**FACTA UNIVERSITATIS**
Series: **Electronics and Energetics** Vol. 30, N° 3, September 2017, pp. 351 - 362
DOI: 10.2298/FUEE1703351S

# LOWPASS FILTERS APPROXIMATION BASED ON THE JACOBI POLYNOMIALS

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**Abstract**. *A case study related to the design the the analog lowpass filter using a set of orthogonal Jacobi polynomials, having four parameters to vary, is considered. The Jacobi polynomial has been modified in order to be used as a filter approximating function. The obtained magnitude response is more general than the response of the classical ultraspherical filter, due to one additional parameter available in orthogonal Jacobi polynomials. This additional parameter may be used to obtain a magnitude response having either smaller passband ripple, smaller group delay variation or sharper cutoff slope. Two methods for transfer function approximation are investigated: the first method is based on the known shifted Jacobi polynomial, and the second method is based on the proposed modification of Jacobi polynomials. The shifted Jacobi polynomials are suitable only for odd degree transfer function. However, the proposed modified Jacobi polynomial filter function is more general but not orthogonal. It is transformed into orthogonal polynomial when orders are equal and then includes the Chebyshev filter of the first kind, the Chebyshev filter of the second kind, the Legendre filter, Gegenbauer (ultraspherical) filter and many other filters, as its special cases.*

**Key words**: *Filters, analog circuits, approximation, filter characteristic function, Jacobi polynomial, orthogonal polynomials.*

## 1 Introduction

The very classical orthogonal polynomials Jacobi, Laguerre and Hermite [1] and their special cases i.e Gegenbauer, Chebyshev and Legendre are widely used in communication theory and particularly in the synthesis transfer function of electric filters. The coefficients of the Bessel-Thomson filters, which provide maximally flatness of the group delay response in the passband without any ripple, are related to the Bessel polynomials [2]. However, the Bessel type polynomials are not orthogonal on an interval of the $x$-axis, but in certain cases are orthogonal on a unit circle.

Received June 14, 2016; received in revised form November 18, 2016
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Apart from Chebyshev polynomials, which are of utmost importance in the synthesis of filters exhibiting a sharp increase in attenuation as the frequency increases above corner frequency, other classes of above mentioned orthogonal polynomials have found many useful applications in the synthesis of electrical filters. In particular, the approximation problem in the synthesis of electrical filters consists of finding a physical realizable function of frequency that shall meet a prescribed set of specifications with regard to its magnitude and/or group delay characteristics.

It is known that, for a given filter degree, there is always a trade-off between the magnitude and group delay characteristics. By considering the whole frequency band, the better group delay characteristic is generally associated with the better time domain characteristic [3]. The better time domain characteristic leads to smaller time delay or smaller values of the overshoot in the step response.

There are approximations that have a very good magnitude characteristic in detriment of their group delay characteristic, as for example, Butterworth [4], Chebyshev [5], [6], Bernstein [7], Legendre [8] [9] [10] and their derivatives by Ku and Drubin [11]. Converse case occurs with other approximations, as for example, Bessel [12], Gauss [13], Hermite [11] and least-squares monotonic [14] [15], all those filters present optimized characteristics in specific points.

Transitional filters are alternative filter solutions that perform a trade-off between the magnitude and group delay characteristics. Transitional Butterworth-Chebyshev [16] filters are considered with magnitude characteristics that vary gradually from those of the Butterworth filter to those of the Chebyshev filter as a number of pass-band ripples (or the degree of flatness at the origin) is varied. Three degrees of freedom are available for transitional Butterworth-Chebyshev filters: the degree $n$, the ripple factor $\varepsilon$ and the degree of flatness at the origin. The smooth transition is accomplished using the method proposed of Peless and Murakami [17] by finding each pole of the transitional Butterworth-Thompson filter as an interpolation between a pole of the Butterworth filter and a corresponding pole of the Thompson filter.

A special class of filter functions of odd order providing monotonic magnitude characteristic of the resulting filter has first been investigated by Papoulis [18] by means of Legendre polynomials. Subsequently these results have been extended so as to include filters of even degree [19], [20], and also some other functions leading to the same class of filtering networks whose magnitude response is bounded to be monotonic have been derived using a different approach based on the applications of Jacobi polynomials [21].

In this paper, the concept of magnitude response synthesis techniques is extended for orthogonal Jacoby lowpass filters. Simple modification of orthogonal Jacobi polynomial, suitable for the continuous-time lowpass filter design, is proposed in this paper. If the degree of the filter is given, both indexes (order) of the Jacobi polynomial can be used for smoothly adjusting the filter performance. The magnitude response obtained is more general than the continuous-time response of the Chebyshev filter because of two additional parameters available with the modified Jacobi polynomials. It should be noted, the proposed Jacobi approximation covers many of the above-mentioned all-pole filter functions.

## 2 Filter magnitude function

In lowpass filter design, assuming all the zeros of the system function are at infinity, the squared magnitude function (insertion loss) can be written as

$$|H_n(j\omega)|^2 = \frac{1}{1+\varepsilon^2\phi_n^2(\omega)} \tag{1}$$

where $\omega$ is frequency variable, $\varepsilon$ is a parameter that controls the passband attenuation tolerance, $n$ denotes the degree of the filter and the polynomial $\phi_n(\omega)$ is the characteristic (or approximating) function of the filter which is to be selected to give desired magnitude characteristic. The characteristic function is normalized to unity at the pass-band edge frequency $\omega_p$, which is also normalized to $\omega_p = 1$, then can be written as $\phi_n(1) = 1$.

This conventional procedure for filter design using the insertion loss method includes the design of a lumped element LC ladder lowpass filter known as the lowpass prototype. A more modern procedure uses this network synthesis technique to design filters with a completely specified frequency response. The design is simplified by beginning with low-pass filter prototypes that are normalized in terms of impedance and frequency. Transformations are then applied to convert the prototype designs to the desired frequency range and impedance level.

In filter design, the characteristic frequency use for frequency normalization is the cutoff frequency known as the filter passband corner frequency, and therefore normalized cutoff frequency is equal to 1. For this application, the function $\phi_n^2(x)$ is required to be an even polynomial $\psi_n(\omega^2) = \phi_n^2(x)$. If $\phi_n(x)$ is even or odd, then $\phi_n^2(x)$ is always even, as is required. Polynomials $\phi_n(x)$, which are neither even nor odd, may be also be used in magnitude functions if $\phi_n(x)$ is replaced by $\phi_n(x^2)$. Therefore it is necessary that no terms of the form $x^{2k+1}$ appear in the characteristic function.

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ have $n$ distinct zeros for $\alpha \neq \beta$ but they are neither even nor odd. Such type of polynomials are not suitable to be a filter characteristic function. However, Jacobi orthogonal polynomials can be adapted for use in the low-pass filter magnitude functions, as will be shown in the next section.

## 3 Jacobi polynomial

The Jacobi polynomials [22], denoted by $P_n^{(\alpha,\beta)}(x)$ of the degree $n$, are orthogonal on the interval $[-1,1]$ with respect to the Jacobi weight function $w^{(\alpha,\beta)} = (1-x)^\alpha(1+x)^\beta$ when $\alpha,\beta \geq -1$. We shall refer to $\alpha$ and $\beta$ as the orders of the Jacobi polynomial. Namely,

$$\int_{-1}^{1} P_m^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(x)\,w^{(\alpha,\beta)}(x)\,\mathrm{d}x = h_n^{(\alpha,\beta)}\delta_{n.m}, \tag{2}$$

where

$$h_n^{(\alpha,\beta)} = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\,\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}, \tag{3}$$

$\delta_{n.m}$ is Kronecker delta symbol and $\Gamma(\cdot)$ is well known Gamma function.

The Jacobi polynomials are generated by the three-term recurrence relation:

$$\begin{aligned} &P_0^{(\alpha,\beta)}(x) = 1, \\ &P_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta), \\ &P_{n+1}^{(\alpha,\beta)}(x) = (a_n^{(\alpha,\beta)}\, x - b_n^{(\alpha,\beta)})P_n^{(\alpha,\beta)}(x) - c_n^{(\alpha,\beta)} P_{n-1}^{(\alpha,\beta)}(x),\ n \geq 1 \end{aligned} \tag{4}$$

where

$$\begin{aligned} a_n^{(\alpha,\beta)} &= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{2(n+1)(n+\alpha+\beta+1)} \\ b_n^{(\alpha,\beta)} &= \frac{(\beta^2-\alpha^2)(2n+\alpha+\beta+1)}{2(n+1)(n+\alpha+\beta+1)1)(2n+\alpha+\beta)} \\ c_n^{(\alpha,\beta)} &= \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)1)(2n+\alpha+\beta)} \end{aligned}$$

Matlab is an inexpensive an easi-to-use software package and widely available comercial product that is in widespread in both academia and industry [23]. A Matlab script for evaluating Jacobi polynomials using the above procedure is given in `JacobiPoly.m`. In addition to Jacobi polynomial, proposed Matlab program also evaluates Gegenbauer and Legendre polynomials.

JacobyPoly.m

```
function P=JacobiPoly(n,a,b)
% Coefficients P of the Jacobi polynomial
% They are stored in decending order of powers
if nargin == 1,
    a=0; b=0;
elseif nargin == 2,
    b=a;
end
p0 = 1;
p1 = [(a+b)/2+1,(a-b)/2];
if n == 0,
    P=p0;
elseif n == 1,
    P=p1;
else
    for k=2:n,
        d=2*k*(k+a+b)*(2*k-2+a+b);
        A=(2*k+a+b-1)*(2*k+a+b-2)*(2*k+a+b)/d;
        B=(2*k+a+b-1)*(a^2-b^2)/d;
        C=2*(k-1+a)*(k-1+b)*(2*k+a+b)/d;
        P=conv([A B],p1)-C*[0,0,p0];
        p0 = p1;
        p1 = P;
    end
end
end
```

Some properties of the Jacobi polynomials, which are needed here, are as follows

$$P_n^{(\alpha,\beta)}(1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1)} \tag{5}$$

and

$$P_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(\beta+1)} \tag{6}$$

Jacobi polynomials have symmetry

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(x) \tag{7}$$

The following important derivative relation is

$$\frac{\mathrm{d}}{\mathrm{d}x}P_n^{(\alpha,\beta)}(x) = \frac{1}{2}(n+\alpha+\beta+1)P_{n-1}^{(\alpha+1,\beta+1)}(x) \tag{8}$$

### 3.1 Shifted Jacobi polynomials

In order to use Jacobi polynomials on the interval $x \in [0,1]$ we define the so-called shifted Jacobi polynomials by introducing the change of variable $x \mapsto 2x-1$. Let the shifted Jacobi polynomials $P_n^{(\alpha,\beta)}(2x-1)$ be denoted by $\mathcal{J}_n^{(\alpha,\beta)}(x)$. The shifted Jacobi polynomials are orthogonal with respect to the weight function $w_s^{(\alpha,\beta)} = (1-x)^\alpha x^\beta$ in the interval $[0,1]$ with the orthogonality property:

$$\int_0^1 w_s^{(\alpha,\beta)}\mathcal{J}_m^{(\alpha,\beta)}(x)\mathcal{J}_n^{(\alpha,\beta)}(x)\mathrm{d}x = \frac{1}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}\delta_{n,m} \tag{9}$$

The shifted Jacobi polynomials are generated from the three-term recurrence relations [24]:

$$\begin{aligned}
&\mathcal{J}_0^{(\alpha,\beta)}(x) = 1,\\
&\mathcal{J}_1^{(\alpha,\beta)}(x) = (\alpha+\beta+2)y-(\beta+1),\\
&\mathcal{J}_{n+1}^{(\alpha,\beta)}(x) = (a_n^{(\alpha,\beta)}\,x - b_n^{(\alpha,\beta)})\mathcal{J}_n^{(\alpha,\beta)}(x) - c_n^{(\alpha,\beta)}\mathcal{J}_{n-1}^{(\alpha,\beta)}(x),\ n \geq 1
\end{aligned} \tag{10}$$

where the recursion coefficients are

$$\begin{aligned}
a_n^{(\alpha,\beta)} &= \frac{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}{(n+1)(n+\alpha+\beta+1)}\\
b_n^{(\alpha,\beta)} &= \frac{(2n+\alpha+\beta+1)(2n^2+(1+\beta)(\alpha+\beta)+2n(\alpha+\beta+1))}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}\\
c_n^{(\alpha,\beta)} &= \frac{(2n+\alpha+\beta+2)(n+\alpha)(n+\beta)}{(n+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}
\end{aligned} \tag{11}$$

The shifted Jacobi polynomial $\mathcal{J}_n^{(\alpha,\beta)}(x)$ can be obtained in the polynomial standard form as

$$\mathcal{J}_n^{(\alpha,\beta)}(x) = \sum_{i=0}^{n}(-1)^{n-i}\frac{\Gamma(n+\alpha+\beta+i+1)}{\Gamma(i+1)\Gamma(n+\alpha+\beta+1)}\frac{\Gamma(n+\beta+1)}{\Gamma(n-i+1)\Gamma(\beta+i+1)}x^i \tag{12}$$

Suppose the Jacobi polynomials should be normalized soo' that $\phi_n(1) = 1$. According to the polynomial (12), the normalization constant is $k_n^{(\alpha,\beta)} = \sum_{i=0}^{n} a_i^{(n)}$, where $a_i^{(n)}$ are corresponding polynomial coefficients.

As an example, Fig. 1 shows the characteristic functions based on the shifted Jacobi polynomials for $n = 1,2,\ldots,5$ in the form $\phi_n(x) = x^\nu \mathcal{J}_m^{(\alpha,\beta)}(x^2)/k_n^{(\alpha,\beta)}$, where $n = \lfloor m/2 \rfloor + \nu$, the floor function $\lfloor m/2 \rfloor$ rounds the value of $m/2$ to the nearest integers towards zero, $\nu = 0$ and $\nu = 1$ for $n$ even and odd, respectively.

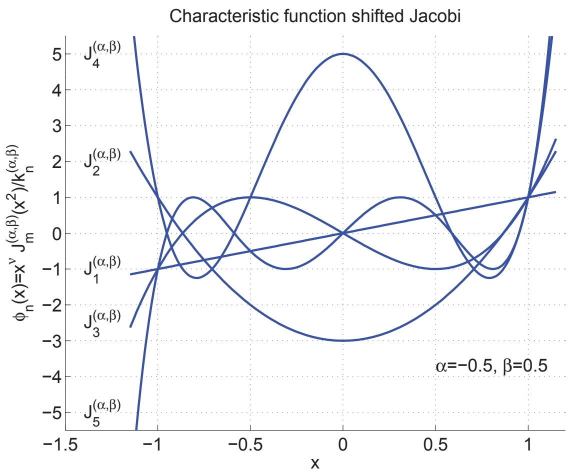


Fig. 1. The normalized shifted Jacobi polynomials $\phi_n(x) = x^\nu \mathcal{J}_m^{(\alpha,\beta)}(x^2)$ for $\nu = 0$ and $\nu = 1$ for $n$ even and odd, respectively, used in place characteristic function, $\alpha = -0.5$ and $\beta = 0.5$, $n = 2m+\nu$, $m = 0,1$ and 2.

As shown in Fig. 1, the hump at $x = 0$ occurs when the filter degree is even. Using (6) size of the hump can be obtained as

$$\phi_m^{(\alpha,\beta)}(0) = \frac{1}{k_n^{(\alpha,\beta)}} P_m^{(\alpha,\beta)}(-1) = \frac{1}{k_n^{(\alpha,\beta)}} \frac{(-1)^m \Gamma(m+\beta+1)}{\Gamma(m+1)\Gamma(\beta+1)} \tag{13}$$

because $\mathcal{J}_n^{(\alpha,\beta)}(0) = P_n^{(\alpha,\beta)}(-1)$. One can easily show that the size of the hump increases when the degree of the filter increases. For example, for $n = 4$, ($m = 2$ and $\nu = 0$) from (13) follow $P_2^{(-0.5,0.5)}(-1) = 1.875$ and from (12) is $k_2^{(-0.5,0.5)} = 0.3750$ then value for hump is $\phi_2(0) = 5$. For $n = 6$ ($m = 3$ and $\nu = 0$) follow $P_3^{(-0.5,0.5)}(-1) = -2.1875$, $k_3^{(-0.5,0.5)} = 0.3125$ then $\phi_3(0) = -7$. Thus, the even degree of the shifted Jacobi polynomial is not suitable as the filter characteristic function.

Other definitions of the monic shifted Jacobi polynomials are given in [22, Chapter 22], $G_n(p,q,x)$, which are also orthogonal in the interval $[0,1]$ with respect to weight function $w(x) = (1-x)^{p-q}x^{q-1}$ (with $q > 0$ and $p > q-1$), are used for the construction magnitude of the filter's transfer function [25] [26] [27]. Shifted Jacobi polynomials [22] are related to the Jacobi Polynomials $P_n^{(\alpha,\beta)}(x)$ as [28]

$$G_n(p,q,x) = \frac{\Gamma(n+1)\Gamma(n+p)}{\Gamma(2n+p)} P_n^{(p-q,q-1)}(2x-1) \tag{14}$$

It can be concluded, the shifted Jacobi polynomials $\mathcal{J}_n^{(\alpha,\beta)}(x)$ have $n$ distinct positive real zeros in the interval $(0,1)$ but they are neither even nor odd then it can not be used as a characteristic function in the equation (1). However, $[x\mathcal{J}_n^{(\alpha,\beta)}(x^2)]^2$ or $[xG(p,q,x^2)]^2$ could be used in (1) in place of squared characteristic function $\phi_n^2(\omega)$.

### 3.2 Modified Jacobi polynomials

We propose the following modified Jacobi polynomials, based on the summation of two Jacobi orthogonal polynomials which have the same degree $n$, as

$$\mathbb{J}_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x) + P_n^{(\beta,\alpha)}(x) \tag{15}$$

where $P_n^{(\alpha,\beta)}(x)$ is above mentioned classical Jacoby orthogonal polynomial in $x$. One can easily show that modified Jacobi polynomial (15) is not orthogonal polynomial except in the case when $\alpha = \beta$ is. Since Jacobi polynomials $P_n^{(\beta,\alpha)}(x) = (-1)^n P_n^{(\alpha,\beta)}(-x)$ are not orthogonal polynomials with the respect to the weight function $w^{(\alpha,\beta)}(x)$ over the interval $[-1,1]$, then the modified orthogonal Jacobi polynomials (15) are not orthogonal polynomials as the shifted Jacobi polynomials are. However, the resulting degree of modified Jacobi polynomial is $n$, which is pure odd or pure even polynomial in $x$, and hence the realization of the lowpass filter is possible for all specifications if they are used as characteristic function.

Many of the aforementioned polynomials are special cases of modified Jacobi polynomials. For $\alpha = \beta$, one can obtain the ultraspherical polynomials (symmetric Jacobi polynomials) [29]. For $\alpha = \beta = \mp 1/2$, the Chebyshev polynomials of first and second kinds. For $\alpha = \beta = 0$, one can obtain the Legendre polynomials. For the two important special cases $\alpha = -\beta \pm 1/2$, the Chebyshev polynomials of third and fourth kinds are also obtained.

Finally, the constants $C_n^{(\alpha,\beta)} = \mathbb{J}_n^{(\alpha,\beta)}(1)$ have to be chosen in such a way that normalization criterion $\phi_n(1) = 1$ is satisfied, i.e.

$$\phi_n(\omega) = \frac{\mathbb{J}_n^{(\alpha,\beta)}(\omega)}{C_n^{(\alpha,\beta)}}, \tag{16}$$

where

$$C_n^{(\alpha,\beta)} = \frac{1}{\Gamma(n+1)}\left[\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} + \frac{\Gamma(n+\beta+1)}{\Gamma(\beta+1)}\right]. \tag{17}$$

Modified Jacobi polynomials are symmetrical in relation to the orders $\alpha$ and $\beta$, i.e. $\mathbb{J}_n^{(\alpha,\beta)}(\omega) = \mathbb{J}_n^{(\beta,\alpha)}(\omega)$. Table 1 contains the modified Jacobi polynomials for $\alpha = -0.5$ and $\beta = 0.5$ up to the ninth degree.

Table 1. The modified orthogonal Jacobi polynomials $\mathbb{J}_n^{(\alpha,\beta)}(x)$, $\alpha=-0.5$, $\beta=0.5$, and $n=0,1,\ldots,10$.

| $n$ | $\mathbb{J}_n^{(-0.5,0.5)}(x)$ |
|---|---|
| 1 | $2x$ |
| 2 | $3x^2-\frac{3}{4}$ |
| 3 | $5x^3-\frac{5}{2}x$ |
| 4 | $\frac{35}{4}x^4-\frac{105}{16}x^2+\frac{35}{64}$ |
| 5 | $\frac{63}{4}x^5-\frac{63}{4}x^3+\frac{189}{64}x$ |
| 6 | $\frac{231}{8}x^6-\frac{1155}{32}x^4+\frac{693}{64}x^2-\frac{231}{512}$ |
| 7 | $\frac{429}{8}x^7-\frac{1287}{16}x^5+\frac{2145}{64}x^3-\frac{429}{128}x$ |
| 8 | $\frac{6435}{64}x^8-\frac{45045}{256}x^6+\frac{96525}{1024}x^4-\frac{32175}{2048}x^2+\frac{6435}{16384}$ |
| 9 | $\frac{12155}{64}x^9-\frac{12155}{32}x^7+\frac{255255}{1024}x^5-\frac{60775}{1024}x^3+\frac{60775}{16384}x$ |
| 10 | $\frac{46189}{128}x^{10}-\frac{415701}{512}x^8+\frac{323323}{512}x^6-\frac{1616615}{8192}x^4+\frac{692835}{32768}x^2-\frac{46189}{131072}$ |

It is important to know where the roots of the modified Jacobi polynomials are located. The fastest way to calculate the zeros of the modified Jacobi polynomials is by using mathematical programs such as Matlab, Mathematica and Maple. It can be concluded that the modified Jacobi polynomials, $\mathbb{J}_n^{(\alpha,\beta)}(x)$, have $n$ simple real zeros in the closed interval $[-1,1]$. For example, the zeros of the modified Jacobi polynomial of degree 8 with $\alpha=-0.5$ and $\beta=0.5$ are:

$$\{-0.9396926,-0.7660444,-0.5000000,-0.1736482,0.1736482,0.5000000,0.7660444,0.9396926\}.$$

The zeros of $\mathbb{J}_n^{(\alpha,\beta)}(x)$ are located symmetrically about $x=0$ in the interval $-1<x<1$.

Note that modified Jacobi polynomials are the only non orthogonal polynomials which are suitable for the synthesis of the filter function given in a closed form.

The characteristic functions $\phi_n(x)$ based on the modified Jacobi polynomials $\mathbb{J}_n^{\alpha,\beta)}(x)$ are illustrated in Figure 1 for $x$ in $[-1,1]$ and $n=1,2,\ldots,5$. They satisfy the following relationships: for $|x|<1$, the characteristic polynomial oscillates around zero and they ripples are bounded by $\pm1$ for $\alpha,\beta\geq-0.5$, $\phi_n(0)\neq0$ for $n$ even and $\phi_n(0)=0$ for $n$ odd. For $|x|>1$, the polynomials magnitude increase (decrease) monotonically.

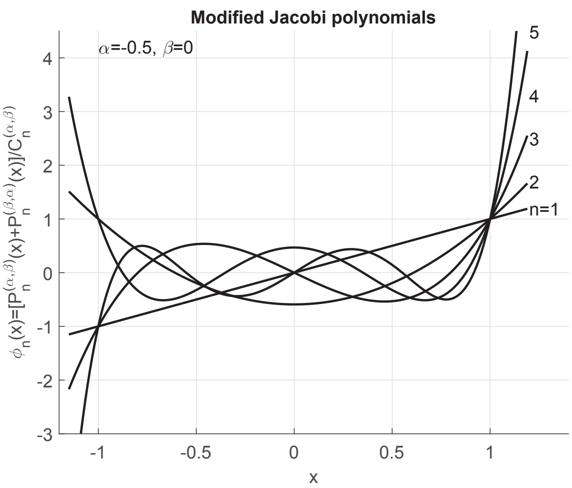


Fig. 2. The normalized modified orthogonal Jacobi polynomials $\mathbb{J}_n^{(\alpha,\beta)}(\omega)/C_n^{(\alpha,\beta)}$ used in place characteristic function $\phi_n(x)$, $\alpha=-0.5$ and $\beta=0.5$, $n=1,\ldots,5$.

An example is given in Figure 3, which shows the ninth-order modified Jacobi lowpass filter and its three partial filters with their individual orders $\alpha$ and $\beta$. As mentioned earlier, Jacobi orthogonal polynomial corresponds to the Chebyshev polynomial if $\alpha=\beta=-0.5$ which have 3dB ripples in the pass-band. In general, passband ripples are being undesirable, but a value less than 0.5 dB is acceptable in many applications. If $\alpha=-0.5$ and order $\beta$ increases, the ripples in the passband decrease smoothly to be unequal and smaller in magnitude. For $\beta>1.5$ the passband response is nearly flat, but the cutoff slope is much steeper than a Butterworth filter cutoff slope. On the other hand, for $-1<\beta<-0.5$ the passband ripples are unequal, but in magnitude are larger than 1, but these values of $\beta$ (also for $\alpha$) have no practical significance. It is shown that the passband ripple can be adjusted to improve the linearity of the group delay response near the $\omega=0$.

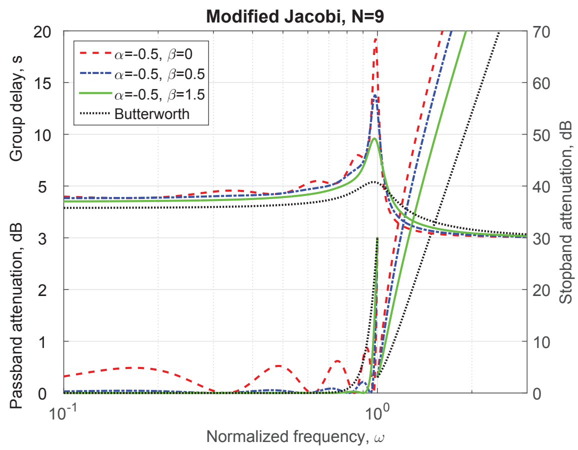


Fig. 3. The frequency responses of the 9th degree modified Jacobi filters.

Generally, for microwave applications modified orthogonal Jacobi as filter function may be also used. The most widely used filters in microwave applications are a band-pass filters [30]. Using lowpass to bandpass frequency transformation of lumped element lowpass filter, the series inductor converts to the series resonator and parallel capacitor converts to the parallel resonator. Richards transformation can be used to emulate the inductive and capacitive behaviour of the lumped circuit elements into distributive element consist the transmission line sections, and Kuroda's identities can be used to facilitate the conversion between the various transmission line realizations.

In the application where approximation of the filter magnitude function based on the Christofel-Darboux formula for classical orthonormal Jacobi polynomials gives excellent results [31] [32], this method cannot be applied to the modified Jacobi filters, because it is non orthogonal. In this case, it should either generate the sum of the product modified Jacobi polynomial, or Christoffel-Darboux formula be applied separately to the both orthonormal Jacobi polynomials as:

$$\begin{aligned} A_{2n}(\omega^2) = &[p_0^{(\alpha,\beta)}(\omega)]^2 + [p_1^{(\alpha,\beta)}(\omega)]^2 + \cdots + [p_n^{(\alpha,\beta)}(\omega)]^2 \\ &+ [p_0^{(\beta,\alpha)}(\omega)]^2 + [p_1^{(\beta,\alpha)}(\omega)]^2 + \cdots + [p_n^{(\beta,\alpha)}(\omega)]^2 \end{aligned} \tag{18}$$

where $p_i^{(\alpha,\beta)}(\omega)$, $i = 1,2,\dots,n$ are orthonormal Jacobi polynomials with respect to the weight function $w^{(\alpha,\beta)}(\omega) = (1-\omega)^\alpha(1+\omega)^\beta$ and $p_i^{(\beta,\alpha)}(\omega)$, $i = 1,2,\dots,n$ are also orthonormal Jacobi polynomials but with respect to the other weight function $w^{(\beta,\alpha)}(\omega) = (1-\omega)^\beta(1+\omega)^\alpha$. The orthonormal Jacobi plynomials are:

$$p_n^{(\alpha,\beta)}(\omega) = \sqrt{\frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}}\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}P_n^{(\alpha,\beta)}(\omega) \tag{19}$$

and

$$p_n^{(\beta,\alpha)}(\omega) = \sqrt{\frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}}\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}P_n^{(\beta,\alpha)}(\omega) \tag{20}$$

where $P_n^{(\beta,\alpha)}(\omega)$ and $P_n^{(\beta,\alpha)}(\omega)$ are the orthogonal Jacobi polynomials which can be evaluated by the proposed Matlab program.

By using Christoffel-Darboux formula equation (18) is reduced to:

$$\begin{aligned} A_{2n}(\omega^2) = &\frac{k_n^{(\alpha,\beta)}}{k_{n+1}^{(\alpha,\beta)}}\Big[\frac{\mathrm{d}p_{n+1}^{(\alpha,\beta)}}{\mathrm{d}\omega}p_n^{(\alpha,\beta)} - \frac{\mathrm{d}p_n^{(\alpha,\beta)}}{\mathrm{d}\omega}p_{n+1}^{(\alpha,\beta)}\Big] \\ &+ \frac{k_n^{(\beta,\alpha)}}{k_{n+1}^{(\beta,\alpha)}}\Big[\frac{\mathrm{d}p_{n+1}^{(\beta,\alpha)}}{\mathrm{d}\omega}p_n^{(\beta,\alpha)} - \frac{\mathrm{d}p_n^{(\beta,\alpha)}}{\mathrm{d}\omega}p_{n+1}^{(\beta,\alpha)}\Big] \end{aligned} \tag{21}$$

where $k_n^{(\alpha,\beta)}$ and $k_n^{(\beta,\alpha)}$ are leading coefficients of the orthonormal Jacobi polynomials $p_n^{(\alpha,\beta)}(\omega)$ and $p_n^{(\beta,\alpha)}(\omega)$, respectively.

The following filter approximating function for $n = 5$, $\alpha = -0.5$ and $\beta = 0.5$ is given as an example:

$$A_{10}(\omega^2) = 325.9493\omega^{10} - 488.9240\omega^8 + 244.4620\omega^6 - 40.7437\omega^4 + 3.8197\omega^2 + 2.5$$

According to the definition, the characteristic function should be normalized so that is unit, $A_{10}(1) = 1$, at the cutoff frequency, $\omega_p = 1$.

## 4 Conclusion

In this paper, we intended to illuminate the usage of Jacobi orthogonal polynomials in the design of time-continuous low-pass filter transfer function. Since Jacobi polynomial cannot be directly used as filter characteristic function, we suggested shifted Jacobi polynomials and proposed a simple modification of Jacobi polynomials to use as a filter characteristic function.

The modified Jacobi polynomials are not orthogonal, but they are suitable for the filter transfer function approximation. Filter degree, maximum passband attenuation and two indexes of Jacobi polynomials are four parameters that adjust the performance of the filter. The new modified Jacobi polynomials are implemented to approximate the lowpass filter transfer function in such a way that they are used directly as filter characteristic function (as standard orthogonal polynomials: Chebyshev or Legendre). These methods of approximation can be used to provide filters with adjustment of the passband ripple, group delay deviation or cutoff slope.

**Acknowledgment** This work is supported by Serbian Ministry of Education, Science and Technological Development, Project No. 32009TR.

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