2$. ∎

Corollary 2.27. *Let $g_1, g_2 : \mathbb{C}^2 \to \mathbb{C}$ be two analytic functions. Set $u(z,w) = |w_1 - g_1(z)|^2 + |w_2 - \overline{g_2}(z)|^2$, $v(z,w) = |w_1 - g_1(z)|^2 + |w_2 - g_2(z)|^2$, $\varphi(z,\zeta) = |\zeta - g_1(z)|^2 + |\zeta - \overline{g_2}(z)|^2$, where $z \in \mathbb{C}^2$, $w = (w_1, w_2) \in \mathbb{C}^2$ and $\zeta \in \mathbb{C}$. Then $u$ and $v$ are not strictly plurisubharmonic functions in $\mathbb{C}^2 \times \mathbb{C}^2$.*

*We have, $\varphi$ is strictly psh in $\mathbb{C}^2 \times \mathbb{C}$ if and only if $|g_1|^2 + |g_2|^2$ (or $|g_1 + \overline{g_2}|^2$) is strictly psh on $\mathbb{C}^2$.*

*Proof.* We have the fundamental decomposition (complex structure)

$$u(z,w) = |w_1 - g_1(z)|^2 + |w_2|^2 + |g_2(z)|^2 - w_2 g_2(z) - \overline{w_2 g_2(z)},$$

for any $(z,w) = (z, w_1, w_2) \in \mathbb{C}^2 \times \mathbb{C} \times \mathbb{C}$ where $z = (z_1, z_2) \in \mathbb{C}^2$.

Put $u_1(z,w) = w_2 g_2(z) + \overline{w_2 g_2(z)}$; $u_1$ is a pluriharmonic function in $\mathbb{C}^2 \times \mathbb{C}^2$. Therefore the Levi hermitian form of this function is equal to 0 over $\mathbb{C}^4$.

Let $u_2(z,w) = |w_1 - g_1(z)|^2$; $u_2$ is a function of class $C^\infty$ in $\mathbb{C}^2 \times \mathbb{C}$, $u_2(z,w) = |w_1|^2 + |g_1(z)|^2 - \overline{w_1} g_1(z) - w_1 \overline{g_1}(z)$. Then the Levi hermitian form of $u_2$ is now

$$L(u_2)(z,w)(\alpha_1,\alpha_2,\beta_1,\beta_2) = \left|\beta_1 - \left[\frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2\right]\right|^2$$

for each $z=(z_1,z_2)\in\mathbb{C}^2$, $w=(w_1,w_2)\in\mathbb{C}^2$ and $(\alpha_1,\alpha_2,\beta_1,\beta_2)\in\mathbb{C}^4$.

Let $u_3(z,w)=|w_2|^2+|g_2(z)|^2$; $u_3$ is a function of class $C^\infty$ in $\mathbb{C}^4$. The Levi hermitian form of $u_3$ is

$$L(u_3)(z,w)(\alpha_1,\alpha_2,\beta_1,\beta_2)=\left|\beta_2\right|^2+\left|\frac{\partial g_2}{\partial z_1}(z)\alpha_1+\frac{\partial g_2}{\partial z_2}(z)\alpha_2\right|^2.$$

The function $u$ is of class $C^\infty$ in $\mathbb{C}^2\times\mathbb{C}^2$. We have

$$\begin{aligned}
&L(u)(z,w)(\alpha_1,\alpha_2,\beta_1,\beta_2)\\
&\quad=-L(u_1)(z,w)(\alpha_1,\alpha_2,\beta_1,\beta_2)+L(u_2)(z,w)(\alpha_1,\alpha_2,\beta_1,\beta_2)\\
&\qquad+L(u_3)(z,w)(\alpha_1,\alpha_2,\beta_1,\beta_2)\\
&\quad=\left|\beta_1-\left[\frac{\partial g_1}{\partial z_1}(z)\alpha_1+\frac{\partial g_1}{\partial z_2}(z)\alpha_2\right]\right|^2+\left|\beta_2\right|^2\\
&\qquad+\left|\frac{\partial g_2}{\partial z_1}(z)\alpha_1+\frac{\partial g_2}{\partial z_2}(z)\alpha_2\right|^2.
\end{aligned}$$

Case 1: $|g_1|^2+|g_2|^2$ (or equivalently $|g_1+\overline{g_2}|^2$) is not strictly psh on $\mathbb{C}^2$. Note that $|g_1|^2$ and $|g_2|^2$ are functions of class $C^\infty$ in $\mathbb{C}^2$. The Levi hermitian form (in $\mathbb{C}^2$) of $|g_1|^2$ is

$$\begin{aligned}
L\big(|g_1|^2\big)(z)(\delta_1,\delta_2)&=\sum_{j,k=1}^{2}\frac{\partial^2\big(|g_1|^2\big)}{\partial z_j\partial\overline{z_k}}\delta_j\overline{\delta_k}=\sum_{j,k=1}^{2}\frac{\partial g_1}{\partial z_j}(z)\frac{\partial\overline{g_1}}{\partial\overline{z_k}}(z)\delta_j\overline{\delta_k}\\
&=\left(\sum_{j=1}^{2}\frac{\partial g_1}{\partial z_j}(z)\delta_j\right)\left(\sum_{k=1}^{2}\overline{\frac{\partial g_1}{\partial z_k}(z)\delta_k}\right)=\left|\sum_{j=1}^{2}\frac{\partial g_1}{\partial z_j}(z)\delta_j\right|^2,
\end{aligned}$$

where $z=(z_1,z_2)\in\mathbb{C}^2$ and $(\delta_1,\delta_2)\in\mathbb{C}^2$. Therefore

$$\begin{aligned}
&L\big(|g_1|^2+|g_2|^2\big)(z_1,z_2)(\alpha_1,\alpha_2)\\
&\quad=\left|\frac{\partial g_1}{\partial z_1}(z)\alpha_1+\frac{\partial g_1}{\partial z_2}(z)\alpha_2\right|^2+\left|\frac{\partial g_2}{\partial z_1}(z)\alpha_1+\frac{\partial g_2}{\partial z_2}(z)\alpha_2\right|^2
\end{aligned}$$

for each $(\alpha_1,\alpha_2)\in\mathbb{C}^2$.

Now fix $z=(z_1,z_2)\in\mathbb{C}^2$. Since $|g_1|^2+|g_2|^2$ is not strictly psh in $\mathbb{C}^2$, then there exists $(\alpha_1,\alpha_2)\in\mathbb{C}^2\backslash\{0\}$ such that

$$\begin{cases}\frac{\partial g_1}{\partial z_1}(z)\alpha_1+\frac{\partial g_1}{\partial z_2}(z)\alpha_2=0\\[4pt] \frac{\partial g_2}{\partial z_1}(z)\alpha_1+\frac{\partial g_2}{\partial z_2}(z)\alpha_2=0.\end{cases}$$

Fix $w = (w_1, w_2) \in \mathbb{C}^2$. Take now $\beta_1 = \beta_2 = 0 \in \mathbb{C}$. Then we have $L(u)(z,w)(\alpha_1,\alpha_2,\beta_1,\beta_2) = 0$ but $(\alpha_1,\alpha_2,\beta_1,\beta_2) \in \mathbb{C}^4\backslash\{0\}$. Consequently, $u$ is not strictly psh in $\mathbb{C}^2 \times \mathbb{C}^2$.

CASE 2: $|g_1|^2 + |g_2|^2$ (or equivalently $|g_1 + \overline{g_2}|^2$) is strictly psh in $\mathbb{C}^2$. $L(u)(z,w)(\alpha_1,\alpha_2,\beta_1,\beta_2) = 0$ if and only if $\beta_1 = \frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2, \beta_2 = 0$ and $\frac{\partial g_2}{\partial z_1}(z)\alpha_1 + \frac{\partial g_2}{\partial z_2}(z)\alpha_2 = 0$.

Fix $(\alpha_1,\alpha_2) \in \mathbb{C}^2\backslash\{0\}$ such that $\frac{\partial g_2}{\partial z_1}(z)\alpha_1 + \frac{\partial g_2}{\partial z_2}(z)\alpha_2 = 0$. Define $\beta_1 \in \mathbb{C}$ by $\beta_1 = \frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2$ ($\beta_2 = 0$). Then $(\alpha_1,\alpha_2,\beta_1,\beta_2) \in \mathbb{C}^4\backslash\{0\}$ and we have $L(u)(z,w)(\alpha_1,\alpha_2,\beta_1,\beta_2) = 0$. Consequently, $u$ is not strictly psh on any domain $D \subset \mathbb{C}^4$.

Concerning the function $v$, we have $v$ is defined on $\mathbb{C}^4$ and of class $C^\infty$. The Levi hermitian form of $v$ is

$$\begin{aligned} &L(v)(z,w)(\alpha_1,\alpha_2,\beta_1,\beta_2) \\ &\quad = \left|\beta_1 - \left[\frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2\right]\right|^2 + \left|\beta_2 - \left[\frac{\partial g_2}{\partial z_1}(z)\alpha_1 + \frac{\partial g_2}{\partial z_2}(z)\alpha_2\right]\right|^2, \end{aligned}$$

where $z = (z_1,z_2) \in \mathbb{C}^2$, $w = (w_1,w_2) \in \mathbb{C}^2$ and $(\alpha_1,\alpha_2,\beta_1,\beta_2) \in \mathbb{C}^4$.

Fix $(z,w) \in \mathbb{C}^2 \times \mathbb{C}^2$. Let $(\alpha_1,\alpha_2) \in \mathbb{C}^2\backslash\{0\}$ such that $\frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2 = 0$. Put $\beta_1 = 0$, $\beta_2 = \left[\frac{\partial g_2}{\partial z_1}(z)\alpha_1 + \frac{\partial g_2}{\partial z_2}(z)\alpha_2\right]$. Then $(\alpha_1,\alpha_2,\beta_1,\beta_2) \in \mathbb{C}^4\backslash\{0\}$ and $L(v)(z,w)(\alpha_1,\alpha_2,\beta_1,\beta_2) = 0$. Consequently, $v$ is not strictly psh on any open of $\mathbb{C}^4$.

Now we have the decomposition

$$\begin{aligned} \varphi(z,\zeta) &= |\zeta - g_1(z)|^2 + |\zeta - \overline{g_2}(z)|^2 \\ &= |\zeta - g_1(z)|^2 + |\zeta|^2 + |g_2(z)|^2 - \zeta g_2(z) - \overline{\zeta g_2(z)}\,, \end{aligned}$$

for every $\zeta \in \mathbb{C}$ and $z = (z_1,z_2) \in \mathbb{C}^2$; $\varphi$ is a function of class $C^\infty$ in $\mathbb{C}^3$. Put $\varphi_1(z,\zeta) = \zeta g_2(z) + \overline{\zeta g_2(z)}$. Then $\varphi_1$ is pluriharmonic in $\mathbb{C}^3$ and consequently, the Levi hermitian form of this function is 0.

Let $\varphi_2(z,\zeta) = |\zeta - g_1(z)|^2$; $\varphi_2$ is a function of class $C^\infty$ in $\mathbb{C}^3$ and

$$L(\varphi_2)(z,\zeta)(\alpha_1,\alpha_2,\beta) = \left|\beta - \left[\frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2\right]\right|^2$$

for each $(z,\zeta) \in \mathbb{C}^2 \times \mathbb{C}$ and $(\alpha_1,\alpha_2,\beta) \in \mathbb{C}^3$.

Let $\varphi_3(z,\zeta) = |\zeta|^2 + |g_2(z)|^2$; $\varphi_3$ is a function of class $C^\infty$ in $\mathbb{C}^3$ and

$$L(\varphi_3)(z,\zeta)(\alpha_1,\alpha_2,\beta) = \left|\beta\right|^2 + \left|\frac{\partial g_2}{\partial z_1}(z)\alpha_1 + \frac{\partial g_2}{\partial z_2}(z)\alpha_2\right|^2$$

for every $(z, \zeta) \in \mathbb{C}^2 \times \mathbb{C}$ and $(\alpha_1, \alpha_2, \beta) \in \mathbb{C}^3$.

It follows that

$$
\begin{aligned}
L(\varphi)(z,\zeta)(\alpha_1, \alpha_2, \beta)& \\
&= L(\varphi_2)(z,\zeta)(\alpha_1, \alpha_2, \beta) + L(\varphi_3)(z,\zeta)(\alpha_1, \alpha_2, \beta) \\
&= \left| \beta - \left[ \frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2 \right] \right|^2 + |\beta|^2 + \left| \frac{\partial g_2}{\partial z_1}(z)\alpha_1 + \frac{\partial g_2}{\partial z_2}(z)\alpha_2 \right|^2 .
\end{aligned}
$$

Therefore $L(\varphi)(z,\zeta)(\alpha_1, \alpha_2, \beta) = 0$ if and only if $\beta = 0$, $\frac{\partial g_2}{\partial z_1}(z)\alpha_1 + \frac{\partial g_2}{\partial z_2}(z)\alpha_2 = 0$ and $\frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2 = 0$. Observe now that $\varphi$ is strictly psh in $\mathbb{C}^3$ if and only if $|g_1|^2 + |g_2|^2$ is strictly psh in $\mathbb{C}^2$. ∎

Corollary 2.28. *Let $g_1, g_2 : \mathbb{C}^2 \to \mathbb{C}$ be two pluriharmonic functions. Put $g_1 = f_1 + \overline{k_1}$, where $g_2 = f_2 + \overline{k_2}$, $f_1, f_2, k_1, k_2 : \mathbb{C}^2 \to \mathbb{C}$ be four analytic functions. Let*

$$
\begin{aligned}
u(z, w_1, w_2) &= |w_1 - g_1(z)|^2 + |w_2 - g_2(z)|^2 , \\
v(z, w_1, w_2) &= \left|w_1 - \overline{k_1}(z)\right|^2 + \left|w_2 - \overline{k_2}(z)\right|^2,
\end{aligned}
$$

*where $(z, w_1, w_2) \in \mathbb{C}^2 \times \mathbb{C} \times \mathbb{C}$. The following conditions are equivalent*

(a) *$u$ is strictly psh in $\mathbb{C}^4$;*

(b) *$v$ is strictly psh in $\mathbb{C}^4$.*

*That is the strict plurisubharmonicity of $u$ is independent of the choice of the analytic functions $f_1$ and $f_2$.*

Corollary 2.29. *Let $g_j, k_j : D \to \mathbb{C}$ be analytic functions, where $1 \leq j \leq N$ and $D$ is a domain of $\mathbb{C}^n$, $n, N \geq 1$. Put*

$$
u = \sum_{j=1}^{N} \left|g_j + \overline{k_j}\right|^2 \qquad \text{and} \qquad v = \sum_{j=1}^{N} \left|g_j\right|^2 + \sum_{j=1}^{N} \left|k_j\right|^2 .
$$

*Then $u$ is strictly psh in $D$ if and only if $v$ is strictly psh in $D$.*

Corollary 2.30. *Let $g_1, \dots, g_N : D \to \mathbb{C}$ be $N$ analytic functions, where $N \geq 1$ and $D$ is a domain of $\mathbb{C}^n$ ($n \geq 1$). Set $u(z, w) = \sum_{j=1}^{N} |w - g_j(z)|^2$, where $(z, w) \in D \times \mathbb{C}$. If $N \leq n$, then $u$ is not strictly psh on any domain of $D \times \mathbb{C}$.*

*Proof.* Fix $z = (z_1, \dots, z_n) \in D$ and $w \in \mathbb{C}$. Let $u_j(z,w) = |w - g_j(z)|^2$, $1 \leq j \leq N$. Then $u_j$ is a function of class $C^\infty$ in $D \times \mathbb{C}$. If now $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ and $\beta \in \mathbb{C}$, we have the Levi hermitian form of $u_j$ is

$$L(u_j)(z,w)(\alpha_1, \dots, \alpha_n, \beta) = \left| \beta - \sum_{s=1}^{n} \frac{\partial g_j}{\partial z_s}(z)\alpha_s \right|^2.$$

Therefore, the Levi form of $u$ is

$$\begin{aligned} L(u)(z,w)(\alpha_1, \dots, \alpha_n, \beta) &= \sum_{j=1}^{N} L(u_j)(z,w)(\alpha_1, \dots, \alpha_n, \beta) \\ &= \sum_{j=1}^{N} \left| \beta - \sum_{s=1}^{n} \frac{\partial g_j}{\partial z_s}(z)\alpha_s \right|^2. \end{aligned}$$

Let $v = [|g_1|^2 + \cdots + |g_N|^2]$; $v$ is a function of class $C^\infty$ on $D$.

CASE 1: $v$ is strictly psh on $D$. We have $L(v)(z)(\alpha_1, \dots, \alpha_n) = 0$ imply that $(\alpha_1, \dots, \alpha_n) = (0, \dots, 0)$. The Levi form of $v$ is

$$L(v)(z)(\alpha_1, \dots, \alpha_n) = \left| \sum_{s=1}^{n} \frac{\partial g_1}{\partial z_s}(z)\alpha_s \right|^2 + \cdots + \left| \sum_{s=1}^{n} \frac{\partial g_N}{\partial z_s}(z)\alpha_s \right|^2.$$

Since $L(v)(z)(\alpha_1, \dots, \alpha_n) = 0$ then $(\alpha_1, \dots, \alpha_n) = 0$. Thus the system of equations in $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ satisfies

$$\begin{cases} \frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \cdots + \frac{\partial g_1}{\partial z_n}(z)\alpha_n = 0 \\ \qquad \vdots \\ \frac{\partial g_N}{\partial z_1}(z)\alpha_1 + \cdots + \frac{\partial g_N}{\partial z_n}(z)\alpha_n = 0 \end{cases}$$

if and only if $(\alpha_1, \dots, \alpha_n) = (0, \dots, 0)$. Since $N \leq n$, then $N = n$. Thus the matrix $(\lambda_{jk})_{1 \leq j,k \leq n}$ is invertible; where $\lambda_{jk} = \frac{\partial g_j}{\partial z_k}(z)$. Now we have $L(u)(z,w)(\alpha_1, \dots, \alpha_n, \beta) = 0$ if and only if

$$\begin{cases} \frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \cdots + \frac{\partial g_1}{\partial z_n}(z)\alpha_n = \beta \\ \qquad \vdots \\ \frac{\partial g_n}{\partial z_1}(z)\alpha_1 + \cdots + \frac{\partial g_n}{\partial z_n}(z)\alpha_n = \beta. \end{cases}$$

Fix $\beta \in \mathbb{C}\backslash\{0\}$; the above system has a unique solution $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n\backslash\{0\}$. Consequently, $(\alpha_1, \dots, \alpha_n, \beta) \in \mathbb{C}^{n+1}\backslash\{0\}$ and $L(u)(z,w)(\alpha_1, \dots, \alpha_n, \beta) = 0$.

CASE 2: $v$ is not strictly psh on $D$. Then there exists $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n\backslash\{0\}$ such that $L(v)(z)(\alpha_1, \dots, \alpha_n) = 0$. Take $\beta = 0 \in \mathbb{C}$. Then

$$(\alpha_1, \dots, \alpha_n, \beta) \in \mathbb{C}^{n+1}\backslash\{0\} \qquad \text{and} \qquad L(u)(z,w)(\alpha_1, \dots, \alpha_n, \beta) = 0\,.$$

Consequently, $u$ is not strictly psh in $D \times \mathbb{C}$. ∎

EXAMPLE. Let $(z_1, z_2) \in \mathbb{C}^2$ and $w \in \mathbb{C}$. Put $g_1(z) = z_1$, $g_2(z) = z_2$, $g_3(z) = z_1 + z_2$; $g_1, g_2, g_3$ are analytic functions in $\mathbb{C}^2$. Put $u(z,w) = \sum_{j=1}^{3} |w - g_j(z)|^2$. Then $u$ is a function of class $C^\infty$ and strictly psh in $\mathbb{C}^2 \times \mathbb{C}$. If $(w_1, w_2, w_3) \in \mathbb{C}^3$, we put $v(z_1, z_2, w_1, w_2, w_3) = \sum_{j=1}^{3} A_j |w_j - g_j(z)|^2$, where $(A_1, A_2, A_3 \in \mathbb{R}_+\backslash\{0\})$. Then $v$ is not strictly psh on any domain of $\mathbb{C}^2 \times \mathbb{C}^3$.

In fact $v$ is a function of class $C^\infty$ in $\mathbb{C}^2 \times \mathbb{C}^3$ and the Levi form of $v$ is

$$\begin{aligned} L(v)(z, w_1, w_2, w_3)(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) &= A_1 \left| \beta_1 - \left[ \frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2 \right] \right|^2 \\ + A_2 \left| \beta_2 - \left[ \frac{\partial g_2}{\partial z_1}(z)\alpha_1 + \frac{\partial g_2}{\partial z_2}(z)\alpha_2 \right] \right|^2 &+ A_3 \left| \beta_3 - \left[ \frac{\partial g_3}{\partial z_1}(z)\alpha_1 + \frac{\partial g_3}{\partial z_2}(z)\alpha_2 \right] \right|^2, \end{aligned}$$

for every $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) \in \mathbb{C}^5$. Let $(\alpha_1, \alpha_2) \in \mathbb{C}^2\backslash\{0\}$ such that $\frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \frac{\partial g_1}{\partial z_2}(z)\alpha_2 = 0$. Put $\beta_1 = 0$, $\beta_2 = \left[\frac{\partial g_2}{\partial z_1}(z)\alpha_1 + \frac{\partial g_2}{\partial z_2}(z)\alpha_2\right]$, $\beta_3 = \left[\frac{\partial g_3}{\partial z_1}(z)\alpha_1 + \frac{\partial g_3}{\partial z_2}(z)\alpha_2\right]$. Then $(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) \in \mathbb{C}^5\backslash\{0\}$ and

$$L(v)(z, w_1, w_2, w_3)(\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3) = 0\,.$$

Therefore $v$ is not strictly psh on any not empty open of $\mathbb{C}^2 \times \mathbb{C}^3$.

Observe that here in fact we have for all $k_1, \dots, k_N : \mathbb{C}^n \to \mathbb{C}$ analytic functions, where $n, N \in \mathbb{N}$, if $N \geq n$, then $v_1$ is not strictly psh in any domain of $\mathbb{C}^n \times \mathbb{C}^N$, where

$$v_1(z, w_1, \dots, w_N) = \sum_{j=1}^{N} B_j |w_j - k_j(z)|^2 \qquad \big(B_1, \dots, B_N \in \mathbb{R}_+\backslash\{0\}\big)\,,$$

$z \in \mathbb{C}^n$ and $(w_1, \dots, w_N) \in \mathbb{C}^N$.

A fundamental application concerning analytic functions and the complex structure is now the following extension.

THEOREM 2.31. *Let $g_1, \dots, g_N : D \to \mathbb{C}$ be $N$ analytic functions, $D$ is a domain of $\mathbb{C}^n$, $(n \geq 1)$ and $(N \geq 1)$. Put*

$$u(z,w) = |w - g_1(z)|^2 + \cdots + |w - g_N(z)|^2,$$
$$v(z,w) = |w - \overline{g_1}(z)|^2 + \cdots + |w - \overline{g_N}(z)|^2,$$
$$v_1(z,w) = |w - h_1(z)|^2 + \cdots + |w - h_N(z)|^2,$$

*where $(z,w) \in D \times \mathbb{C}$, and $h_j = \mathrm{Re}(g_j)$, for $1 \leq j \leq N$.*

(a) *Suppose that $u$ is strictly psh in $D \times \mathbb{C}$. Then $v$ and $v_1$ are strictly psh in $D \times \mathbb{C}$ and $N \geq n+1$, but the converse is false.*

(b) *In fact, $v$ is strictly psh in $D \times \mathbb{C}$ if and only if $v_1$ is strictly psh in $D \times \mathbb{C}$.*

For the proof of this theorem, we use Lemma 2.2.

EXAMPLE. Let $g_1(z) = z$, $g_2(z) = z^2$, where $z \in \mathbb{C}$. $u_1(z,w) = |w-z|^2 + |w-z^2|^2$, $u_2(z,w) = |w - \overline{z}|^2 + |w - \overline{z^2}|^2$, for $(z,w) \in \mathbb{C}^2$; $u_1$ is not strictly psh on any domain of the form $D\big(\frac{1}{2}, r\big) \times \mathbb{C}$ (for every $r > 0$); $u_2$ is strictly psh on $\mathbb{C}^2$.

On the other hand, the minimal number $N$ of analytic functions $k_1, \dots, k_N : \mathbb{C}^n \to \mathbb{C}$ $(n \geq 1)$ such that if $u_1(z,w) = \big|w - \overline{k_1}(z)\big|^2 + \cdots + \big|w - \overline{k_N}(z)\big|^2$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ is in fact $N = n$. But for all $\varphi_1, \dots, \varphi_N : \mathbb{C}^n \to \mathbb{C}$ be $N$ analytic functions, $u_2(z,w) = |w - \varphi_1(z)|^2 + \cdots + |w - \varphi_N(z)|^2$ satisfies $u_2$ is not strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if $N \leq n$.

Now there are a great differences between the class of functions defined analogues to $u_1$ and the class of functions defined similar of $u_2$.

Now we are in position to prove the following result.

THEOREM 2.32. *Let $g_1, \dots, g_n : D \to \mathbb{C}$, $D$ is a domain of $\mathbb{C}^n$, $n \in \mathbb{N}$. Set $v(z,w) = |w_1 - g_1(z)|^2 + \cdots + |w_n - g_n(z)|^2$, where $(z,w) \in D \times \mathbb{C}^n$, $w = (w_1, \dots, w_n)$. The following conditions are equivalent:*

(a) *$v$ is strictly psh in $D \times \mathbb{C}^n$;*

(b) *$g_1, \dots, g_n$ are prh functions in $D$ and for all $z = (z_1, \dots, z_n) \in D$ (fixed), the system*

$$\begin{cases} \frac{\partial g_1}{\partial z_1}(z)\alpha_1 + \cdots + \frac{\partial g_1}{\partial z_n}(z)\alpha_n = 0 \\ \vdots \\ \frac{\partial g_n}{\partial z_1}(z)\alpha_1 + \cdots + \frac{\partial g_n}{\partial z_n}(z)\alpha_n = 0 \end{cases}$$

*has only the solution $(\alpha_1, \dots, \alpha_n) = (0, \dots, 0)$. That is strictly plurisubharmonic functions and partial differential equations have a rigid relation to discover here for example.*

QUESTION 2.33. Let $g_1 : \mathbb{C}^2 \to \mathbb{C}$ be a prh function. Find a condition satisfied by $g_1$ such that there exists $g_2 : \mathbb{C}^2 \to \mathbb{C}$ prh and satisfying $u$ is strictly psh on $\mathbb{C}^2 \times \mathbb{C}^2$, where $u(z, w) = A_1|w_1 - g_1(z)|^2 + A_2|w_2 - g_2(z)|^2$, $z \in \mathbb{C}^2$, $w = (w_1, w_2) \in \mathbb{C}^2$ and $A_1, A_2 \in \mathbb{R}_+\backslash\{0\}$. In general this problem have no solution and an affirmative answer is given by the following result.

PROPOSITION 2.34. *Let $g : \mathbb{C}^2 \to \mathbb{C}$, $g(z_1, z_2) = k_1(z_1)k_2(z_2)$, where $(z_1, z_2) \in \mathbb{C}^2$, $k_1, k_2 : \mathbb{C} \to \mathbb{C}$ be two analytic not constant functions, $k_1(0) = k_2(0) = 0$. For all $A_1, A_2 \in \mathbb{R}_+\backslash\{0\}$, there does not exists a function $k : \mathbb{C}^2 \to \mathbb{C}$ be analytic such that $v = A_1|g|^2 + A_2|k|^2$ is strictly psh on $\mathbb{C}^2$.*

*Proof.* Let $k : \mathbb{C}^2 \to \mathbb{C}$ be a analytic function. Put $v = A_1|g|^2 + A_2|k|^2$. $v$, $|g|^2$ and $|k|^2$ are functions smooth of class $C^\infty$ in $\mathbb{C}^2$. The Levi hermitian form of $|g|^2$ is

$$L(|g|^2)(z_1, z_2)(\alpha_1, \alpha_2) = \left| \frac{\partial g}{\partial z_1}(z_1, z_2)\alpha_1 + \frac{\partial g}{\partial z_2}(z_1, z_2)\alpha_2 \right|^2$$

for each $z = (z_1, z_2)$ and $(\alpha_1, \alpha_2) \in \mathbb{C}^2$. Therefore, the Levi hermitian form of $v$ is

$$\begin{aligned} L(v)(z_1, z_2)(\alpha_1, \alpha_2) = A_1 &\left| k_1'(z_1)k_2(z_2)\alpha_1 + k_1(z_1)k_2'(z_2)\alpha_2 \right|^2 \\ &+ A_2 \left| \frac{\partial k}{\partial z_1}(z)\alpha_1 + \frac{\partial k}{\partial z_2}(z)\alpha_2 \right|^2. \end{aligned}$$

Take $z_1 = z_2 = 0$. $L(v)(0,0)(\alpha_1, \alpha_2) = A_2 \left| \frac{\partial k}{\partial z_1}(0)\alpha_1 + \frac{\partial k}{\partial z_2}(0)\alpha_2 \right|^2$. Now take $(\alpha_1, \alpha_2) \in \mathbb{C}^2\backslash\{0\}$ such that $\frac{\partial k}{\partial z_1}(0)\alpha_1 + \frac{\partial k}{\partial z_2}(0)\alpha_2 = 0$. Then $L(v)(0,0)(\alpha_1, \alpha_2) = 0$. Consequently, $v$ is not strictly psh on $\mathbb{C}^2$. ∎

It follows that, for all $k : \mathbb{C}^2 \to \mathbb{C}$ be analytic, for all $A_1, A_2 \in \mathbb{R}_+\backslash\{0\}$, if $u_1(z,w) = A_1|w_1 - g(z)|^2 + A_2|w_2 - k(z)|^2$, where $z = (z_1, z_2) \in \mathbb{C}^2, w = (w_1, w_2) \in \mathbb{C}^2$. Then $u_1$ is not strictly psh on $\mathbb{C}^2 \times \mathbb{C}^2$.

Consequently, the above question globally has a negative answer. But locally we have a positive answer. Because in fact, by using all the notation of the question 2.33, we have if $g_2$ exists, then $|g_1|^2 + |g_2|^2$ is strictly psh on $\mathbb{C}^2$. By the above proposition, there exists a function $g : \mathbb{C}^2 \to \mathbb{C}$ be analytic such that $A_1|g|^2 + A_2|k|^2$ is not strictly psh on $\mathbb{C}^2$, for any $k : \mathbb{C}^2 \to \mathbb{C}$ be analytic, for every $A_1, A_2 \in \mathbb{R}_+\backslash\{0\}$.

Now locally, if $z^0 = (z_1^0, z_2^0) \in \mathbb{C}^2$ we can write $g_1 = f_1 + \overline{k_1}$, $g_2 = f_2 + \overline{k_2}$, where $f_1, f_2, k_1, k_2 : \mathbb{C}^2 \to \mathbb{C}$ be 4 analytic functions. In fact we can prove that, the functions $f_1$ and $f_2$ do not have any role on the subject of the strict plurisubharmonicity of $u$; $u$ is a function of class $C^\infty$ in $\mathbb{C}^2 \times \mathbb{C}^2$. The Levi hermitian form of $u$ is

$$
\begin{aligned}
&L(u)\big(z^0, w_1, w_2\big)(\alpha_1, \alpha_2, \beta_1, \beta_2)\\
&\quad= A_1\left|\beta_1 - \left[\frac{\partial f_1}{\partial z_1}\big(z^0\big)\alpha_1 + \frac{\partial f_1}{\partial z_2}\big(z^0\big)\alpha_2\right]\right|^2 + A_1\left|\frac{\partial k_1}{\partial z_1}\big(z^0\big)\alpha_1 + \frac{\partial k_1}{\partial z_2}\big(z^0\big)\alpha_2\right|^2\\
&\qquad+ A_2\left|\beta_2 - \left[\frac{\partial f_2}{\partial z_1}\big(z^0\big)\alpha_1 + \frac{\partial f_2}{\partial z_2}\big(z^0\big)\alpha_2\right]\right|^2 + A_2\left|\frac{\partial k_2}{\partial z_1}\big(z^0\big)\alpha_1 + \frac{\partial k_2}{\partial z_2}\big(z^0\big)\alpha_2\right|^2,
\end{aligned}
$$

where $(w_1, w_2) \in \mathbb{C}^2$, $(\alpha_1, \alpha_2, \beta_1, \beta_2) \in \mathbb{C}^4$. If $u$ is strictly psh on a neighborhood $G$ of $(z^0, w_0), w_0 \in \mathbb{C}^2$, then $L(u)(z,w)(\alpha_1, \alpha_2, \beta_1, \beta_2) = 0$ implies that $(\alpha_1, \alpha_2, \beta_1, \beta_2) = 0$, for every $(z,w) \in G = G_1 \times G_2$, $G_1$ and $G_2$ are convex domains of $\mathbb{C}^2$, where $z^0 \in G_1$, $w_0 \in G_2$. But $L(u)(z,w)(\alpha_1, \alpha_2, \beta_1, \beta_2) = 0$ has only the solution $(\alpha_1, \alpha_2, \beta_1, \beta_2) = 0$ (for every $(z,w) \in G$), if and only if the system

$$
\begin{cases}
\frac{\partial k_1}{\partial z_1}(z)\alpha_1 + \frac{\partial k_1}{\partial z_2}(z)\alpha_2 = 0\\
\frac{\partial k_2}{\partial z_1}(z)\alpha_1 + \frac{\partial k_2}{\partial z_2}(z)\alpha_2 = 0
\end{cases}
$$

(where $z$ is fixed on $G_1$ and $(\alpha_1, \alpha_2)$ is the variable in $\mathbb{C}^2$) has only the solution $(\alpha_1, \alpha_2) = (0,0)$.

Observe that this condition is independent of $w_0 \in \mathbb{C}^2$. Therefore if $\left(\frac{\partial k_1}{\partial z_1}(z^0), \frac{\partial k_1}{\partial z_2}(z^0)\right) \neq (0,0)$, there exists a ball $B(z^0, r) \subset \mathbb{C}^2$ $(r > 0)$ such that $\left(\frac{\partial k_1}{\partial z_1}(z), \frac{\partial k_1}{\partial z_2}(z)\right) \neq (0,0)$ for each $z \in B(z^0, r)$. Suppose for example that $\frac{\partial k_1}{\partial z_1}(z) \neq 0$, for every $z \in B(z^0, t)$, where $0 < t < r$. Let $k_2(z_1, z_2) = z_2$, where $(z_1, z_2) \in \mathbb{C}^2$; $k_2$ is analytic on $\mathbb{C}^2$. Put $g_2 = \overline{k_2}$; $g_2$ is pluriharmonic on

$\mathbb{C}^2$. We have $\frac{\partial k_2}{\partial z_1}(z) = 0$, $\frac{\partial k_2}{\partial z_2}(z) = 1$. The above system has only the solution $(\alpha_1, \alpha_2) = (0, 0)$. Then $u$ is strictly psh on $B(z^0, t) \times \mathbb{C}^2$.

PROPOSITION 2.35. *Let $g_1, g_2 : \mathbb{C}^2 \to \mathbb{C}$. Put $u(z, w_1, w_2) = A_1|w_1 - g_1(z)|^2 + A_2|w_2 - g_2(z)|^2$, where $z \in \mathbb{C}^2$, $(w_1, w_2) \in \mathbb{C}^2$, $A_1, A_2 \in \mathbb{R}_+\backslash\{0\}$; $u_1(z, w_1) = |w_1 - g_1(z)|^2 + |g_2(z)|^2$, $u_2(z, w_2) = |w_2 - g_2(z)|^2 + |g_1(z)|^2$. The following conditions are equivalent:*

(a) *$u$ is strictly psh on $\mathbb{C}^2 \times \mathbb{C}^2$;*

(b) *$u_1$ and $u_2$ are strictly psh functions on $\mathbb{C}^2 \times \mathbb{C}$;*

(c) *$g_1$ and $g_2$ are prh functions over $\mathbb{C}^2$, $g_1 = f_1 + \overline{k_1}$, $g_2 = f_2 + \overline{k_2}$ ($f_1, k_1, f_2, k_2 : \mathbb{C}^2 \to \mathbb{C}$ be analytic) and the antiholomorphic parts of $g_1$ and $g_2$ satisfies $|k_1|^2 + |k_2|^2$ is stictly psh on $\mathbb{C}^2$.*

*Moreover observe that if the holomorphic parts of $g_1$ and $g_2$ satisfies $|f_1|^2+|f_2|^2$ is strictly psh on $\mathbb{C}^2$ (therefore here $|g_1|^2 + |g_2|^2$ is strictly psh on $\mathbb{C}^2$) but we can not conclude that $u$ is strictly psh on $\mathbb{C}^2$.*

EXAMPLE. Let $g_1 : \mathbb{C}^2 \to \mathbb{C}$ be a prh function and let $N \in \mathbb{N}$, $N \geq 2$. Prove that there exists $g_2, \dots, g_{N+1} : \mathbb{C}^2 \to \mathbb{C}$ be $N$ prh functions such that if $u(z, w_1, w_2, \dots, w_{N+1}) = \sum_{j=1}^{N+1} |w_j - g_j(z)|^2$, where $z \in \mathbb{C}^2$, then $u$ is strictly psh on $\mathbb{C}^2 \times \mathbb{C}^{N+1}$. In fact, the answer is very simple, if we consider the family of prh functions $g_2(z) = \overline{z_1}$, $g_3(z) = \overline{z_2}$, $g_4(z) = \cdots = g_{N+1}(z) = 0$, where $z = (z_1, z_2) \in \mathbb{C}^2$. We have in this case $|w_2 - g_2|^2 + |w_3 - g_3|^2$ is strictly psh in $\mathbb{C}^2 \times \mathbb{C}^2$. Then $u$ is strictly psh in $\mathbb{C}^2 \times \mathbb{C}^{N+1}$.

## 3. CONVEX AND STRICTLY PLURISUBHARMONIC FUNCTIONS

We consider in this section a classical family of psh functions, that is the class of convex and strictly psh functions.

THEOREM 3.1. *Let $g_1, g_2 : \mathbb{C} \to \mathbb{C}$ be two analytic functions. Assume that*

$$u(z, w, w_1, w_2) = A|w_1 - g_1(w - \overline{z})|^2 + B|w_2 - g_2(w - \overline{z})|^2,$$
$$v(z, w_1, w_2) = A|w_1 - g_1(z)|^2 + B|w_2 - g_2(z)|^2,$$

*where $(z, w, w_1, w_2) \in \mathbb{C}^4$, $A, B \in \mathbb{R}_+\backslash\{0\}$. The following statements are equivalent:*

(a) *$u$ is psh on $\mathbb{C}^4$;*

(b) *$g_1$ and $g_2$ are analytic affine functions;*

(c) *$v$ is convex on $\mathbb{C}^3$.*

*Proof.* (a) $\Rightarrow$ (b) Fix $w_1, w_2 \in \mathbb{C}$. Put $u_1(z,w) = |w_1 - g_1(w - \overline{z})|^2 + |w_2 - g_2(w - \overline{z})|^2$, where $(z,w) \in \mathbb{C}^2$. Since $u_1$ is of class $C^\infty$ and psh on $\mathbb{C}^2$, then the Levi hermitian form of $u_1$ is

$$L(u_1)(z,w)(\alpha,\beta) = \big(|g_1'(z)|^2 + |g_2'(z)|^2\big)\alpha\overline{\alpha} + \big(|g_1'(z)|^2 + |g_2'(z)|^2\big)\beta\overline{\beta} \\ + 2\operatorname{Re}\Big(\big[(w_1 - g_1(z))g_1''(z) + (w_2 - g_2(z))g_2''(z)\big]\alpha\overline{\beta}\Big) \geq 0\,,$$

for all $(\alpha,\beta) \in \mathbb{C}^2$. Thus

$$\big|(w_1 - g_1(z))g_1''(z) + (w_2 - g_2(z))g_2''(z)\big| \leq |g_1'(z)|^2 + |g_2'(z)|^2,$$

for all $z \in \mathbb{C}$ and all $(w_1, w_2) \in \mathbb{C}^2$.

Now fix $z \in \mathbb{C}$. If $g_1''(z) \neq 0$. Fix $w_2 = g_2(z) \in \mathbb{C}$. Then $|(w_1 - g_1(z))g_1''(z)| \leq |g_1'(z)|^2 + |g_2'(z)|^2$, for any $w_1 \in \mathbb{C}$. It follows that $\mathbb{C}$ is bounded. A contradiction. Consequently, $g_1'' = 0$, $g_2'' = 0$ over $\mathbb{C}$. Therefore $g_1$ and $g_2$ are analytic affine functions over $\mathbb{C}$. ∎

Comparison theorems. We prove in this context that there exists an infinite number $F_1$ of $C^\infty$ functions defined on $\mathbb{C}^2$, such that for each $F \in F_1$, the function $F$ satisfy $F$ has a fixed type, $F$ is convex and strictly psh on $\mathbb{C}^2$, but $F$ is not strictly convex on $\mathbb{C}^2$. Denote by $< .,. >$ the habitual hermitian product over $\mathbb{C}^n$ in all of this section.

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a analytic function. Set $u(z,w) = |w - f(z)|^2$, $v(z,w) = |w - \overline{f}(z)|^2$, $u_1(z,w) = A_1|w - (< z, a > + b)|^2 + A_2|w - (\overline{< z, a >} + \overline{b})|^2$, where $(z,w) \in \mathbb{C}^n \times \mathbb{C}$, $a \in \mathbb{C}^n$, $b \in \mathbb{C}$, $A_1, A_2 \in \mathbb{R} + \backslash\{0\}$. We study now the structure of the functions $u$, $v$ and $u_1$. We have the following 3 assertions:

(a) $u$ is psh in $\mathbb{C}^n \times \mathbb{C}$, but $u$ is not strictly psh on any domain of $\mathbb{C}^n \times \mathbb{C}$.

(b) $v$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}$ if and only if $n = 1$ and $|f'| > 0$ over $\mathbb{C}$. But $v$ is not strictly convex in all not empty convex domain of $\mathbb{C}^n \times \mathbb{C}$ for every $n \geq 1$ and for any $f : \mathbb{C}^n \to \mathbb{C}$ be analytic.

(c) $u_1$ is not strictly convex in all not empty euclidean open ball subset of $\mathbb{C}^n \times \mathbb{C}$, for $A_1, A_2 \in \mathbb{R}_+\backslash\{0\}$ and $(a,b) \in \mathbb{C}^n \times \mathbb{C}$.

But if we consider $u_2(z,w) = |w - f(z)|^2 + |w - \overline{f}(z)|^2 + |w - g(z)|^2$, where $g : \mathbb{C} \to \mathbb{C}$ be analytic, $n = 1$, $(z,w) \in \mathbb{C}^2$, we have the following result.

PROPOSITION 3.2. *$u_2$ is strictly convex in $\mathbb{C}^2$ if and only if $f$ and $g$ are analytic affine functions, $f(z) = a_1 z + b_1$, $g(z) = a_2 z + b_2$, for $z \in \mathbb{C}$, where $a_1, a_2, b_1, b_2 \in \mathbb{C}$ such that $((a_1, a_2 \in \mathbb{C}\backslash\{0\}$ and $\frac{a_2}{a_1} \neq 1)$ or $(a_1 = 0,\ a_2 \neq 0)$ or $(a_1 \neq 0,\ a_2 = 0))$.*

*Proof.* Suppose that $u_2$ is strictly convex in $\mathbb{C}^2$. Recall that if $\varphi : \mathbb{C}^m \to \mathbb{R}$ be a function of class $C^2$ $(m \geq 1)$, then $\varphi$ is strictly convex in $\mathbb{C}^m$ if and only if

$$\left| \sum_{j,k=1}^{m} \frac{\partial^2 \varphi}{\partial z_j \partial z_k}(z)\alpha_j\alpha_k \right| < \sum_{j,k=1}^{m} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z_k}}(z)\alpha_j\overline{\alpha_k}$$

for each $z \in \mathbb{C}^m$ and all $(\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m\backslash\{0\}$. We have

$$\begin{aligned} u_2(z,w) = w\overline{w} + f(z)\overline{f}(z) - w\overline{f}(z) - \overline{w}f(z) + w\overline{w} + f(z)\overline{f}(z) \\ - wf(z) - \overline{w}\overline{f}(z) + w\overline{w} + g(z)\overline{g}(z) - w\overline{g}(z) - \overline{w}g(z)\,, \end{aligned}$$

where $(z,w) \in \mathbb{C}^2$. Let $(\alpha, \beta) \in \mathbb{C}^2$; $u_2$ is strictly convex in $\mathbb{C}^2$, then $u_2$ is convex in all $\mathbb{C}^2$. Now since $u_2$ is of class $C^\infty$ in $\mathbb{C}^2$, then we have

$$\begin{aligned} \big|\big[f''\overline{f} - \overline{w}f'' + f''\overline{f} - wf'' + g''\overline{g} - \overline{w}g''\big]\alpha^2 - 2\alpha\beta f'\big| \\ \leq |\beta - f'\alpha|^2 + |\beta|^2 + |f'\alpha|^2 + |\beta - g'\alpha|^2 \end{aligned}$$

is valid over $\mathbb{C}$ for each $(\alpha, \beta) \in \mathbb{C}^2$ and $w \in \mathbb{C}$. If $w \in \mathbb{R}$, then $|w(2f''(z) + g''(z)) + \varphi(z)| \leq \varphi_1(z)$, where $\varphi : \mathbb{C} \to \mathbb{C}$ and $\varphi_1 : \mathbb{C} \to \mathbb{R}_+$ be two functions. The condition $2f'' + g'' \neq 0$, imply that $\mathbb{R}$ is bounded, which is a contradiction. Thus $2f'' + g'' = 0$ over $\mathbb{C}$.

Now put $w = it$, where $t \in \mathbb{R}$. Therefore for each $t \in \mathbb{R}$, $|tg'' + \theta| \leq \theta_1$, where $\theta : \mathbb{C} \to \mathbb{C}$ and $\theta_1 : \mathbb{C} \to \mathbb{R}_+$ be two functions. Then $g'' = 0$ in $\mathbb{C}$. It follows that $f'' = 0$. Consequently, $f$ and $g$ are analytic affine functions over $\mathbb{C}$; $f(z) = a_1 z + b_1$, $g(z) = a_2 z + b_2$ for $z \in \mathbb{C}$, where $a_1, a_2, b_1, b_2 \in \mathbb{C}$.

CASE 1: $a_1 = 0$. In this situation $u_2(z,w) = |w - b_1|^2 + |w - \overline{b_1}|^2 + |w - g_2(z)|^2$; $u_2$ is a smooth function over $\mathbb{C}^2$. Let $(z,w) \in \mathbb{C}^2$ and $(\alpha, \beta) \in \mathbb{C}^2\backslash\{0\}$. We have

$$\left| \frac{\partial^2 u_2}{\partial z^2}(z,w)\alpha^2 + \frac{\partial^2 u_2}{\partial w^2}(z,w)\beta^2 + 2\frac{\partial^2 u_2}{\partial z \partial w}(z,w)\alpha\beta \right| = 0\,,$$

$$\frac{\partial^2 u_2}{\partial z \partial \overline{z}}(z,w)|\alpha^2| + \frac{\partial^2 u_2}{\partial w \partial \overline{w}}(z,w)|\beta|^2 + 2\,\mathrm{Re}\left[\frac{\partial^2 u_2}{\partial z \partial \overline{w}}(z,w)\alpha\overline{\beta}\right]$$
$$= |\beta - a_2\alpha|^2 + 2|\beta|^2,$$

and then

$$0 < |\beta - a_2\alpha|^2 + 2|\beta|^2$$

for each $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{0\}$. If $\beta = 0$, then $\alpha \neq 0$ and $0 < |a_2\alpha|^2$. It follows that $a_2 \neq 0$. In this case we have

$$2|\beta|^2 + |\beta - a_2\alpha|^2 > 0$$

for each $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{0\}$.

Case 2: $a_2 = 0$. In this situation $u_2(z,w) = |w - f(z)|^2 + |w - \overline{f}(z)|^2 + |w - b_2|^2$. Let $(z,w) \in \mathbb{C}^2$ and $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{0\}$; $u_2$ is a function of class $C^\infty$ in $\mathbb{C}^2$. We have

$$\left|\frac{\partial^2 u_2}{\partial z^2}(z,w)\alpha^2 + \frac{\partial^2 u_2}{\partial w^2}(z,w)\beta^2 + 2\frac{\partial^2 u_2}{\partial z \partial w}(z,w)\alpha\beta\right| = |-2\alpha\beta f'(z)|\,,$$

$$\frac{\partial^2 u_2}{\partial z \partial \overline{z}}(z,w)|\alpha^2| + \frac{\partial^2 u_2}{\partial w \partial \overline{w}}(z,w)|\beta|^2 + 2\,\mathrm{Re}\left[\frac{\partial^2 u_2}{\partial z \partial \overline{w}}(z,w)\alpha\overline{\beta}\right]$$
$$= |\beta - a_1\alpha|^2 + |\beta|^2 + |a_1\alpha|^2 + |\beta|^2 > |2\alpha\beta a_1|\,.$$

Assume that $a_1 = 0$. We take $\beta = 0$ and $\alpha = 1$. We obtain $0 > 0$, which is a contradiction. It follows that $a_1 \neq 0$. In this case we have

$$|2\alpha\beta a_1| \leq |\beta|^2 + |a_1\alpha|^2.$$

But also we have$|\beta - a_1\alpha|^2 + |\beta|^2 > 0$ for each $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{0\}$. Thus

$$|\beta - a_1\alpha|^2 + 2|\beta|^2 + |a_1\alpha|^2 > 2|\alpha\beta a_1|$$

for every $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{0\}$.

Case 3: $a_1 \neq 0$ and $a_2 \neq 0$. By the above development it follows that

$$|2\alpha\beta a_1| < |\beta - a_1\alpha|^2 + |\beta|^2 + |a_1\alpha|^2 + |\beta - a_2\alpha|^2,$$

for all $(\alpha, \beta) \in \mathbb{C}^2 \backslash \{0\}$. Thus

$$t_1(\alpha, \beta) = |\beta - a_1\alpha|^2 + |\beta - a_2\alpha|^2 + (|\beta| - |a_1\alpha|)^2 > 0\,.$$

Assume that $\alpha a_1 = \beta\delta$, where $\delta \in \partial D(0,1)$. Then $|\alpha a_1| = |\beta|$. In this case, $|1-\delta|^2 + \left|1 - \frac{a_2}{a_1}\delta\right|^2 > 0$. Consequently, $t_1(\alpha,\beta) > 0$ for each $(\alpha,\beta) \in \mathbb{C}^2\backslash\{0\}$ if and only if $|1-\delta|^2 + \left|1 - \frac{a_2}{a_1}\delta\right|^2 > 0$ for every $\delta \in \partial D(0,1)$. Thus $t_1(\alpha,\beta) > 0$ for each $(\alpha,\beta) \in \mathbb{C}^2\backslash\{0\}$ if and only if $\frac{a_2}{a_1} \neq 1$. ∎

Finally, we resume the above development by the following result. Let $\varphi_1, \varphi_2 : \mathbb{C} \to \mathbb{C}$ be 2 analytic functions. Set

$$\varphi(z,w) = A_1|w-\varphi_1(z)|^2 + A_2|w-\overline{\varphi_1}(z)|^2 + A_3|w-\varphi_2(z)|^2,$$

where $(z,w) \in \mathbb{C}^2$, $A_1, A_2, A_3 \in \mathbb{R}_+\backslash\{0\}$. Then $\varphi$ is strictly convex in $\mathbb{C}^2$ if and only if $\varphi_1(z) = a_1 z + b_1$, $\varphi_2(z) = a_2 z + b_2$, for $z \in \mathbb{C}$ and $a_1, a_2, b_1, b_2 \in \mathbb{C}$ with $a_1 \neq a_2$ (the stictly convexity of $\varphi$ is independent of $b_1$ and $b_2$). Note that, for all $B_1, B_2 \in \mathbb{R}_+\backslash\{0\}$, $\psi$ is not strictly convex in $\mathbb{C}^2$, where $\psi(z,w) = B_1|w-\varphi_1(z)|^2 + B_2|w-\overline{\varphi_1}(z)|^2$, $(z,w) \in \mathbb{C}^2$. But there exists several possible cases (of the analytic function $\varphi_1$ defined over $\mathbb{C}$) such that $\psi$ is strictly psh over all $\mathbb{C}^2$.

Question 3.3. Prove that there exists an analytic function $g : \mathbb{C} \to \mathbb{C}$ such that for all $A_0, A_1, A_2, A_3 \in \mathbb{R}_+\backslash\{0\}$, the function

$$u = A_0|g|^2 + A_1|g'|^2 + A_2|g''|^2 + A_3|g'''|^2$$

is not convex over $\mathbb{C}$. We can in fact generalize this question for every fixed order $m$ of the derivative of $g$ denoted $\frac{\partial^m g}{\partial z^m}$ or over analytic functions defined on $\mathbb{C}^n$, where $n \geq 2$.

*Remark* 3.4. Let $g_1, \dots, g_N : \mathbb{C}^n \to \mathbb{C}$ be $N$ analytic functions, where $n, N \in \mathbb{N}$. Assume that $|g_1|^2 + \cdots + |g_N|^2 = u$ is convex and strictly psh in $\mathbb{C}^n$. We can not conclude that $u$ is strictly convex in $\mathbb{C}^n$. But we have the next statement.

Theorem 3.5. *Let $g_1, \dots, g_N : \mathbb{C}^n \to \mathbb{C}$ be $N$ analytic functions and $n, N \in \mathbb{N}$. Put*

$$u(z,w) = |g_1(w_1 - \overline{z_1})|^2 + \cdots + |g_N(w_N - \overline{z_N})|^2,$$
$$v(w_1, \dots, w_N) = |g_1(w_1)|^2 + \cdots + |g_N(w_N)|^2,$$

*where $(z_j, w_j) \in \mathbb{C}^n \times \mathbb{C}^n$, $1 \leq j \leq N$ and $(z,w) = (z_1, \dots, z_N, w_1, \dots, w_N)$. The following conditions are equivalent:*

(a) *$u$ is strictly psh in $(\mathbb{C}^n \times \mathbb{C}^n)^N$;*

(b) *$n = 1$ and $|g_1|^2, \dots, |g_N|^2$ are strictly convex functions over $\mathbb{C}$;*

(c) *$n = 1$ and $v$ is strictly convex in $\mathbb{C}^N$.*

*Proof.* Recall that by Abidi [2], we have for every function $K : \mathbb{C}^n \to \mathbb{C}$ be analytic, if we put $\varphi(z,w) = |K(w - \overline{z})|^2$, where $(z,w) \in \mathbb{C}^n \times \mathbb{C}^n$. Then $\varphi$ is psh on $\mathbb{C}^n \times \mathbb{C}^n$ if and only if $(K(z) = (< z, a > +b)^m$ for each $z \in \mathbb{C}^n$, where $a \in \mathbb{C}^n$, $b \in \mathbb{C}$ and $m \in \mathbb{N} \cup \{0\})$ or $(K(z) = e^{(<z,\lambda>+\mu)}$, for every $z \in \mathbb{C}^n$, where $\lambda \in \mathbb{C}^n$ and $\mu \in \mathbb{C})$. Note that $\varphi$ is psh on $\mathbb{C}^n \times \mathbb{C}^n$ if and only if $|K|^2$ is convex on $\mathbb{C}^n$.

(a) $\Rightarrow$ (b) Let $u_1(z,w) = |g_1(w_1 - \overline{z_1})|^2, \dots, u_N(z,w) = |g_N(w - \overline{z})|^2$, where $(z,w) = ((z_1,w_1),\dots,(z_N,w_N)) \in (\mathbb{C}^n \times \mathbb{C}^n)^N$. $u$ is strictly psh on $(\mathbb{C}^n \times \mathbb{C}^n)^N$ if and only if $u_1,\dots,u_N$ are strictly psh on $\mathbb{C}^n \times \mathbb{C}^n$. For example by Abidi [2], $u_1$ is strictly psh on $\mathbb{C}^n \times \mathbb{C}^n$ if and only if $n = 1$ and $g_1$ is an affine bijective function over $\mathbb{C}$. Therefore $|g_1|^2$ is strictly convex on $\mathbb{C}$. It follows that $|g_1|^2, \dots, |g_N|^2$ are strictly convex functions over $\mathbb{C}$.

The remainder of the proof of this theorem follows from the above development. ∎

Claim 3.6. Let $k_1, \dots, k_n : D \to \mathbb{C}$ be $n$ analytic functions, $D$ is a domain of $\mathbb{C}^n$, $n \geq 1$. The system

$$\begin{cases} \alpha_1 \frac{\partial k_1}{\partial z_1}(z) + \cdots + \alpha_n \frac{\partial k_1}{\partial z_n}(z) = 0 \\ \vdots \\ \alpha_1 \frac{\partial k_n}{\partial z_1}(z) + \cdots + \alpha_n \frac{\partial k_n}{\partial z_n}(z) = 0 \end{cases}$$

has only the solution $(\alpha_1,\dots,\alpha_n) = 0 \in \mathbb{C}^n$ (for all $z$ fixed in $D$), if and only if $u$ is strictly psh in $D \times \mathbb{C}^n$, where

$$u(z,w) = A_1 \left|w_1 - \overline{k_1}(z)\right|^2 + \cdots + A_n \left|w_n - \overline{k_n}(z)\right|^2,$$

for $z = (z_1,\dots,z_n) \in D$, $w = (w_1,\dots,w_n) \in \mathbb{C}^n$ and $A_1,\dots,A_n \in \mathbb{R}_+ \backslash \{0\}$.

Now fix $f_1,\dots,f_n : D \to \mathbb{C}$ be n arbitrary analytic functions. The above system has only the solution $(\alpha_1,\dots,\alpha_n) = (0,\dots,0)$ for all $z \in D$ if and only if $v$ is strictly psh in $D \times \mathbb{C}^n$, where

$$v(z,w) = A_1 \left|w_1 - f_1(z) - \overline{k_1}(z)\right|^2 + \cdots + A_n \left|w_n - f_n(z) - \overline{k_n}(z)\right|^2,$$

for $z = (z_1,\dots,z_n) \in D$, $w = (w_1,\dots,w_n) \in \mathbb{C}^n$ and $A_1,\dots,A_n \in \mathbb{R}_+ \backslash \{0\}$.

That is we have a rigid relation between strictly plurisubharmonic functions and holomorphic or antiholomorphic partial differential equations in $\mathbb{C}^n$, $n \geq 1$. Observe that we have a good relation between the algebraic method for the resolution of a system of holomorphic partial differential equations and the study of the strictly plurisubharmonic of a only one function in complex analysis and conversely.

In the case of a power of analytic equations, we have the following result.

Theorem 3.7. *Let $g_1, g_2, , k_1, k_2 : \mathbb{C}^2 \to \mathbb{C}$ be four analytic functions, and let $m_1, s_1, m_2, s_2 \in \mathbb{N}$. The system*

$$\begin{cases} \left(\alpha_1 \frac{\partial k_1}{\partial z_1}(z) + \alpha_2 \frac{\partial k_1}{\partial z_2}(z)\right)^{2m_1} + \left(\alpha_1 \frac{\partial g_1}{\partial z_1}(z) + \alpha_2 \frac{\partial g_1}{\partial z_2}(z)\right)^{2s_1} = 0 \\ \left(\alpha_1 \frac{\partial k_2}{\partial z_1}(z) + \alpha_2 \frac{\partial k_2}{\partial z_2}(z)\right)^{2m_2} + \left(\alpha_1 \frac{\partial g_2}{\partial z_1}(z) + \alpha_2 \frac{\partial g_2}{\partial z_2}(z)\right)^{2s_2} = 0 \end{cases}$$

*has only the solution $(\alpha_1, \alpha_2) = (0, 0)$ for each $z = (z_1, z_2) \in \mathbb{C}^2$, if and only if $u$ is strictly psh on $\mathbb{C}^2 \times \mathbb{C}$, where*

$$u(z, w) = \left|w - \overline{k_1}(z)\right|^2 + \left|w - \overline{k_2}(z)\right|^2 + \left|w - \overline{g_1}(z)\right|^2 + \left|w - \overline{g_2}(z)\right|^2$$

*for each $(z, w) \in \mathbb{C}^2 \times \mathbb{C}$.*

*Proof.* Define $v$ by $v(z, w) = 4|w|^2 + |k_1(z)|^2 + |k_2(z)|^2 + |g_1(z)|^2 + |g_2(z)|^2$, where $(z, w) \in \mathbb{C}^2 \times \mathbb{C}$; $u$ and $v$ are functions of class $C^\infty$ on $\mathbb{C}^2 \times \mathbb{C}$. In fact $u$ and $v$ have the same hermitian Levi form over $\mathbb{C}^2 \times \mathbb{C}$. Now the proof is easy to describe. ∎

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