A link between the peak stresses and the averaged strain energy density for cracks under mixed-mode (I+II) loading

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ABSTRACT. In this work, a link between the averaged strain energy density (SED) approach and the peak stress method in the case of cracks subjected to mixed mode (I+II) loading has been investigated. Some closed-form expressions of the strain energy density, averaged in a volume of radius R_0, as function of the stress intensity factors are provided for plane strain conditions under mixed mode I+II loadings.

On the basis of the peak stress method (PSM) some expressions useful to estimate the mode I and mode II stress intensity factors (SIFs) have been recently derived. These relationships take advantage of the elastic peak stresses from FE analyses carried out by using a given mesh pattern where the element size and type are kept constants. The evaluation of the SIFs from a numerical analysis of the local stress field usually requires very refined meshes and then large computational effort. The usefulness of the PSM-based expressions is that (i) only the elastic peak stresses numerically evaluated at the crack tip are needed and not a set of stress–distance data; (ii) the employed meshes are rather coarse if compared to those necessary for the evaluation of the whole local stress field.

By substituting the PSM-based relationships in the closed-form expressions of the averaged SED it appears that the latter can be directly estimated by means of the elastic peak stresses evaluated at the crack tip.

Several FE analyses have been carried out on cracked plates subjected to tension loading considering different geometrical combinations, varying the length 2a and the inclination ϕ of the crack (i.e. the mode mixity) as well as the size d of the adopted finite elements, with the aim to evaluate the local SED and the elastic peak stress components σ_{peak} and τ_{peak}. In all cases the numerical values of the SED derived from the FE analyses have been compared with those analytically obtained by using the expressions for the SED based on the elastic peak stresses, in order to verify the range of applicability of the proposed relationships.

KEYWORDS. Local approaches; Stress intensity factors; Strain energy density; Finite elements; Coarse mesh.

INTRODUCTION

Notch stress intensity factors (NSIFs) play an important role in static strength assessments of components made of brittle or quasi-brittle materials and weakened by sharp V-shaped notches [1]. This holds true also for components made of structural materials undergoing high cycle fatigue loading [2] as well as for welded joints

[3,4]. In plane problems, the mode I and mode II NSIFs for sharp V-notches, which quantify the intensity of the asymptotic stress distributions in the close neighbourhood of the notch tip, can be expressed by means of the Gross and Mendelson's definitions [5]:

$$K_1 = \sqrt{2\pi} \cdot \lim_{r \to 0} \left[\left(\sigma_{\theta\theta} \right)_{\theta=0} \cdot r^{1-\lambda_1} \right] \qquad (1)$$

$$K_2 = \sqrt{2\pi} \cdot \lim_{r \to 0} \left[\left(\tau_{r\theta} \right)_{\theta=0} \cdot r^{1-\lambda_2} \right] \qquad (2)$$
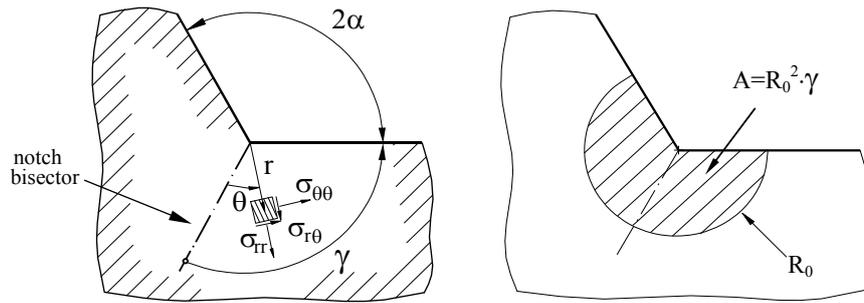

Figure 1: (a) Polar coordinate system centred at the notch tip. (b) Control volume (area) of radius R_0 surrounding the V-notch tip.

where (r,θ) is a polar coordinate system centred at the notch tip (Fig. 1a), $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ are the stress components according to the coordinate system and λ_1 and λ_2 are the mode I and mode II first eigenvalues in William's equations [6], respectively. The condition $\theta = 0$ characterizes all points of the notch bisector line. When the V-notch angle 2α is equal to zero, λ_1 and λ_2 equal 0.5 and K_1 and K_2 match the conventional stress intensity factors of a crack, K_I and K_{II}, according to the Linear Elastic Fracture Mechanics (LEFM).

The main practical disadvantage in the application of the NSIF-based approach is that very refined meshes are needed to calculate the NSIFs by means of definitions (1) and (2). The modelling procedure becomes particularly time-consuming for components that cannot be analysed by means of two-dimensional models.

Recently, Nisitani and Teranishi [7,8] presented a new numerical procedure suitable for estimating K_I for a crack emanating from an ellipsoidal cavity. Such a procedure is based on the usefulness of the linear elastic stress σ_{peak} calculated at the crack tip by means of FE analyses characterized by a mesh pattern having a constant element size. In particular Nisitani and Teranishi [7,8] were able to show that the ratio K_I/σ_{peak} depends only on the element size, so that the σ_{peak} value can be used to rapidly estimate K_I, provided that the adopted mesh pattern has been previously calibrated on geometries for which the exact value of K_I is known. This approach has been theoretically justified and extended also to sharp V-shaped notches subject to mode I loading [9] giving rise to the so-called Peak Stress Method (PSM), which can be regarded as an approximate FE-based method to estimate the NSIFs. Later on, the PSM has been extended to cracks subjected to mode I as well as mode II stresses [10]. The element size required to evaluate K_1 and K_2 from σ_{peak} and τ_{peak}, respectively, is several orders of magnitude greater than that required to directly evaluate the local stress field. The second advantage of using σ_{peak} and τ_{peak} is that only a single stress value is sufficient to estimate K_1 and K_2, respectively, instead of a number of stress-distance FE data, as usually made by applying definitions (1) and (2).

Since the units of the mode I and mode II NSIFs, K_1 and K_2, depend on the notch opening angle, a direct comparison of the NSIF values cannot be performed. This problem was overcome by Lazzarin and Zambardi [11], who proposed to use the total elastic strain energy density (SED) averaged over a sector of radius R_0 (Fig. 1b) for static [11-14] and fatigue [11,15,16] strength assessments. With reference to plane strain conditions, the SED value can be evaluated as follows:

$$\overline{W} = \frac{e_1}{E} \left[\frac{K_1}{R_0^{1-\lambda_1}} \right]^2 + \frac{e_2}{E} \left[\frac{K_2}{R_0^{1-\lambda_2}} \right]^2 \qquad (3)$$

where e_1 and e_2 [11] are two parameters which depend on the notch opening angle 2α and the Poisson's ratio ν. In principle, Eq. (3) is valid when the influence of higher order, non-singular terms can be neglected inside the control volume. In the case of short cracks or thin welded lap joints, for example, the T-stress must be included in the local SED evaluation [17].

Aims of the present contribution are as follows:
- to recall the fundamental concepts of the PSM for pure modes of loading;
- to present the extension of the PSM to the case of mixed mode (I+II) loading;
- to investigate a link between the SED approach and the PSM in the case of mixed mode (I+II) loading.

THE PEAK STRESS METHOD FOR PURE MODES OF LOADING

The Peak Stress Method (PSM) is a simplified numerical method to estimate the NSIFs parameters. Originally it was formulated for cases where only mode I singular stresses exists (i.e. $K_2 = 0$ or mode II stresses are negligible). It has been based on a link between the exact value of mode I NSIF K_1, see Eq. (1), and the linear elastic opening peak stress σ_{peak} calculated at the V-notch tip according to the following expression [9]:

$$K_{FE}^* = \frac{K_1}{\sigma_{peak} \cdot d^{1-\lambda_1}} \cong 1.38 \tag{4}$$

The PSM according to Eq. (4) was applied to correlate the fatigue strength of fillet-, full penetration and butt welded joints subjected to mode I loading [18-20].
Recently the Peak Stress Method has been extended also to mode II crack problems, linking the exact value of mode II NSIF K_2, see Eq. (2) with $2\alpha = 0°$ and $\lambda_2 = 0.5$, and the linear elastic sliding peak stress τ_{peak} calculated at the crack tip according to the following expression [10]:

$$K_{FE}^{**} = \frac{K_2}{\tau_{peak} \cdot d^{1-\lambda_2}} \cong 3.38 \tag{5}$$

In previous expressions d is the mean finite element size adopted when using the free mesh generation algorithm available in Ansys numerical code, while "exact NSIF values" must be meant as the values obtained using very refined FE mesh patterns in the numerical analyses and applying definitions (1) and (2) to the numerical results. Eqs. (4) and (5) are useful in practical applications because if the mean element size d is kept constant, then also K_1/σ_{peak} and K_2/τ_{peak} ratios are constant. Eqs. (4) and (5) are valid under the following conditions:
- use of 4-node linear quadrilateral elements, as implemented in ANSYS® numerical code (PLANE 42 of Ansys element library or alternatively PLANE 182 with K-option 1 set to 3);
- the pattern of finite elements around the V-notch tip must be that shown in Fig. 2b (see also [9, 10]); in particular, four elements share the node located at the crack tip;
- concerning Eq. (4), V-notches characterised by an opening angle 2α ranging from 0° to 135°;
- the ratio a/d must be greater than 3 in order to obtain $K_{FE}^* = 1.38 \pm 3\%$, a being the semi-crack length (or the notch depth when dealing with open V-notches). When mode II (sliding) stresses are of interest, meshes are more refined such that the ratio a/d must be greater than 14 in order to obtain $K_{FE}^{**} = 3.38 \pm 3\%$.

THE PEAK STRESS METHOD FOR MIXED MODE (I+II) LOADING

In the present paragraph the Peak Stress Method is extended to mixed mode (I+II) crack problems. Consider a crack ($2\alpha = 0°$) centred in a plate having the geometry reported in Fig. 2a and subjected to tensile loading. By varying the inclination angle ϕ of the crack it is possible to obtain different mode mixities, from pure mode I ($\phi = 0°$) to mixed mode I+II ($\phi > 0°$) loading. Different geometrical combinations have been considered, varying the projected crack length 2h (from 5 to 80 mm) and the inclination ϕ (from 0° to 60°) as well as the size of the element d, with the aim to investigate to what extent the PSM holds true. Finite element analyses have been performed by using the commercial code Ansys® and 4-node quadrilateral element (PLANE 42). The free mesh algorithm has been used in all numerical analyses and the sole control parameter set to generate the mesh has been the so-called 'global element size', i.e. the mean element size of the finite elements, which ranged from 0.5 mm to 10 mm. With the purpose of obtaining the pattern of finite

elements oriented along the crack bisector line (see Fig. 2b), the geometry of the plate has been divided into six areas, such that each crack tip is shared by four areas, as shown in Fig 2a. By so doing four elements (each one belonging to a different area) share the node located at the crack tip.

For the considered case, $K_1 = K_I$, $K_2 = K_{II}$, $\lambda_1 = \lambda_2 = 0.5$, while σ_{peak} and τ_{peak} represent the maximum elastic normal and tangential stress referred to the bisector line and evaluated at the crack tip according to Fig. 2a. The exact values of the mode I and mode II SIF, K_I and K_{II}, have been evaluated by means of further finite element analyses performed on the same geometries, but adopting very refined meshes (size of the smallest element of the order of 10^{-5} mm) in the close neighbourhood of the crack tip.

Fig. 3 plots the results of the numerical analyses in terms of the non-dimensional parameters K^{*}_{FE} and K^{**}_{FE} defined in Eqs. (4) and (5). The results for the cases $\phi = 10°$, $45°$, $30°$ and $60°$ have been reported. From Fig. 3, K^{*}_{FE} and K^{**}_{FE} are seen to converge to the previously calibrated values, that is 1.38 [9] and 3.38 [10], respectively, within a scatter band of the numerical results of $\pm3\%$ also in the case of mixed mode (I+II) loading.

This occurs for a ratio a/d greater than a value between 3 and 6, for mode I loading, and between 12 and 21, for mode II loading. It can be observed that the minimum a/d ratios to assure the validity of PSM under mixed mode (I+II) loading are only slightly greater than the results obtained in [9], in the case of pure mode I (a/d $\cong$ 3), and the results reported in [10] with reference to pure mode II (a/d $\cong$ 14). Furthermore, as highlighted in [10], it should be noted that the mode II loading is more critical to analyse with the PSM than the mode I loading because it requires more refined finite element mesh patterns.
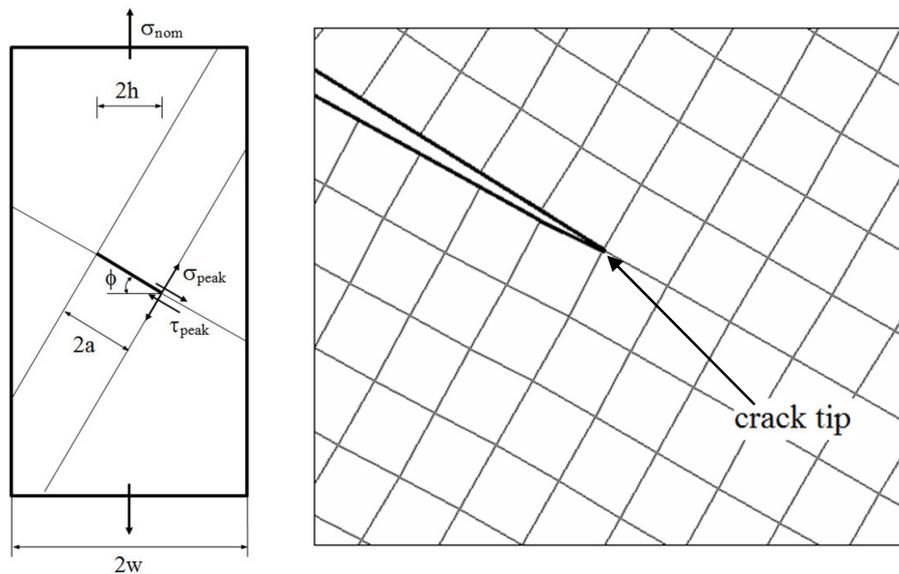

Figure 2: (a) Geometry and loading condition of the analysed mixed mode crack problem. 2W = 200 mm. (b) Pattern of finite elements around the singularity point: four elements share the node located at the crack tip.

A LINK BETWEEN THE PEAK STRESSES AND THE AVERAGED VALUE OF THE LOCAL STRAIN ENERGY DENSITY

In the present paragraph, a link between the averaged SED [11,12] and the peak stresses [9,10] in the case of cracks subjected to mixed mode (I+II) loading is investigated. By substituting the PSM-based relationships, Eqs. (4) and (5), in the closed-form expression of the averaged SED, Eq. (3), it appears that the latter can be directly estimated by means of the elastic peak stresses evaluated at the crack tip, σ_{peak} and τ_{peak}:

$$\overline{W}_{PSM} = \frac{e_1}{E}\left[K^{*}_{FE} \cdot \sigma_{peak} \cdot \left(\frac{d}{R_0}\right)^{1-\lambda_1} \right]^2 + \frac{e_2}{E}\left[K^{**}_{FE} \cdot \tau_{peak} \cdot \left(\frac{d}{R_0}\right)^{1-\lambda_2} \right]^2 \tag{6}$$

Several refined FE analyses have been carried out on the same cracked plates taken into consideration in the previous paragraph, with the aim to evaluate the local SED averaged over a control volume centred at the crack tip. Different geometrical combinations have been considered, varying the length 2a and the inclination ϕ of the crack (i.e. the mode mixity), while the radius of the control volume R_0 has been kept constant and equal to 0.1 mm.
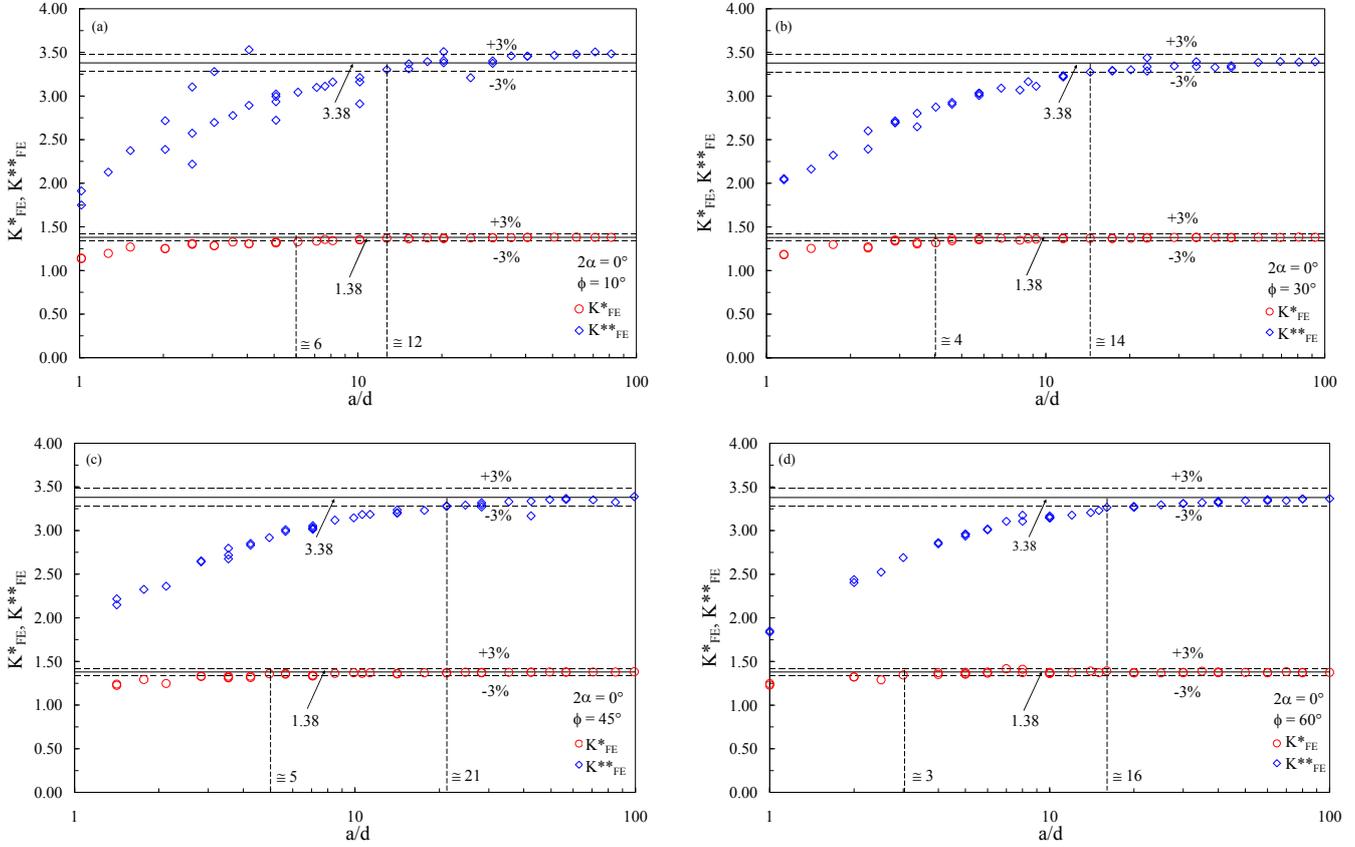

Figure 3: Calibration of the PSM approach for a crack (2α = 0°) under mixed mode (I+II) loading: (a) ϕ = 10°; (b) ϕ = 30°; (c) ϕ = 45°; (d) ϕ = 60°. Non-dimensional SIFs related to mode I and mode II.

The mode mixity ratio (MM) has been evaluated according to the following definition:

$$MM = \frac{K_{II}}{K_I + K_{II}} \tag{7}$$

Eq. (7) provides as master cases MM = 0 for pure mode I with ϕ = 0°, MM = 0.5 for mixed mode with ϕ = 45° and MM = 1 for pure mode II loading.

In all cases the numerical values of the SED calculated from the FE analyses have been compared with those analytically obtained by using the expressions for the SED based on the elastic peak stresses, Eq. (6), in order to verify the range of applicability of the proposed method.

Being the exact values of the SIFs available, the mean value of the SED has been evaluated also according to Eq. (3). In particular the maximum difference between the SED parameter evaluated analytically (Eq. (3)) and numerically (by FEM) resulted about 5%, that means that the influence of higher order terms, as the T-stress, can be neglected in these cases, at least from an engineering point of view.

The ratio between the SED based on the elastic peak stresses (Eq. 6, $\overline{W}_{PSM}$) and the SED calculated from the FE analyses ($\overline{W}_{FEM}$) has been reported in Fig. 4, with reference to an inclination ϕ of the crack equal to 0°, 30°, 45° and 60°. From Fig. 4, it can be observed that the ratio $\overline{W}_{PSM} / \overline{W}_{FEM}$ converges to unity, within a scatter band of ±10% for all different mode mixities taken into consideration. This occurs for a ratio a/d greater than 3 for the case MM = 0 (ϕ = 0°), 8.50 for

MM = 0.37 (ϕ = 30°), 11 for MM = 0.50 (ϕ = 45°) and 16 for MM = 0.63 (ϕ = 60°). In particular the minimum a/d ratio to assure the validity of the proposed method increases as the mode II loading becomes dominant, that is as the mode mixity ratio (MM) defined by Eq. (7) increases. A similar behavior has been observed in the previous paragraph and in [10], with reference to K^*_{FE} and K^{**}_{FE}.
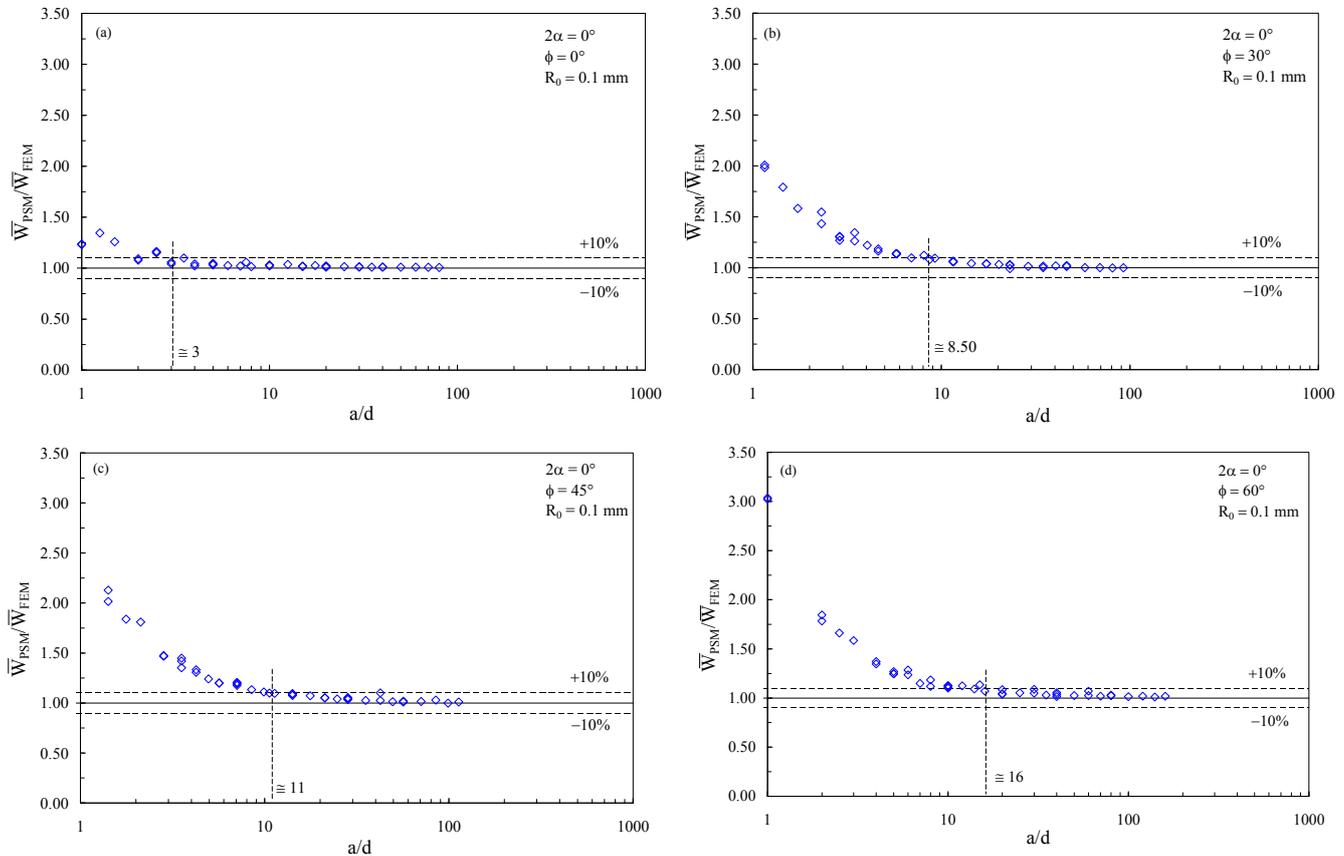

Figure 4: Ratio between the SED based on the elastic peak stresses ($\overline{W}_{PSM}$) and the SED derived from the FE analyses ($\overline{W}_{FEM}$) for a crack (2α = 0°) under mixed mode (I+II) loading: (a) ϕ = 0° (MM = 0), (b) ϕ = 30° (MM = 0.37), (c) ϕ = 45° (MM = 0.50) and (d) ϕ = 60° (MM = 0.63).

CONCLUSIONS

In the present contribution (*), a link between the averaged SED approach and the peak stress method in the case of cracks subjected to mixed mode (I+II) loading has been investigated:

- On the basis of the peak stress method, some expressions useful to estimate the mode I and mode II SIFs, recently derived for pure modes of loading, have been verified also in the case of mixed mode (I+II) crack problems.
- Since the normal and tangential peak stresses are proportional to the mode I and mode II SIFs, a link can immediately be established with the SED parameter by means of Eqs. (3), (4) and (5). By substituting the PSM-based relationships in the closed-form expressions of the averaged SED it appears that the latter can be directly estimated by means of the elastic peak stresses evaluated at the crack tip.
- The ratio between the SED based on the elastic peak stresses and the SED derived from the FE analyses converge to a unit value for a ratio a/d greater than a value between 3 (MM = 0) and 16 (MM = 0.63). The minimum a/d ratio to assure the validity of the proposed method increases with increasing the mode mixity ratio (MM).
- The usefulness of the SED expression based on the elastic peak stresses is that (i) only the elastic peak stresses numerically evaluated at the crack tip are needed and the definition of a control volume is no longer required;

(ii) the employed meshes are rather coarse if compared to those necessary for SED calculation, because the mean element size can be significantly greater than the radius of the control volume adopted for SED evaluation.

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