Estimation of the crack propagation direction in a mixed-mode geometry via multi-parameter fracture criteria

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ABSTRACT. The presented work introduces a numerical parametric study on the crack propagation direction under mixed-mode conditions (mode I + II). It is conducted for the geometry of an eccentric asymmetric four-point bending of a single edge notched beam specimen; various levels of mode-mixity are ensured by modifications in the crack length and crack eccentricity. The direction of crack propagation is estimated semi-analytically using both the maximum tangential stress criterion and the strain energy density criterion (implemented as a procedure within the used finite element computational code) as well as numerically (from verification reasons). Multi-parameter fracture mechanics is employed in the presented work for precise analytical evaluation of the stress field in the cracked specimen. This theory is based on description of the stress and deformation fields in the cracked body by means of their approximation using several initial terms of the Williams power series. Recent studies show that utilization of only first term of the series, which corresponds to the stress intensity factor (SIF), the single controlling parameter for the crack initiation and propagation assessment in brittle materials, is insufficient in many crack problems. It appears also in this study that the higher-order terms of the asymptotic crack-tip field are of great relevance for the conducted analysis, similarly to a number of other fracture phenomena (near-crack-tip stress field approximation, non-linear zone extent estimation, etc.).

KEYWORDS. Near-crack-tip fields; Williams expansion; Crack propagation direction; Multi-parameter fracture criteria; Finite element analysis.

INTRODUCTION

It is well known that the classical (one-parameter) fracture mechanics concept is based on the existence of the stress intensity factor that expresses the amplitude of the singular term describing the stress distribution in a cracked specimen. Nevertheless, this approach exhibits several limitations and one of them is the extent of the zone of non-linear behavior that has to be small enough in comparison to the typical structural dimensions. It is obvious that this restriction is too strong for a large group of materials, such as the quasi-brittle or the elastic-plastic ones. Fracture processes occurring in these cases are more complicated and many works have been concerned with developing approaches that will allow their precise description, see e.g. [1–3]. It can be shown that it is reasonable to take into account more parameters for the stress field approximation and the so-called multi-parameter fracture mechanics approach seems to be a very useful tool for assessment of fracture behavior of a wide range of materials. The introduced approach is based on the Williams solution of the crack-tip stress field distribution in a cracked specimen, see [4]. Extensive investigations in this field have been performed by the authors, see e.g. [5–8]. It can be noted that what is referred to as the so-called over-deterministic method has been used for estimation of values of the coefficients of the higher-order terms of the Williams expansion (WE). More details about the method can be found for instance in [9], or in some of the authors' works, e.g. [10].

In this work, a parametric study is carried out on a mixed-mode geometry and the initial kink angle of a crack is investigated for a wide range of mode-mixity situations. The multi-parameter form of two commonly used fracture criteria is introduced and tested. The main focus is on the discussion of the influence of considering the higher-order terms of the WE. Note that this work is only a part of the extensive ongoing research of the authors on the application of this multi-parameter approach in more advanced fracture mechanics tasks.

WILLIAMS EXPANSION AND MULTI-PARAMETER FRACTURE MECHANICS CONCEPT

The multi-parameter fracture mechanics concept consists in the idea that the crack-tip stress field is described by means of the Williams expansion [4]. Originally, the infinite power series was derived for a homogeneous elastic isotropic cracked body with an arbitrary remote loading and it can be written in the form:

$$\sigma_{ij} = \sum_{n=1}^{\infty} A_n \frac{n}{2} r^{\frac{n}{2}-1} f_{ij,\sigma}(n,\theta) + \sum_{m=1}^{\infty} B_m \frac{m}{2} r^{\frac{m}{2}-1} g_{ij,\sigma}(m,\theta) \quad , \quad i,j \in \{x,y\} \tag{1}$$

Similarly, the equation for the displacement vector components can be expressed as:

$$u_i = \sum_{n=1}^{\infty} r^{\frac{n}{2}} A_n f_{i,u}(n,\theta,E,\nu) + \sum_{m=1}^{\infty} r^{\frac{m}{2}} B_m g_{i,u}(m,\theta,E,\nu) \quad , \quad i \in \{x,y\} \tag{2}$$

Eq. 2 is important with regard to the method used for evaluation of the series coefficients, see the text below. Note that the approximation of the individual stress tensor and displacement vector components via a truncated series is used. The meaning of the symbols used in Eq. 1 and 2 is: σ_{ij} and u_i represent the stress tensor and displacement vector components, respectively; r, θ symbolize the polar coordinates (provided the centre of the coordinate system at the crack tip and the crack faces lying on the negative x-axis); $f_{ij,\sigma}$,$g_{ij,\sigma}$ and $f_{i,u}$, $g_{j,u}$ stand for the known functions and their expressions can be found in classical textbooks on fracture mechanics; E and ν represent Young's modulus and Poisson's ratio, respectively. The only unknown symbols are the coefficients A_n and B_m – their values depend on the relative crack length or generally, on the cracked specimen configuration. Therefore, it is necessary to calculate these parameters numerically and the method used within this work is described in the following text.

Over-deterministic method

The authors point out that there exist several methods suggested for estimation of the coefficients of the terms of the WE. In this paper, the so-called over-deterministic method (ODM) was chosen, taking into account its appreciable advantages. Whereas the use of hybrid crack elements (HCE, see e.g. [11,12]) or boundary collocation method (BCM, see e.g. [13]) requires advanced numerical procedures and special elements, ODM is based only on the knowledge of the displacement field around the crack tip and therefore, the conventional finite element analysis (FEA) is sufficient. Particularly, the displacements are determined in a set of nodes in the vicinity of the crack tip (usually at a ring around the crack tip) and together with polar coordinates of the nodes are then substituted as inputs into the Eq. 2. Thus, when k nodes are selected, a system of $2k$ algebraic equations for the variables A_n and B_m is created. More details, recommendations and/or restrictions regarding the application of the method can be found for instance in [9,14,15].

Fracture criteria

The literature survey of mixed mode fracture criteria can be find in [16]. For research in this contribution, two fracture criteria were chosen for estimation of the initial crack propagation direction: maximum tangential stress (MTS) criterion [17] and strain energy density (SED) criterion [18,19]. Both of them are well known and often used when the crack path shall be investigated.

The MTS criterion is independent on the plane stress or plane strain conditions (it is stress-based criterion) and it assumes that a crack will propagate in the direction where the tangential stress $\sigma_{\theta\theta}$ is maximal:

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \quad , \quad \frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2} < 0 \tag{3}$$

When the classical one-parameter fracture mechanics concept is preferred, an explicit equation for the crack propagation direction γ can be derived:

$$\gamma = 2 \arctan \frac{-2K_{\mathrm{II}}}{K_{\mathrm{I}} + \sqrt{K_{\mathrm{I}}^2 + 8K_{\mathrm{II}}^2}} \tag{4}$$

K_{I} and K_{II} represent the stress intensity factors and are directly related to the first Williams expansion terms:

$$K_{\mathrm{I}} = A_1\sqrt{2\pi} \quad \text{and} \quad K_{\mathrm{II}} = -B_1\sqrt{2\pi} \tag{5}$$

Note that when the multi-parameter fracture mechanics approach is utilized, the stress component $\sigma_{\theta\theta}$ in Eq. 3 is approximated via the WE and the maximum needs to be sought numerically.
Contrary to the MTS criterion, the SED criterion states that a crack will grow in the direction where the strain energy density S reaches its minimum:

$$\frac{\partial S}{\partial \theta} = 0 \quad , \quad \frac{\partial^2 S}{\partial \theta^2} > 0 \ , \quad \text{where} \quad S = \frac{1}{2\mu}\left[\frac{\kappa+1}{8}\left(\sigma_{rr} + \sigma_{\theta\theta}\right)^2 - \sigma_{rr}\sigma_{\theta\theta} + \sigma_{r\theta}^2\right] \tag{6}$$

Again, the Williams series expansion is used for approximation of the stress tensor components and the minimum is sought through numerical methods. Particularly, a procedure for searching for the extreme of the function is programmed in Wolfram Mathematica code [20]. The relevant quantities are expressed by means of the WE considering both various WE terms numbers (N = M = 1, 2, 3,... ,10) and various distances from the crack tip where the criterion is applied (r_c = 0.2, 0.4, 1.0, 1.5, 1.8 and 3.2 mm). For each configuration, the angle where the derivative of the corresponding quantity is zero is sought via an iterative method (Newton's method is default in the Mathematica code).

SPECIMEN GEOMETRY AND NUMERICAL MODEL

A mixed mode geometry of an eccentric asymmetric four point bending specimen (EA4PB), see a diagram of the geometry in Fig. 1, was chosen for the investigations presented.

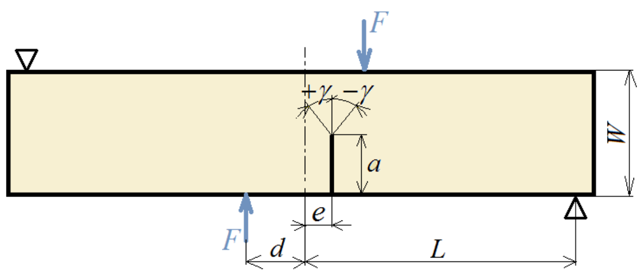

Figure 1: Eccentric asymmetric four point bending of a notched beam specimen under study.

A set of 45 various geometrical configurations was modelled and analyzed; the following dimensions were considered: half specimen length L = 100 mm, half span between the applied forces d = 20 mm, specimen width W = 40 mm, the relative crack length a/W varied from 0.1 up to 0.9 (in steps equal to 0.1) and the ratio between the crack eccentricity and the specimen width e/W was modelled in the range between 0 and 0.4 (with step 0.1). The idea was to cover a large range of mode-mixities; particularly, the ratio between the stress intensity factors of mode I and II varied from approximately 0 up to ca. 12. Note that the relation between the SIFs and the coefficients of the WE terms is described in Eq. 5. Although the $K_{\mathrm{I}}/K_{\mathrm{II}}$ ratio is used very often in fracture mechanics works, its using within this work is not fully correct with regard to the generalized/multi-parameter fracture mechanics concept applied. Nevertheless, no other more relevant parameter is known yet.

All the numerical simulations have been done in ANSYS finite element software [21]. For the two-dimensional linear elastic numerical model created, the following material properties were used: Young's modulus E = 40 GPa, Poisson's ratio ν = 0.2 and the loading force was F = 4 kN. Plane strain conditions were set. The specimen was meshed with PLANE82 elements, only the vicinity of the crack tip was specially refined: one row of special crack elements with shifted mid-side nodes was used around the crack tip and six more regular rings of nodes were modelled at the distances of r_c = 0.2, 0.4, 1.0, 1.5, 1.8 and 3.2 mm from the crack tip. These rings were utilized for the analyzes performed, particularly:

- the displacements and coordinates of the nodes at the defined rings were used for the evaluation of the initial 10 terms of the WE by means of the ODM, as it is described in the text above;
- the corresponding stress tensor components in the nodes at the defined rings were used for the direct evaluation of the fracture criteria.

In the following chapter, the values of the initial crack propagation angles are presented. Particularly, the angles calculated from the multi-parameter form of the fracture criteria (considering various numbers of the initial terms of the WE; $N = M$ = 1, 2, 4, 7 and 10) are compared to the FEM results determined purely numerically from the nodal solution performed in ANSYS. Results for all the cracked specimen configurations investigated are introduced and the influence of the distance from the crack, where the fracture criteria are applied, is discussed. Note that existence of the new/additional length parameter (r_c, the distance from the crack tip, where the criterion is applied) is a phenomenon connected to the application of the multi-parameter fracture mechanics approach. Generally, it is assumed that this value should be a material property, i.e. it should be constant for the particular material and should be related to material characteristics, see several suggestions in [22–24].

RESULTS AND DISCUSSION

This chapter is devoted to the presentation of the results obtained on the introduced mixed-mode cracked geometry. Particularly, the initial kink angle was investigated. The main attention is devoted to description and discussion of potential advantages of the application of the multi-parameter form of the fracture criteria suggested. The results are presented in the form of dependences of the initial crack propagation direction angle γ on the relative crack length ($\alpha = a/W$ = 0.1 ÷ 0.9). The plots are displayed for various relative crack eccentricities (e/W = 0.0, 0.1, 0.2, 0.3 and 0.4) and furthermore, the application of the fracture criteria at various distances (r_c = 0.2, 0.4, 1.0, 1.5, 1.8 and 3.2 mm) is studied. The data presented are of two kinds:

- the purely numerical results, denoted as "FEM": crack propagation direction angle γ is determined directly from the nodal solution of the FE analysis, i.e. the maximum of the tangential stress and minimum of the strain energy density, respectively is being sought at a given distance r_c from the crack tip;
- the semi-analytical results, denoted as "$N = M$ = 1, 2, 4, 7 and 10": crack propagation direction angle γ is determined from the multi-parameter/generalized form of the fracture criteria, i.e. the maximum of the tangential stress and minimum of the strain energy density, respectively is being sought iteratively from Eq. 3 and 6 when the stress tensor components are expressed by means of the WE (see Eq. 1) considering various numbers of the initial terms of the series.

The results obtained from the MTS criterion can be found in Fig. 2, those from the SED criterion in Fig. 3.

The dependences plotted in Fig. 2 and 3 enable to summarize the following statements and conclusions:

1) General results, Fig. 2 and 3:

- The results validate the theory that the fracture criteria in its classical one-parameter form ($N = M$ = 1) are independent on the length parameter, i.e. on the radial distance.
- The dependences obtained via SED criterion are not so sensitive to changing of the parameters (radial distance from the crack tip, crack eccentricity, number of the WE terms in the fracture criterion, etc.) as the dependences obtained via MTS criterion.
- The larger distance from the crack tip, the worse agreement of the kink angles calculated by means of the multi-parameter fracture criteria considering a low number of the WE terms ($N = M$ = 1, 2) with the kink angles determined numerically via FEM.
- Scatter of all the data presented grows with increasing r_c for each approximation. This effect is much more evident for long cracks, which is connected with the influence of the free surface and the necessity of the stress redistribution.

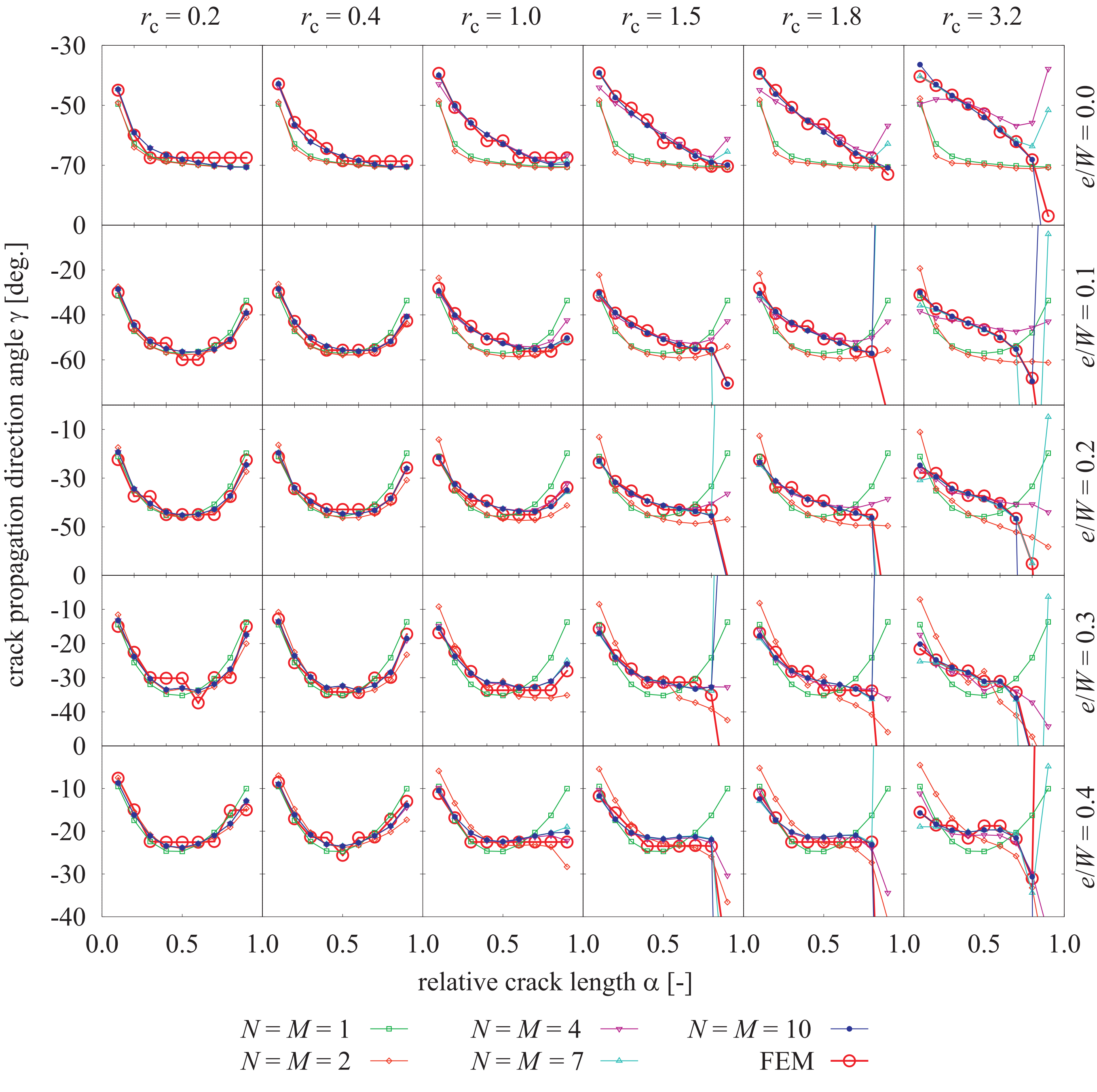

Figure 2: Dependences of the initial crack propagation direction angle γ on the relative crack length α for various relative crack eccentricities e/W obtained from the MTS fracture criterion at various radial distances from the crack tip r_c; the purely numerical results ("FEM") are compared to the semi-analytical results ("$N = M = 1$, 2, 4, 7 and 10") calculated via the multi-parameter form of the fracture criterion.

2) MTS fracture criterion, Fig. 2:

- Whereas the dependences of the kink angle calculated via FEM fluctuate at several points, the results obtained by means of the WE approximation seems to be more stable.
- The kink angle dependence calculated via FEM is changing with increasing r_c and this phenomenon can be well described by means of the multi-parameter form of the fracture criterion; $N = M = 4$ is sufficient up to $r_c = 1.5$ mm, $N = M \geq 7$ needs to be considered for larger distances from the crack tip.

- Prediction of the crack propagation angle by means of one or two terms of the WE is accurate enough if the distance from the crack tip is small enough (let's say up to r_c = 0.4 mm). The deviations from the numerical solution increase with larger distances.

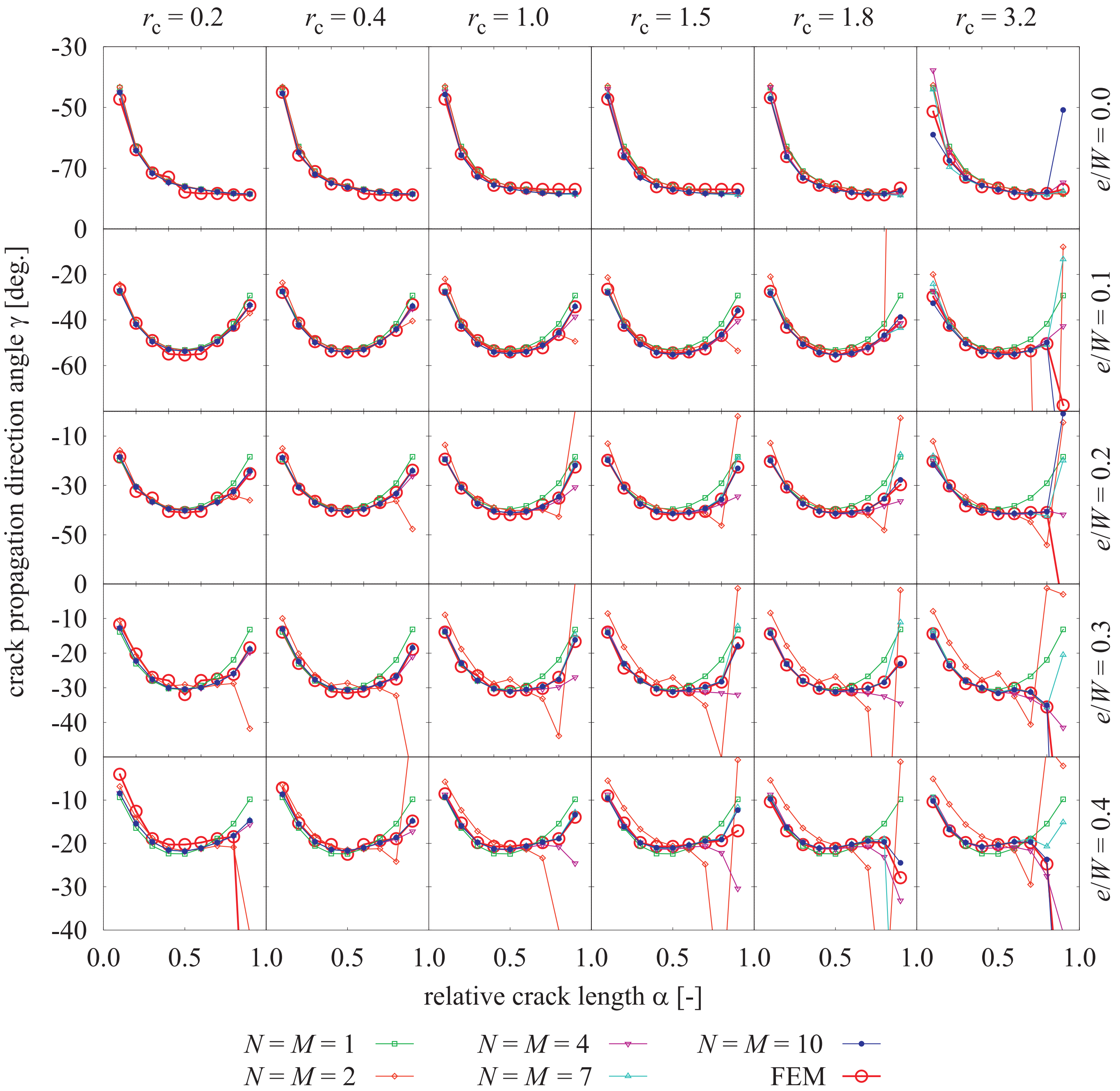

Figure 3: Dependences of the initial crack propagation direction angle γ on the relative crack length α for various relative crack eccentricities e/W obtained from the SED fracture criterion at various radial distances from the crack tip r_c; the purely numerical results ("FEM") are compared to the semi-analytical results ("$N = M$ = 1, 2, 4, 7 and 10") calculated via the multi-parameter form of the fracture criterion.

3) SED fracture criterion, Fig. 3:

- The dependences of the kink angle calculated via FEM behave more stably, value of the angle doesn't change so much as when the MTS criterion is applied.

- If the crack eccentricity is small (e/W = 0.0 or 0.1), the prediction of the crack propagation angle by means of the multi-parameter form of the criterion considering one or two terms of the WE seems to be accurate enough; the differences increase with growing crack eccentricity (especially for larger distances from the crack tip).
- The multi-parameter form of the SED fracture criterion is unavoidable if both the crack eccentricity and the distance from the crack tip are large.

CONCLUSIONS

The paper is focused on the application of the multi-parameter fracture mechanics approach to the task of the determination of the initial crack propagation angle. A parametric study is performed on an eccentric asymmetric four point bending specimen. Various mode-mixities (combinations of the mode I and II) are ensured by means of varying geometrical parameters of the cracked specimen (crack length and crack eccentricity). Two fracture criteria, MTS and SED, are presented. The kink angle calculated exclusively from the numerical solution by means of the FEM is compared to the angle calculated from the fracture criteria considering the WE approximation of the stress tensor components. Especially the influence of assuming various numbers of the initial WE terms is investigated and discussed. It can be concluded that the SED criterion is much less sensitive to variations in all the parameter studied and with the exception of the configuration with large crack eccentricity and large distances from the crack tip (where the criterion is applied) seems to be accurate enough in its classical one-parameter form. On the other hand, the MTS criterion exhibits much larger differences from the numerical results if only the first term of the WE is considered for the crack-tip stress field approximation. Again, this is essential especially if the criterion is applied farther from the crack tip. Therefore, it can be concluded that several more higher-order terms of the WE should be used for the stress field description when the distance from the crack tip is not small enough. This statement can be generalized not only for the crack propagation angle estimation task but also for more complicated issues within the nonlinear fracture mechanics field, when the nonlinear zone extent is comparable to the typical structural dimensions.

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