Fast assessment of the critical principal stress direction for multiple separated multiaxial loadings

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ABSTRACT. The critical plane calculation for multiaxial damage assessment is often a demanding task, particularly for large FEM models of real components. Anyway, in actual engineering requests, sometime, it is possible to take advantage of the specific properties of the investigated case.

This paper deals with the problem of a mechanical component loaded by multiple, but “time-separated”, multi-axial external loads. The specific material damage is dependent from the max principal stress variation with a significant mean stress sensitivity too.

A specifically fitted procedure was developed for a fast computation, at each node of a large FEM model, of the direction undergoing the maximum fatigue damage; the procedure is defined according to an effective stress definition based on the max principal stress amplitude and mean value.

The procedure is presented in a general form, applicable to the similar cases.

KEYWORDS. Multiaxial Fatigue; Principal Stress directions; Critical Plane.

INTRODUCTION

Multi-axial fatigue theories usually address problems characterized by different degrees of complexity. The simplest problems are typically the proportional loading conditions. In these situations, the principal stress directions are constant; they can be calculated at any time value and the variations (mean values, amplitudes or ranges) can be easily computed since, along each principal direction, principal stress is a known time-variable scalar. In this condition, many fatigue criteria based on principal stress value can be used; the most important is probably the Sines criterion [1] which is actually based on octahedral shear stress and such a value requires the calculation of the principal stress direction.

In a less specific way, under non-proportional loading, principal directions change in time; this variation is one of the most tricky problem in the investigation of the multi-axial fatigue damage, since there is not an intrinsic or natural frame of reference where defining the fundamental stress quantities.

Several research efforts, in multiaxial fatigue, deal with the definition and computation of critical plane direction or main principal frame of reference; for instance, just few examples can be found in [2-5].

Specifically, the main principal stress directions can be used to evaluate the consequent shear based critical plane [2]. Otherwise, the principal stress can be directly used for fatigue damage assessment [6], in this case the principal stress amplitudes and mean values are used in the criterion for equivalent stress amplitude definition.

Moreover, the variability of the principal directions, which is null under proportional loading, can be caused by several type of non-proportionalities. In the most complex cases, several independent time-variable component are superimposed. In these cases, the numerical iterative procedures are necessary.

More simple cases are given by a static loading combined with one time-variable component; in these cases, a proportional variability is combined with a non-proportional static loading; this is a generalization of the loading condition initially investigated in [7]. The overall stress condition is non-proportional, but it is less complex than previous general case.
The presented research addresses this last case. It is explicitly established for fatigue damage assessment of materials dependent on principal stress variations. Anyway, its application to more general cases is possible.
More precisely, the following discussion deals with the problem of determining the direction of maximum damage in cases of an external static loading combined with one or several variable loading, ranging from null to a maximum value (in the following called pulsating loadings).
- In the first case, we will deal with the treatment of one single external pulsating loading combined with a static load.
- In the second case, superimposed to a static load, there will be the combination of two pulsating loading; the time variable loading shall act separately, i.e. disjoint in time.

SUPERPOSITION OF INDEPENDENT STATIC AND PULSATING LOADING

Let us consider a point C of a structural component, loaded, in a period 0-T, by a static loading superimposed to an independent time-variable loading. The resulting stress tensor will be the sum of a constant value $[\sigma]_S$ and a time variable component. Assuming that the time variable components ranges from zero to a value $[\sigma]_V$, the overall stress tensor can be indicated as:

$$[\sigma](t)=[\sigma]_S+[\sigma]_V\, f_1(t) \tag{1}$$

Where, the function f_1 varies continuously from zero to 1 and back to zero, in the time interval from t_1 to t_3; elsewhere f_1 is null, see Fig. 1.

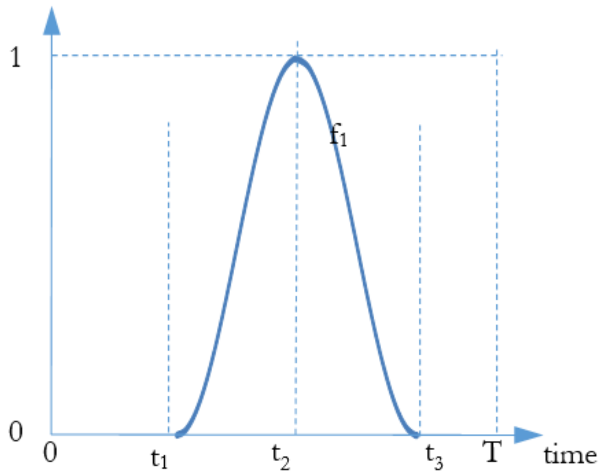

Figure 1: qualitative trend of function f1, related to the case of a single time variable component.

The versor "n" defines a generic direction at point C. The normal stress σ_n, acting on the plane defined by the versor "n", is given by:

$$\sigma_n(t)=n^T[\sigma](t)n=n^T[\sigma]_S\, n+n^T[\sigma]_V\, n\, f_1(t) \tag{2}$$

Due to the properties of the time variable function, the maxima and minima will necessarily occur at t_1 and t_2 alternatively. It is possible to evaluate the mean value and the amplitude of the normal stress:

$$\sigma_{n,m}=\frac{\sigma_{n,\max}+\sigma_{n,\min}}{2}=n^T[\sigma]_S\, n+\frac{1}{2}n^T[\sigma]_V\, n \tag{3}$$

$$\sigma_{n,a} = \frac{\sigma_{n,\max} - \sigma_{n,\min}}{2} = \frac{1}{2}\left| n^T \left[\sigma\right]_V n \right| = \max\left\{ \frac{1}{2} n^T \left[\sigma\right]_V n \,, -\frac{1}{2} n^T \left[\sigma\right]_V n \right\} \tag{4}$$

Where the max{} function identifies the maximum value among the values within the bracket.
At this stage, it is clear that the simple amplitude of the normal stress is dependent only from the time-variable stress tensor. Hence, the maximum value of normal stress amplitude is the half of the maximum eigenvalue of the $[\sigma]_V$ tensor; consequently, the direction "n" loaded by the maximum normal stress amplitude is the related eigenvector. The modulus is conveniently used in order to properly take into account even compressive loading.

$$\sigma_a = \max_n \left\{ \sigma_{n,a} \right\} = \frac{1}{2} \max\left\{ \left| eigvl\left[\sigma\right]_V \right| \right\} \tag{5}$$

Where eigvl[ ] function specifies the eigenvalues of the considered matrix.
Anyway, the problem turns out much more complex if, as usual, the consider material has any kind of sensibility to mean or hydrostatic stress components.
Let us consider a material showing a linear sensibility to mean value, i.e. the fatigue strength decreases linearly by increasing the mean stress value. This assumption is usually called "simplified Goodman" relationship. In this case, it is possible to consider an effective value of the stress amplitude by combining the actual stress amplitude with a portion of the applied mean stress value.

$$\sigma_{a,eq} = \sigma_a + b\,\sigma_m \tag{6}$$

The constant "b" is dependent from the material, it is usually positive and lower than 1.
The critical direction is no longer the direction previously identified, but the direction undergoing the maximum value of the equivalent amplitude.
For a general direction n, the equivalent amplitude value turns out to be:

$$\begin{aligned}
\sigma_{n,a,eq} &= \sigma_{n,a} + b\,\sigma_{n,m} = \\
&= \frac{1}{2}\left| n^T \left[\sigma\right]_V n \right| + b\left(n^T \left[\sigma\right]_S n + \frac{1}{2} n^T \left[\sigma\right]_V n \right) = \\
&= \max\left\{ \frac{1}{2} n^T \left[\sigma\right]_V n \,, -\frac{1}{2} n^T \left[\sigma\right]_V n \right\} + b\left(n^T \left[\sigma\right]_S n + \frac{1}{2} n^T \left[\sigma\right]_V n \right) = \\
&= \max\left\{ n^T \left[\frac{b+1}{2}\left[\sigma\right]_V + b\left[\sigma\right]_S \right] n \,, n^T \left[\frac{b-1}{2}\left[\sigma\right]_V + b\left[\sigma\right]_S \right] n \right\}
\end{aligned} \tag{7}$$

Finally, it is necessary to find the direction with the maximum value of equivalent amplitude. The linear combination of tensors is itself a tensor; hence, the maximum value, by changing the direction n, is again the maximum eigenvalue of the considered tensors.

$$\sigma_{a,eq} = \max_n \left\{ \sigma_{n,a,eq} \right\} = \max\left\{ eigvl\left[\frac{b+1}{2}\left[\sigma\right]_V + b\left[\sigma\right]_S \right], eigvl\left[\frac{b-1}{2}\left[\sigma\right]_V + b\left[\sigma\right]_S \right] \right\} \tag{8}$$

Applicative example
Let us consider a time variable tensile loading, ranging from 0 to 2S, so that its amplitude is S. The equivalent amplitude is simply (b+1)S.
Then, if the variable tensile loading is combined with a static torsional loading, the related tensors are:

$$\left[\sigma\right]_S = \begin{bmatrix} 0 & T & 0 \\ T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \left[\sigma\right]_V = \begin{bmatrix} 2S & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9}$$

If we assume that the tensile loading is actually positive, the terms "b+1" are predominant and the resulting equivalent stress amplitude is:

$$\sigma_{a,eq} = (b+1)\, S \left[\frac{1}{2} + \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{b}{b+1}\right)^2 \left(\frac{T}{S}\right)^2} \right] \tag{10}$$

SUPERPOSITION OF STATIC AND MULTIPLE INDEPENDENT PULSATING LOADINGS

The procedure above described can be extended to multiple time variable components. It is relevant if such components are independent and separated, i.e. if each one is different from zero only when other components are null.
Let us consider, for instance, another time variable function $[\sigma]_{V2}$, compared to the single one of Eq. 1.
With two separated time variable components, the resulting stress condition is:

$$[\sigma](t) = [\sigma]_S + [\sigma]_{V1}\, f_1(t) + [\sigma]_{V2}\, f_2(t) \tag{11}$$

An example of two separated time-functions is given in Fig. 2. If f_1 is different from 0 between t_1 and t_3, then f_2 will be different from 0 elsewhere, for instance from t_3 to t_5.

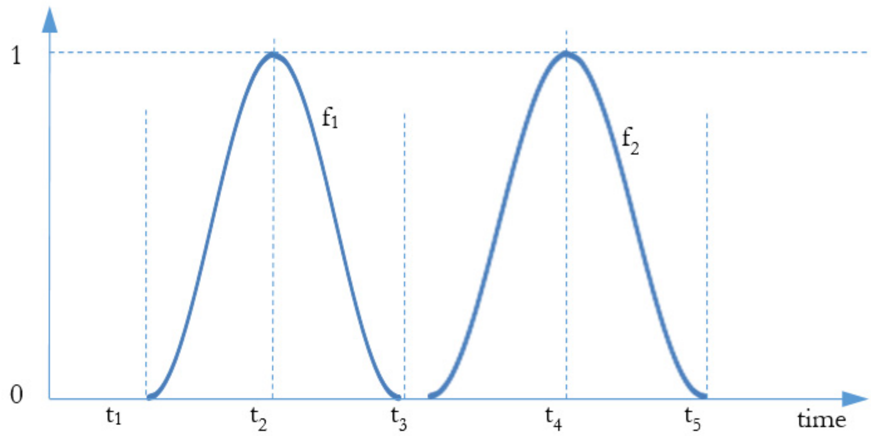

Figure 2: definition of two "separated" time functions.

Similarly to previous equations, the normal stress on the plane defined by tensor n is:

$$\sigma_n(t) = n^T [\sigma]_S\, n + n^T [\sigma]_{V1}\, n\, f_1(t) + n^T [\sigma]_{V2}\, n\, f_2(t) \tag{12}$$

In this case relative maxima and minima will occur at t_1 (or t_3 or t_5), t_2 and t_4. At these instants, necessarily an extreme occurs; generally, they could be local maxima or minima of σ_n.
For fatigue strength assessment, the maximum amplitude shall be computed; such a maximum amplitude is defined by the largest difference between the values obtained at extremes.
For instance, by investigating the extremes at t_2 and t_4, the mean value and relative amplitude shall be computed slightly differently from previous case:

$$\begin{aligned} \sigma_{n,m,2-4} &= \frac{1}{2}\left(\sigma_n(t_2) + \sigma_n(t_4)\right) = n^T [\sigma]_S\, n + \frac{1}{2} n^T [\sigma]_{V1}\, n + \frac{1}{2} n^T [\sigma]_{V2}\, n \\ \sigma_{n,a,2-4} &= \frac{1}{2}\left|\sigma_n(t_2) - \sigma_n(t_4)\right| = \frac{1}{2}\left|n^T [\sigma]_{V1}\, n - n^T [\sigma]_{V2}\, n\right| = \\ &= \max\left\{ \frac{1}{2} n^T \left[[\sigma]_{V1} - [\sigma]_{V2} \right] n,\ -\frac{1}{2} n^T \left[[\sigma]_{V1} - [\sigma]_{V2} \right] n \right\} \end{aligned} \tag{13}$$

According to previous considerations, the equivalent amplitude at a generic direction n, turns out:

$$\begin{aligned}\sigma_{n,a,eq,2-4} &= \sigma_{n,a} + b\,\sigma_{n,m} = \\ &= \max\left\{\frac{1}{2}n^T\left[[\sigma]_{V1} - [\sigma]_{V2}\right]n, -\frac{1}{2}n^T\left[[\sigma]_{V1} - [\sigma]_{V2}\right]n\right\} + \\ &\quad + b\left(n^T[\sigma]_S\, n + \frac{1}{2}n^T[\sigma]_{V1}\, n + \frac{1}{2}n^T[\sigma]_{V2}\, n\right) \\ &= \max\left\{n^T\left[\frac{b+1}{2}[\sigma]_{V1} + \frac{b-1}{2}[\sigma]_{V2} + b[\sigma]_S\right]n, n^T\left[\frac{b-1}{2}[\sigma]_{V1} + \frac{b+1}{2}[\sigma]_{V2} + b[\sigma]_S\right]n\right\}\end{aligned} \tag{14}$$

And the maximum value, by changing the direction n, is:

$$\begin{aligned}\sigma_{a,eq,2-4} &= \max_n\left\{\sigma_{n,a,eq}\right\} = \\ &= \max\left\{eigvl\left[\frac{b+1}{2}[\sigma]_{V1} + \frac{b-1}{2}[\sigma]_{V2} + b[\sigma]_S\right], eigvl\left[\frac{b-1}{2}[\sigma]_{V1} + \frac{b+1}{2}[\sigma]_{V2} + b[\sigma]_S\right]\right\}\end{aligned} \tag{15}$$

However, the relative amplitude between the extremes at t_2 and t_4 is only one of the possible critical amplitudes; the other possibilities are the differences between t_1 and t_2 and, finally, between t_1 and t_4. These two combination are similar to that investigated in the single pulsating loading case. Then the relative equivalent amplitudes are:

$$\sigma_{n,a,eq,1-2} = \max\left\{eigvl\left[\frac{b+1}{2}[\sigma]_{V1} + b[\sigma]_S\right], eigvl\left[\frac{b-1}{2}[\sigma]_{V1} + b[\sigma]_S\right]\right\} \tag{16}$$

$$\sigma_{n,a,eq,1-4} = \max\left\{eigvl\left[\frac{b+1}{2}[\sigma]_{V2} + b[\sigma]_S\right], eigvl\left[\frac{b-1}{2}[\sigma]_{V2} + b[\sigma]_S\right]\right\} \tag{17}$$

The researched equivalent amplitude is, finally, the maximum of each computed amplitude; hence the maximum eigenvalue of the suggested six tensors:

$$\sigma_{n,a,eq} = \max\left\{\begin{array}{cc} eigvl\left[\frac{b+1}{2}[\sigma]_{V1} + \frac{b-1}{2}[\sigma]_{V2} + b[\sigma]_S\right]; & eigvl\left[\frac{b-1}{2}[\sigma]_{V1} + \frac{b+1}{2}[\sigma]_{V2} + b[\sigma]_S\right]; \\ eigvl\left[\frac{b+1}{2}[\sigma]_{V1} + b[\sigma]_S\right]; & eigvl\left[\frac{b-1}{2}[\sigma]_{V1} + b[\sigma]_S\right]; \\ eigvl\left[\frac{b+1}{2}[\sigma]_{V2} + b[\sigma]_S\right]; & eigvl\left[\frac{b-1}{2}[\sigma]_{V2} + b[\sigma]_S\right]; \end{array}\right\} \tag{18}$$

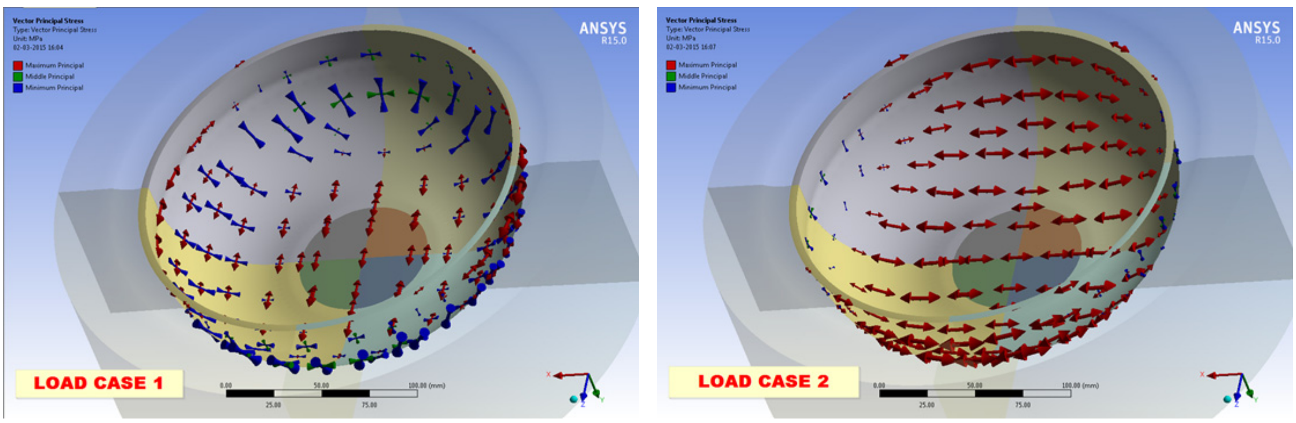

Figure 3: Stress distribution at minimum et maximum of fatigue loading.

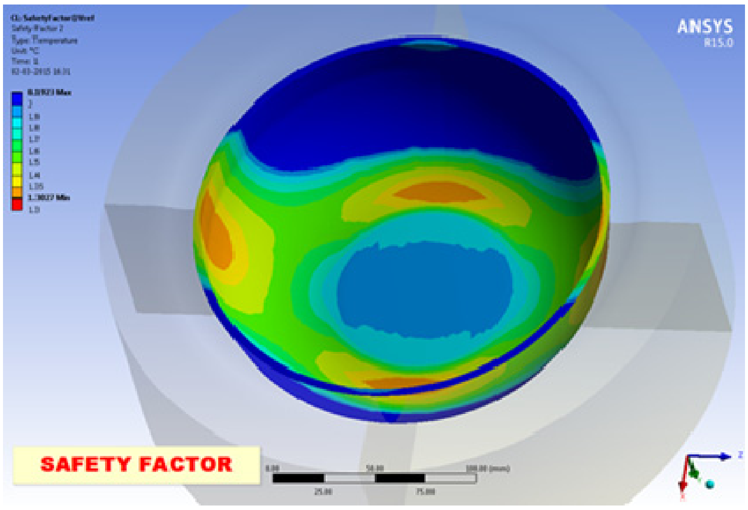

Figure 4: Safety factor and final result of the overall investigation.

NUMERICAL APPLICATION

The proposed procedure is written in an algebraic form in order to use it in a computer-assisted procedure and to apply it in fatigue damage assessment in FE models.

The method computes directly the equivalent stress amplitude, without time consuming iterative or recursive algorithms. This effectiveness is quite useful in large models where effective values shall be evaluated at each node; sometimes millions of nodes shall be processed.

Simply for instance, Fig. 3 and 4 show an example of an industrial application. This groove is a detail of a machine component subjected to a fatigue load history, which has two relevant states. The maximum principal stresses in these load cases are reported. Due to the complex 3D shape, the material is subjected to very different stress conditions, on a multi-axial fatigue perspective, from point to point. In these cases, it is crucial to have an automated multi-axial criterion to properly post-process the FEM results in order to identify the critical locations and the relative safety margin. Fig. 2 shows the safety factor plot according to the presented criterion.

CONCLUSIONS

The paper suggests a procedure for explicit and direct assessment of an equivalent stress amplitude under multiaxial fatigue loading. The procedure is appropriate only for particular loading conditions, specifically under a static loading combined with one or several independent and separated pulsating loadings.

Moreover, the proposed procedure is suitable for materials where fatigue damage is mainly dependent from principal stress values, for instance cast irons or brittle materials.

Anyway, under these particular conditions, the procedure is effective and can be easily used in any FE software, by providing a fast and efficient assessment of an equivalent value for the direct valuation of fatigue damage all over a mechanical component.

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