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CONTROL OF SYSTEMS ON SPATIAL DOMAINS WITH MOVING BOUNDARIES: 3D PRINTING AND TRAFFIC

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Abstract. *Until roughly the year 2000, control algorithms (of the kind that can be physically implemented and provided guarantees of stability and performance) were mostly available only for systems modeled by ordinary differential equations. In other words, while controllers were available for finite-dimensional systems, such as robotic manipulators of vehicles, they were not available for systems like fluid flows. With the emergence of the "backstepping" approach, it became possible to design control laws for systems modeled by partial differential equations (PDEs), i.e., for infinite dimensional systems, and with inputs at the boundaries of spatial domains. But, until recently, such backstepping controllers for PDEs were available only for systems evolving on fixed spatial PDE domains, not for systems whose boundaries are also dynamical and move, such as in systems undergoing transition of phase of matter (like the solid-liquid transition, i.e., melting or crystallization). In this invited article we review new control designs for moving-boundary PDEs of both parabolic and hyperbolic types and illustrate them by applications, respectively, in additive manufacturing (3D printing) and freeway traffic.*

Key words: PDE backstepping, Stefan problem, 3D printing, traffic

1. CONTROL SYSTEMS AND FEEDBACK LAWS

For dynamical systems modeled by ordinary or partial differential equations (PDEs) with significantly fewer input variables than state variables—like a scalar input variable for a PDE with a spatially-distributed or infinite-dimensional state—control theory constructs the input as a function(al) of the state. This achieves stability for the dynamical system, where "stability" in a technically rigorous sense refers to a set of mathematical properties, which includes the property that the state converges to zero as time approaches infinity.

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Constructing such input functions, also called “feedback laws” because the input depends on the measurable state, is part of the design of most technological systems. A simple example is the Segway, whose driver would nosedive or fall backward without the feedback system that feeds the pitch angle measurements into the wheel angle inputs to keep the apparatus and rider upright. Less obvious feedback systems developed through evolution to both keep organisms alive and prevent them from making drastic changes to themselves, regardless of how much they desire said modifications. For instance, feedback systems that regulate metabolism prevent people from achieving significant weight loss by starving themselves over several days. These feedback systems developed in the living organisms in order to maintain—in the case of human organisms—our energy reserves in periods of famine and during strenuous travel.

2. PDE CONTROL ON MOVING DOMAINS

Classical control theory developed for ordinary differential equations (ODEs) requires remarkable sophistication in the design of feedback laws for nonlinear systems. Feedback synthesis for PDEs poses even greater challenges, namely in transitioning from the finite to infinite system dimension. Nonlinear ODE control saw its greatest achievements in the 1980s [1] and 90s [2], whereas PDE control has blossomed during the last two decades [3].

Not all physical systems are modeled by ODEs of a fixed order or PDEs on fixed domains. Some important applications—including traffic, opinion dynamics, and climate science—involve processes whose dimensions or domains depend on the size of the process state. For instance, the state vector dimension can increase with the size of the state. Or a higher temperature in its PDE spatial domain may cause the domain to grow, as in, melting ocean ice.

Classical control techniques are unequipped to deal with such dimension-varying dynamics. In fact, such possibilities have rarely even occurred to the control research community, which has been preoccupied in recent years with already difficult nonlinear, infinite-dimensional, stochastic, and hybrid phenomena in fixed dimension.

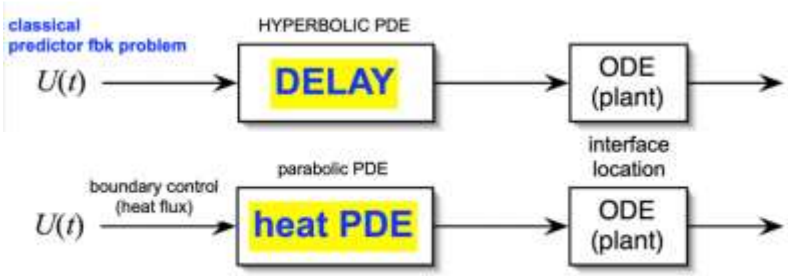

Fig 1 Examples of cascade systems in which a PDE, which is directly controlled, feeds into an ODE. Top: a hyperbolic PDE-ODE cascade, where a pure delay is example of the simplest hyperbolic PDE (example: control of congested traffic). Bottom: a parabolic PDE-ODE traffic (example: additive manufactruring/3D printing).

Among the simplest and most elegant problems with the state's dimension that varies with the state's size are those that involve a connected ODE and PDE, so that the PDE's state acts as an input to the ODE, whose state thus represents the PDE's boundary location. Such PDE-ODE systems may involve either hyperbolic or parabolic PDEs. Figure 1 depicts general PDE-ODE cascade systems in which the ODE is a general stabilizable dynamical system. Control of such PDE-ODE cascade systems is studied in [4]. In this article the ODE considered is a special case—a scalar ODE governing the position of the PDE's boundary.

3. CONTROL OF THE STEFAN SYSTEM (PARABOLIC): EXAMPLE OF ADDITIVE MANUFACTURING WITH LASER ACTUATION

An example of a *parabolic* PDE-ODE system in which the ODE state represents the PDE's boundary locatoin is the so-called *Stefan system*. Developed and analytically solved in the late 1800s by Slovenian-Austrian physicist Josef Stefan (of Stefan-Boltzmann fame), known in former Yugoslavia as Jožef Štefan, the system models melting and freezing [5].

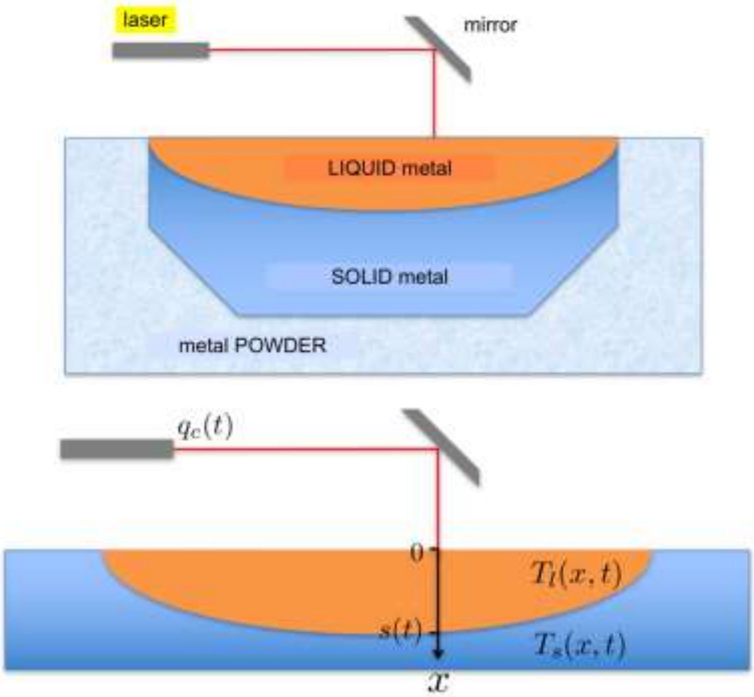

Fig. 2 Diagrams of additive manufacturing through laser-based sintering. Laser melts metal powder, which subsequently solidifies, allowing to build, layer-by-layer, a complex 3D solid form. Top: a diagram of the laser sintering system. Bottom: a notational representation of the temperature fields in the liquid and solid phases, represented in one spatial dimension, denoted by *x*. The heat flux q_c represents a boundary input to the liquid phase.

Researchers have recently used the Stefan system to model numerous other physical phenomena, including additive manufacturing with both polymers and metals, depicted in Figure 2; growth of axons in neurons; tumor growth; cancer treatment via cryosurgeries; spread of invasive species in ecology; lithium-ion batteries; domain walls in ferroelectric thin films; and information propagation in social networks.

Figure 3, shows the image at the bottom of Figure 2 rotated clockwise by 90 degrees, where $T_l(x,t)$ and $T_s(x,t)$ respectively represent the spatiotemporal temperatures in the solid and liquid. Heat PDEs govern the temperatures. A scalar ODE—whose inputs are the heat fluxes at the PDEs' boundary—governs the liquid-solid interface position $s(t)$.

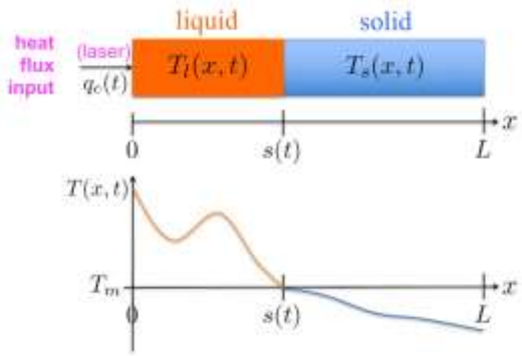

Fig. 3 Temperature profiles and phase interface in a PDE-ODE system involving a liquid, a solid, and rightward melting with the aid of heat flux applied by a laser on the left boundary.

The Stefan model is given by the parabolic (heat equation) PDE

$$\begin{aligned} T_t(x,t) &= \alpha T_{xx}(x,t), \quad 0 < x < s(t) < L \\ T_x(0,t) &= -q_c(t)/k \\ T(s(t),t) &= T_m \end{aligned}$$

in which $T(x,t)$ represents the spatiotemporal distribution of temperature, at location x and at time t, the heat flux q_c represents a boundary input at $x = 0$, and the liquid-solid interface s is governed by the ODE

$$\dot{s}(t) = -\beta T_x(s(t),t)$$

Even though the heat equation above, for T, appears linear, the scalar ODE governing s is clearly nonlinear because its right-hand side is a nonlinear function of s, where the nonlinearity is the heat flux function at the liquid-solid interface. This nonlinearity, along with the non-constancy of the PDE's domain, is what makes control of this seemingly simple system quite challenging and entirely unconventional.

Stefan's PDE-ODE model gives rise to several control and state estimation problems. The early efforts on control of the Stefan problem are [6, 7, 8, 9]. Here we focus on a control problem that is both simple and difficult. The goal is to regulate the liquid-solid interface position $s(t)$ to a setpoint $s_r > 0$. This goal is depicted in Figure 4. The non-obvious thing to note is that, as the liquid-solid interface position $s(t)$ is regulated to its equilibrium value s_r, the temperature in both the liquid and the solid phases is being regulated to the melting/freezing temperature T_m. If this were not the case, namely, if the liquid were the be regulated substantially above, and the solid substantially below T_m, the liquid-solid interface position $s(t)$ would keep on moving, either melting more of the solid, or freezing more of the liquid.

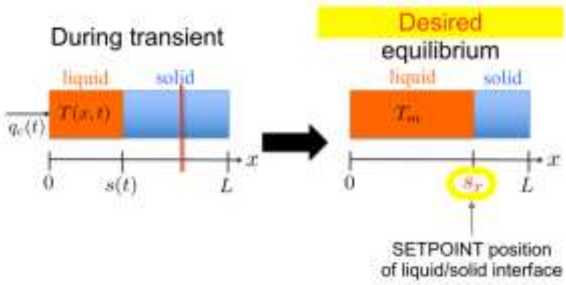

Fig. 4 A depiction of the control objective in the Stefan problem. The liquid-solid interface is regulated to the setpoint, while, at the same time, the temperature fields of both the liquid and the solid phases are being regulated to the melting/freezing temperature, which represents the thermal equilibrium in this problem.

Using the backstepping approach for PDE-ODE systems [4], we design and implement a feedback law $q_c(s, T)$ by using a laser to apply a heat flux to the liquid. This backstepping feedback is given by

$$q_c(t) = -ck\left(\frac{1}{\alpha}\int_0^{s(t)} (T(x,t) - T_m)dx + \frac{1}{\beta}(s(t) - s_r)\right)$$

where c is a positive gain constant. This backstepping control law is proportional to the error between the measured thermal energy and the thermal energy at the melting/freezing point, plus the interface tracking error $s - s_r$. The feedback law appears linear but it is not. The dependence of the upper limit of integration in x on the solid-liquid interface s is what makes this controller nonlinear, for the system which is nonlinear.

The backstepping approach entails construction of a Volterra transformation of the temperature state and a Lyapunov functional based on the transformed temperature state [10, 11].[1]

[1] http://a2c2.org/awards/o-hugo-schuck-best-paper-award

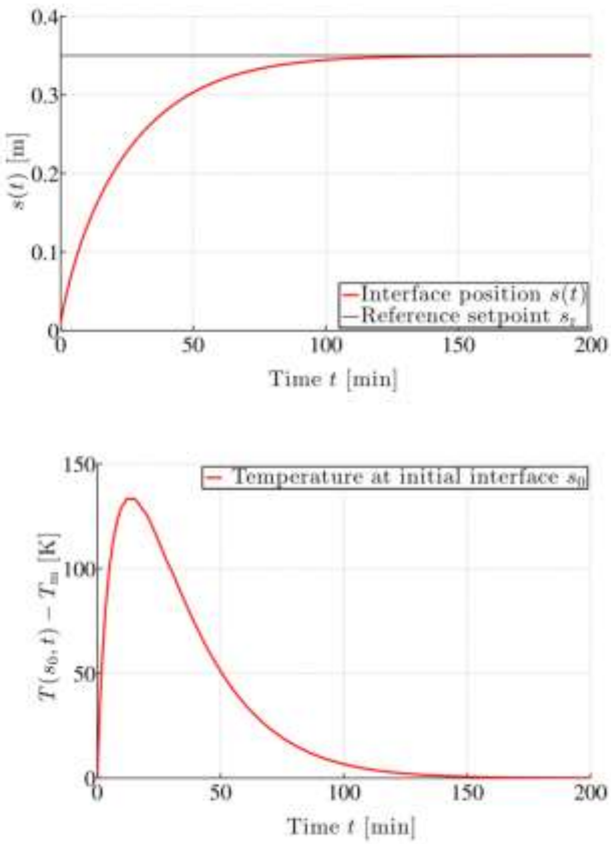

Fig. 5 Time evolution of the liquid-solid interface (top), which approaches its setpoint without an overshoot, and the temperature at the initial location of the liquid-solid interface (bottom) which starts from the melting point, has an upward excursion while the solid gets melted, and returns to the melting point, which is the system's thermal equilibrium. At no point does the temperature in the liquid phase fall below freezing. At no point does the heat flux get negative, which ensures the monotonicity of the motion of the liquid-solid interface and the absence of frozen islands within the liquid.

Figure 5 shows that the controller succeeds in its task. The solid-liquid interface is regulated to its setpoint. The temperature throughout the liquid domain is regulated to the melting point, which is the system's thermal equilibrium.

This control law achieves global stabilization for all initial conditions where the liquid temperature is above melting and the solid temperature is below freezing; both temperatures remain in these states for all time. In physical terms, this means that no solid islands form within the liquid and no pools of liquid form within the solid. The maximum principle for the heat equation establishes this result [12, 13].

4. Control of Moving Shock in Congested Traffic

The analog to the Stefan system's parabolic PDE phenomenon is the hyperbolic PDE phenomenon that arises in traffic. This originates with a moving shock that delineates the free traffic (upstream of shock) from the congested traffic (downstream from shock), as seen in Figure 6.

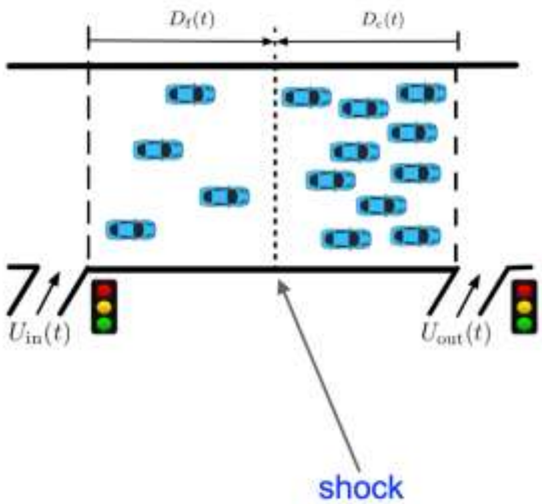

Fig. 6 Free traffic (upstream/left) and congested traffic (downstream/right) are separated by shock, depicted as a sharp increase in density. Modulating the durations of the red and green lights on the on-ramps regulate the shock location to a desired position.

The hyperbolic nonlinear Lighthill-Whitham-Richards PDE [14, 15], which acts as a simple delay for small deviations, models the traffic flow. A scalar ODE governs the shock motion, and the traffic densities of the congested and free traffic at the shock location form the ODE's inputs. This ODE represents the Rankine-Hugoniot jump condition that is common in compressible gas models. The PDE-ODE system is given by

$$\frac{\partial}{\partial t}\rho_f = -v_m \frac{\partial}{\partial x}\left(\rho_f - \frac{\rho_f^2}{\rho_m}\right)$$
$$\frac{\partial}{\partial t}\rho_c = -v_m \frac{\partial}{\partial x}\left(\rho_c - \frac{\rho_c^2}{\rho_m}\right)$$

$$\frac{d}{dt}l(t) = v_m - \frac{v_m}{\rho_m}(\rho_c(l(t),t) + \rho_f(l(t),t))$$

where the first PDE models the density of cars in the free traffic segment, the second PDE models the density in the congested traffic segment, and the ODE at the bottom models the motion of the free-congested interface *l*(*t*), namely, of the shock location.

If left uncontrolled, this system will exhibit the upstream motion of the shock, until the entire freeway is consumed by congestion. This is shown in Figure 7, which shows a simulation of the PDE model on the left and a simulation of a "microscopic" model on the right (where each car's motion is modeled individually).

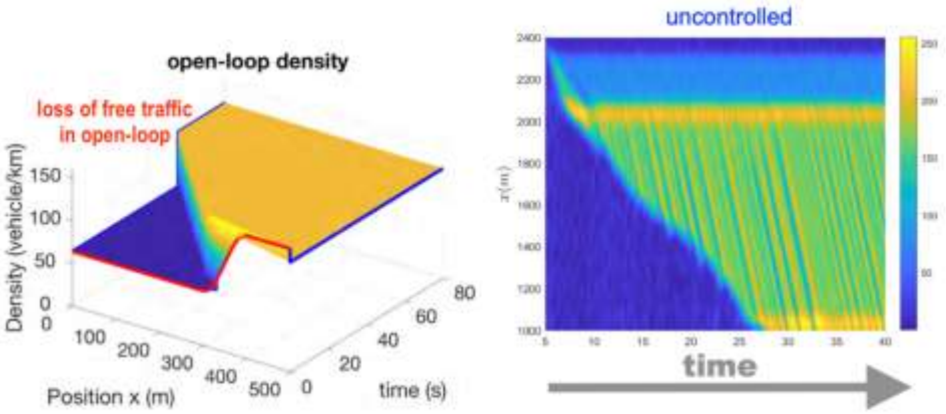

Fig. 7 Shock starting near the downstream end of the freeway segment propagates upstream until the entire freeway segment is consumed by congestion. Left: LWR PDE simulation. Right: "microscopic" simulation showing density of cars where blue denotes low density and yellow/green denotes high density, namely, congestion.

To prevent the loss of free traffic, we again use the PDE backstepping design to devise a feedback law that regulates the moving shock's position to a setpoint. This backstepping controller is given by the formulas

$$U_{\mathrm{in}}(t) = K_{\mathrm{f}} \left[X(t) - \frac{b}{u} \left(\int_0^{l(t)} \bar{\rho}_{\mathrm{f}}(\xi, t) d\xi + \int_{l(t)}^{\min\{L, 2l(t)\}} \bar{\rho}_{\mathrm{c}}(\xi, t) d\xi \right) \right]$$

$$U_{\mathrm{out}}(t) = K_{\mathrm{c}} \left[X(t) - \frac{b}{u} \left(\int_{l(t)}^{L} \bar{\rho}_{\mathrm{c}}(\xi, t) d\xi + \int_{\max\{0, 2l(t) - L\}}^{l(t)} \bar{\rho}_{\mathrm{f}}(\xi, t) d\xi \right) \right]$$

The variable U_{in} denotes the deviation of the density of cars at the inlet of the freeway segment relative to a setpoint, whereas the variable U_{out} denotes the deviation of the density of cars at the outlet of the freeway segment relative to a setpoint. The quantities K_{f} and K_{c} denote positive gain constants, whereas L denotes the length of the freeway segment.

The feedback laws above are implemented via "ramp metering," which involves modulation of the red and green lights on the freeway on-ramps around steady durations that correspond to the desired location of the shock.

Figure 8 illustrates the success of the feedback laws. They "arrest" the upstream drift of the shock and keep the segment of the freeway upstream of the shock in free, i.e., uncongested traffic.

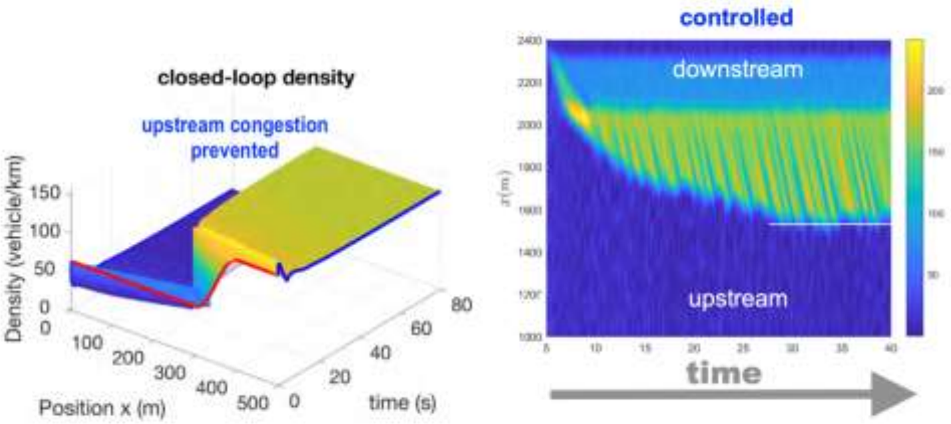

Fig. 8 The controllers implemented through ramp metering at the inlet and outlet of the freeway prevent the drift of the congested traffic beyond the setpoint for the shock. Hence, the upstream portion of the freeway is kept uncongested (blue denotes low density of cars in both pictures). Allowing the downstream portion of the freeway to be congested is important—not doing so would mean that many cars are prevented from entering the freeway and are instead kept on the ramps and on the streets leading to the ramps.

The similarity between the feedback laws for the Stefan (additive manufacturing) and the freeway problems are quite noticeable. Both feedbacks include integrals over varying spatial domains and both feedbacks also include the error between the measured interface position and the reference position.

Analyzing the PDE-ODE system with the feedback law once again employs a backstepping/Volterra transformation of the traffic density PDE's state, along with a resulting Lyapunov functional. Like with the Stefan system, stability occurs in the H_1 Sobolev norm. The details are contained in [16]. However, while stability for the Stefan system holds for all physically-meaningful initial conditions, it only holds locally—for small deviations of the density field around its equilibrium profile—for the traffic problem.

Another important result on control of an LWR-like model of traffic is [17].

4. CONCLUSIONS

In this tutorial exposition of two PDE control designs from distinct domains of physics and engineering, we have illustrated the current state-of-the art in designing controllers for infinite-dimensional systems modeled by PDEs with moving boundaries. These techniques are also applicable to a variety of other phase-change problems, including tumor growth and cancer treatment, lithium-ion batteries, and information propagation in social networks, as well as to multi-phase flows, fluid-structure interactions, and undersea construction using long cables.

Future research needs to advance these techniques from one spatial dimension to two and three spatial dimension, multi-PDE scenarios, and systems in which the interface is not governed by an ODE but by another PDE, possibly from a different class than in the main domain. An example of such a dynamical system is a biological cell whose membrane is governed by an elastic structural PDE model (second-order in time and fourth-order in space), while the interior is governed by a diffusion-dominated parabolic PDE.

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REFERENCES

[1] A. Isidori, *Nonlinear Control Systems*, Springer, 1989.
[2] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic, *Nonlinear and Adaptive Control Design*, Wiley, 1995.
[3] M. Krstić and A. Smyshlyaev, *Boundary Control of PDEs: A Course on Backstepping Designs*, SIAM, 2008.
[4] M. Krstić, *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*, Boston, MA: Birkhauser, 2009.
[5] J. Stefan, "Uber die Theorie der Eisbildung, insbesondere uber die Eisbildung im Polarmeere," *Annalen der Physik*, vol. 278, pp. 269–286, 1891.
[6] A. Armaou and P.D. Christofides, "Robust control of parabolic PDE systems with time-dependent spatial domains," *Automatica*, vol. 37, pp. 61–69, 2001.
[7] N. Petit, "Control problems for one-dimensional fluids and reactive fluids with moving interfaces," In *Advances in the theory of control, signals and systems with physical modeling*, volume 407 of *Lecture notes in control and information sciences*, pages 323–337, Lausanne, Dec 2010.
[8] B. Petrus, J. Bentsman, and B.G. Thomas, "Feedback control of the two-phase Stefan problem, with an application to the continuous casting of steel," In Proceedings of the 49th IEEE Conference on Decision and Control (CDC), 2010, pp. 1731–1736.
[9] M. Izadi and S. Dubljevic, "Backstepping output feedback control of moving boundary parabolic PDEs," *European Journal of Control*, vol. 21, pp. 27–35, 2015.
[10] S. Koga, M. Diagne, and M. Krstić, "Control and state estimation of the one-phase Stefan problem via backstepping design," *IEEE Transactions on Automatic Control*, vol. 64, pp. 510–525, 2019.
[11] S. Koga, I. Karafyllis, and M. Krstić, "Input-to-state stability for the control of Stefan problem with respect to heat loss at the interface," In Proceedings of the 2018 American Control Conference. Milwaukee, WI, 2018.
[12] A. Friedman "Free boundary problems for parabolic equations I. Melting of solids," *Journal of Mathematics and Mechanics*, vol. 8, no. 4, pp. 499–517, 1959.
[13] S. Gupta, *The Classical Stefan Problem. Basic Concepts, Modelling and Analysis*. North-Holland: Applied mathematics and Mechanics, 2003.
[14] M. J. Lighthill and G. B. Whitham, "On kinematic waves. II. A theory of traffic flow on long crowded roads," *Proc. Roy. Soc. London. Ser. A.*, 229 317–345, 1955.
[15] P. I. Richards, "Shock waves on the highway," *Operations Res*., 4, 42–51, 1956
[16] H. Yu, L.-G. Zhang, M. Diagne, and M. Krstic, "Bilateral boundary control of moving traffic shockwave," *IFAC Symposium on Nonlinear Control Systems*, 2019.
[17] I. Karafyllis, N. Bekiaris-Liberis, & M. Papageorgiou. "Feedback Control of Nonlinear Hyperbolic PDE Systems Inspired by Traffic Flow Models". *IEEE Transactions on Automatic Control*, 2018.