



## Crack initiation at V-notch tip subjected to in-plane mixed mode loading: An application of the fictitious notch rounding concept

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**ABSTRACT.** The fictitious notch rounding concept is applied for the first time to V-notches with root hole subjected to in-plane mixed mode loading. Out-of-bisector crack propagation is taken into account. The fictitious notch radius is determined as a function of the real notch radius, the microstructural support length and the notch opening angle. Due to the complexity of the problem, a method based on the simple normal stress failure criterion has been used. It is combined with the maximum tangential stress criterion to determine the crack propagation angle. An analytical method based on Neuber's procedure has been developed. The method provides the values of the microstructural support factor as a function of the mode ratio and the notch opening angle. The support factor is considered to be independent of the microstructural support length. Finally, for comparison, the support factor is determined on a purely numerical basis by iterative analysis of FE models.

**KEYWORDS.** Fictitious notch rounding; Mixed mode loading; V-notches with root hole; Microstructural support.

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### INTRODUCTION

All previous publications on the fictitious notch rounding (FNR) concept deal with the pure loading modes 1, 2 and 3. The difficulty of considering mixed mode loading conditions is due to the fact that the most critical direction, in which cracks might provisionally initiate and propagate, varies as a function of the mode ratio. This direction varies from the notch bisector line in the case of pure mode 1 loading, to a direction substantially out of the notch bisector line in the case of pure mode 2 loading. The present work extends the FNR concept to mixed mode 1 and 2 loading conditions for the first time and provides a solution to the problem as a function of the mode ratio.

The FNR concept [1-3] refers to the fact that the theoretical maximum notch stress does *not* characterise the static strength or fatigue strength of pointed or sharply rounded notches. The notch stress averaged over a short radial distance at pointed notches or over a small distance normal to the notch edge at rounded notches (real notch radius  $\rho$ ) is the key parameter of the method. The mentioned distance  $\rho^*$  is usually called 'microstructural support length'. In the high-cycle fatigue regime, notch stress averaging should take a path which coincides with the point and direction of fatigue crack initiation and propagation. The basic idea of the FNR concept is to determine the fatigue-effective averaged notch stress directly (i.e. without actual notch stress averaging) by performing the notch stress analysis with a fictitiously enlarged notch radius  $\rho_f$  given by

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$$\rho_f = \rho + s \rho^* \quad (1)$$

By taking advantage of the analytical frame provided by Filippi *et al.* [4] and Neuber [1], the FNR approach has been applied to V-notches under mode 1 and mode 3 loading, respectively [5, 6]. The support factor  $s$  was found to be highly dependent on the notch opening angle  $2\alpha$ . A satisfactory agreement was found between the theoretical stress concentration factor  $K_t(\rho_f)$  evaluated at the fictitiously rounded notch and the averaging stress concentration factor  $\bar{K}_t$  obtained by integrating the relevant stress over the distance  $\rho^*$  in the bisector line of the pointed V-notch.

It is worth mentioning that the notch stress averaging method was originally proposed for brittle fracture problems by Wiegardt [7] and later extended by Weiss [8]. The FNR concept was mathematically formalised by Neuber [1, 3, 9], who provided a general theoretical frame of the method. The application of the concept to strength assessments was demonstrated in Ref. [2]. There is also a correspondence of the concept with the ‘critical distance approach’ proposed and successfully applied by Peterson [10]. It was also applied many years later to notched thin plates subjected to fatigue loading [11, 12] using the characteristic length  $a_0$  derived by El-Haddad, Topper and Smith [13] from the conventional endurance limit and the threshold stress intensity factor range.

Referring to Neuber’s concept, Radaj [14-17] proposed to apply the FNR method to assess the high-cycle fatigue strength of welded joints (toe or root failures). A worst case assessment for welded low-strength steels, notch radius  $\rho = 0$  mm, microstructural support length  $\rho^* = 0.4$  mm and support factor  $s = 2.5$ , resulted in  $\rho_f = 1.0$  mm. This reference radius was found to be generally applicable to fatigue strength assessments of welded joints in structural steels and aluminium alloys and has become a standardised procedure within the design recommendations of the International Institute of Welding (IIW) [18].

A recent work has been devoted to the application of the FNR approach to notches with root hole subjected to pure mode 1 loading [19, 20]. Some analytical expressions have been derived for the fictitious notch radius  $\rho_f$  and the support factor  $s$  taking advantage of some closed form expressions specifically derived for V-notches with root hole [21]. An extension to pure mode 3 loading has been carried out in [22] on the basis of the same set of equations [21].

The case of in-plane shear loading (mode 2) is more complex to analyse than mode 1 and mode 3 loading. The reason for the higher complexity is the fact that out-of-bisector crack propagation is observed, because the maximum notch stress occurs outside the notch bisector line. The mode 2 problem was considered from a theoretical point of view resulting in closed-form solutions for elliptical notches and in numerical solutions for keyhole notches [23]. In a recent paper, pointed V-notches subjected to pure mode 2 loading were investigated [24]. Due to the complexity of the analytical developments, the support factor  $s$  was determined numerically using the FE method. The key problem was the choice of the crack path direction for stress averaging over the microstructural support length. Two criteria available from the literature were used to determine the angle of most probable crack propagation, the maximum tangential stress (MTS) criterion according to Erdogan-Sih [25] and the minimum strain energy density (MSED) criterion according to Sih [26].

Taking advantage of the closed form equations provided in Ref. [21] for V-notches with root hole, the FNR concept has been mathematically formalised for this notch shape in the case of in-plane shear loading [27]. Two analytical methods and one numerical method have been proposed in the quoted contribution for determining the fictitious notch radius  $\rho_f$  and therefrom the support factor  $s$  dependent on the notch opening angle  $2\alpha$ . In all three methods, the crack propagation angle has been determined based on the MTS criterion.

A state-of-the art review of the FNR approach comprising also the recent developments just mentioned has been carried out in Refs [28, 29]. While the application of the FNR approach to the pure loading modes is well developed, in-plane mixed mode loading remains an open problem which is difficult to solve for various reasons. The main difficulty is that the most probable crack propagation angle depends on the mode ratio ranging from pure mode 1 to pure mode 2 loading. The aim of the present paper is to provide a theoretically founded basis for the application of the FNR approach to in-plane mixed mode loading. Using the newly developed analytical frame [21], the values of the support factor  $s$  as a function of the mode ratio  $M$  and the notch opening angle  $2\alpha$  are obtained. As implied by Eq. (1), the support factor  $s$  is considered to be independent of the microstructural support length  $\rho^*$ , which is only approximately the case. But all the  $s$  values reported in the present contribution are on the safe side when used for strength assessments.

The analytical procedure given by Neuber [1, 3] for determining the factor  $s$  is applied in edge-normal directions outside the bisector line. This is an extension of what has been done by the authors [5, 6] in the cases of pure mode 1, mode 2 and mode 3 loading. By this method, V-notches with root hole, both in the real and in the fictitious configuration, are considered. Pointed notches are the limit case obtained when the real notch root radius tends to zero. The proposed method requires a numerical solution of the rather complex governing equation to determine the values of  $s$ , but easy-to-survey tables and diagrams present these values as a function of the mode ratio  $M$  for various notch opening angles  $2\alpha$  in



order to provide a simple tool for the engineering application of the FNR approach in the case of mixed mode loading. Finally, for verification of the method, the factor  $s$  is also determined on a purely numerical basis by iteration of FE models.

### ANALYTICAL FRAME FOR V-NOTCHES WITH ROOT HOLE SUBJECTED TO MIXED MODE LOADING CONDITIONS

Using the normal stress criterion in combination with the MTS criterion (for the crack propagation angle), an analytical method has been developed for evaluating the fictitious notch radius  $\rho_f$  and the support factor  $s$  therefrom as a function of the mode ratio  $M$  defined below. The method refers to Fig. 1 where the real root radius  $\rho$  is substituted by the fictitious notch radius  $\rho_f$ .

The V-notch with root hole subjected to mode 1 loading is considered. Taking the relevant boundary conditions into account, the stress components for the symmetric mode result from Ref. [21]:

$$\sigma_{\theta\theta} = \frac{K_{1\rho}}{\sqrt{2\pi}} \frac{r^{\lambda_1-1}}{(1+\lambda_1)+\varphi_1(\gamma)} \left\{ \cos(1-\lambda_1)\theta \left[ (1+\lambda_1) + \tilde{\psi}_{11}(\theta) \left(\frac{\rho}{r}\right)^{2\lambda_1} + \tilde{\psi}_{12}(\theta) \tilde{\chi}_{11}(\theta) \left(\frac{\rho}{r}\right)^{2\lambda_1+1} \right] + \varphi_1(\gamma) \cos(1+\lambda_1)\theta \left[ 1 + (1-\lambda_1) \left(\frac{\rho}{r}\right)^{2\lambda_1} + (2+\lambda_1) \left(\frac{\rho}{r}\right)^{2(\lambda_1+1)} \right] \right\} \quad (2.1)$$

$$\sigma_{rr} = \frac{K_{1\rho}}{\sqrt{2\pi}} \frac{r^{\lambda_1-1}}{(1+\lambda_1)+\varphi_1(\gamma)} \left\{ \cos(1-\lambda_1)\theta \left[ (3-\lambda_1) - \tilde{\psi}_{11}(\theta) \left(\frac{\rho}{r}\right)^{2\lambda_1} - \tilde{\psi}_{12}(\theta) \tilde{\chi}_{11}(\theta) \left(\frac{\rho}{r}\right)^{2\lambda_1+1} \right] + \varphi_1(\gamma) \cos(1+\lambda_1)\theta \left[ (3+\lambda_1) \left(\frac{\rho}{r}\right)^{2\lambda_1} - 1 - (2+\lambda_1) \left(\frac{\rho}{r}\right)^{2(\lambda_1+1)} \right] \right\} \quad (2.2)$$

$$\tau_{r\theta} = \frac{K_{1\rho}}{\sqrt{2\pi}} \frac{r^{\lambda_1-1}}{(1+\lambda_1)+\varphi_1(\gamma)} \left\{ \sin(1-\lambda_1)\theta \left[ (1-\lambda_1) + \tilde{\psi}_{11}(\theta) \left(\frac{\rho}{r}\right)^{2\lambda_1} + \tilde{\psi}_{12}(\theta) \tilde{\chi}_{12}(\theta) \left(\frac{\rho}{r}\right)^{2\lambda_1+1} \right] + \varphi_1(\gamma) \sin(1+\lambda_1)\theta \left[ (1+\lambda_1) \left(\frac{\rho}{r}\right)^{2\lambda_1} + 1 - (2+\lambda_1) \left(\frac{\rho}{r}\right)^{2(\lambda_1+1)} \right] \right\} \quad (2.3)$$

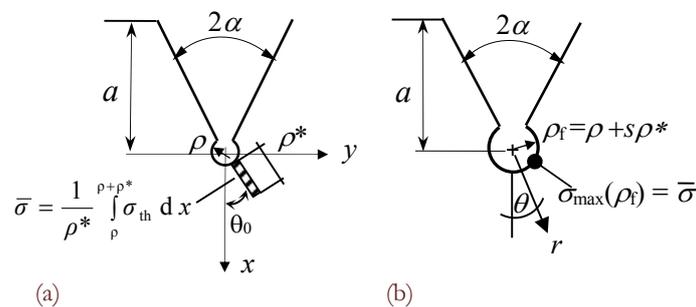


Figure 1: Fictitious notch rounding concept applied to mixed mode loading (mode 1+2): real root hole notch with stress averaged over  $\rho^*$  in direction of crack propagation (a) and substitute root hole notch with fictitious notch radius  $\rho_f$  producing  $\sigma_{max} = \bar{\sigma}$  (b).

The parameter  $\lambda_1$  is Williams' [30] mode 1 eigenvalue, which is dependent on the notch opening angle  $2\alpha$ .

The generalised mode 1 notch stress intensity factor  $K_{1\rho}$  can be expressed as follows:



$$K_{1\rho} = \frac{\sqrt{2\pi} r^{1-\lambda_1} \sigma_{\theta\theta}(r,0) [(1+\lambda_1) + \varphi_1(\gamma)]}{\left\{ g_1 + g_2 \left(\frac{\rho}{r}\right)^{2\lambda_1} + g_3 \left(\frac{\rho}{r}\right)^{2\lambda_1+1} + g_4 \left(\frac{\rho}{r}\right)^{2\lambda_1+2} \right\}} \quad (3)$$

When the notch radius  $\rho$  tends to zero,  $K_{1\rho}$  tends to the mode 1 notch stress intensity factor  $K_1$  defined according to Gross and Mendelson [31]:

$$K_1 = \sqrt{2\pi} \lim_{r \rightarrow 0} \sigma_{\theta} r^{1-\lambda_1} \quad (4)$$

Considering the V-notch with root hole under mode 2 loading, the stress components for the antisymmetric mode result also from Ref. [21]:

$$\begin{aligned} \sigma_{\theta\theta} = & -\frac{K_{2\rho}}{\sqrt{2\pi}} \frac{r^{\lambda_2-1}}{(1-\lambda_2) + \varphi_2(\gamma)} \left\{ \sin(1-\lambda_2)\theta \left[ (\lambda_2+1) - \tilde{\psi}_{21}(\theta) \left(\frac{\rho}{r}\right)^{2\lambda_2} - \tilde{\psi}_{22}(\theta) \tilde{\chi}_{22}(\theta) \left(\frac{\rho}{r}\right)^{2\lambda_2+1} \right] + \right. \\ & \left. + \varphi_2(\gamma) \sin(1+\lambda_2)\theta \left[ 1 + (1-\lambda_2) \left(\frac{\rho}{r}\right)^{2\lambda_2} + (2+\lambda_2) \left(\frac{\rho}{r}\right)^{2(\lambda_2+1)} \right] \right\} \end{aligned} \quad (5.1)$$

$$\begin{aligned} \sigma_{rr} = & -\frac{K_{2\rho}}{\sqrt{2\pi}} \frac{r^{\lambda_2-1}}{(1-\lambda_2) + \varphi_2(\gamma)} \left\{ \sin(1-\lambda_2)\theta \left[ (3-\lambda_2) + \tilde{\psi}_{21}(\theta) \left(\frac{\rho}{r}\right)^{2\lambda_2} + \tilde{\psi}_{22}(\theta) \tilde{\chi}_{22}(\theta) \left(\frac{\rho}{r}\right)^{2\lambda_2+1} \right] + \right. \\ & \left. - \varphi_2(\gamma) \sin(1+\lambda_2)\theta \left[ 1 - (3+\lambda_2) \left(\frac{\rho}{r}\right)^{2\lambda_2} + (2+\lambda_2) \left(\frac{\rho}{r}\right)^{2(\lambda_2+1)} \right] \right\} \end{aligned} \quad (5.2)$$

$$\begin{aligned} \tau_{r\theta} = & -\frac{K_{2\rho}}{\sqrt{2\pi}} \frac{r^{\lambda_2-1}}{(1-\lambda_2) + \varphi_2(\gamma)} \left\{ \cos(1-\lambda_2)\theta \left[ (1-\lambda_2) - \tilde{\psi}_{21}(\theta) \left(\frac{\rho}{r}\right)^{2\lambda_2} - \tilde{\psi}_{22}(\theta) \tilde{\chi}_{21}(\theta) \left(\frac{\rho}{r}\right)^{2\lambda_2+1} \right] + \right. \\ & \left. + \varphi_2(\gamma) \cos(1+\lambda_2)\theta \left[ 1 + (1+\lambda_2) \left(\frac{\rho}{r}\right)^{2\lambda_2} - (2+\lambda_2) \left(\frac{\rho}{r}\right)^{2(\lambda_2+1)} \right] \right\} \end{aligned} \quad (5.3)$$

The parameter  $\lambda_2$  is Williams' [30] mode 2 eigenvalue, which is dependent on the notch opening angle  $2\alpha$ . The generalised mode 2 notch stress intensity factor  $K_{2\rho}$  can be expressed as follows:

$$K_{2\rho} = \frac{\sqrt{2\pi} r^{1-\lambda_2} \tau_{r\theta}(r,0)}{\left[ 1 + b_1 \left(\frac{\rho}{r}\right)^{2\lambda_2} + b_2 \left(\frac{\rho}{r}\right)^{2\lambda_2+1} + b_3 \left(\frac{\rho}{r}\right)^{2\lambda_2+2} \right]} \quad (6)$$

When the notch root radius tends to zero,  $K_{2\rho}$  tends to the mode 2 notch stress intensity factor  $K_2$  defined according to the following expression:

$$K_2 = \sqrt{2\pi} \lim_{r \rightarrow 0} \tau_{r\theta} r^{1-\lambda_2} \quad (7)$$

It is important to underline that the property of  $K_{2\rho}$  to converge to the stress intensity factor of the pointed V-notch,  $K_2$ , when the notch root radius tends to zero, is a characteristic of the set of equations provided in Ref. [21] for V-notches with root hole. Other sets of equations [32, 33] applicable to rounded V-notches of different shape do not have this property as discussed in Ref. [34]. In that paper it was also documented the stable trend of the generalized notch stress



intensity factors given in Eqs (4) and (8) as a function of the distance  $r$ . The problem of the oscillating trend of  $K_{1\rho}$  and  $K_{2\rho}$  described in [33] for blunt V-notches with flanks tangent to the notch root radius is overcome by employing the set of equations reported in [21]. Fictitious notch rounding is considered for V-notches with root hole subjected to mode 1 and mode 2 loading based on Eqs. (2.1) and (5.1). Based on the MTS criterion, the crack propagation angle  $\theta_0$  is obtained from the following condition

$$d\sigma_\theta/d\theta = 0 \tag{8}$$

The hoop stress field  $\sigma_{\theta\theta}$  depends on the position variables  $r$  and  $\theta$  in such a way the condition  $d\sigma_\theta/d\theta = 0$  does not determine the angle of crack propagation unless a value of  $r$  is specified. The angle  $\theta_0$  increases by increasing the distance  $r$  due to the higher contribution of Mode 2 with respect to Mode 1 loading at greater distances from the notch tip. In this paper the values of  $\rho=0$  (worst case configuration) and  $r=0.005$  mm (early crack propagation) have been set to evaluate the crack initiation angle  $\theta_0$ .

It will be shown in the following that the specific value of  $r$  allows us to match the values of  $s$  obtained for pure mode 1 [20] and pure mode 2 [27]. The normal stress criterion is now introduced for the averaged stress  $\bar{\sigma}$  and the MTS criterion for the crack propagation angle  $\theta_0$ . Based on Eqs. (2.1) and (5.1) the maximum theoretical notch stress  $\sigma_{th}(r, \theta_0)$  is obtained and the averaged stress in the direction of  $\theta_0$  is determined:

$$\bar{\sigma} = \frac{1}{Q^*} \int_{\rho}^{\rho+\rho^*} \sigma_{th}(r, \theta_0) dr \tag{9}$$

The determinate integral in Eq. (9) has first been solved in its indeterminate form separated into mode 1 and mode 2 components (termed ii1 and ii2), and only then in its determinate form (termed di1 and di2).

□ The determinate integrals referring to mode 1 and mode 2, respectively, result in the following form:

$$di1(\rho, \rho^*, \theta_0) = ii1(\rho + \rho^*, \theta_0) - ii1(\rho, \theta_0) \tag{10}$$

$$di2(\rho, \rho^*, \theta_0) = ii2(\rho + \rho^*, \theta_0) - ii2(\rho, \theta_0) \tag{11}$$

For the sake of brevity of presentation and due to the length of the final expressions, the explicit forms of di1 and di2 are omitted here. The limit value of the average stress for  $\rho^* \rightarrow 0$  (and  $\rho = \rho_f$ ) is:

$$\lim_{\rho^* \rightarrow 0} di1(\rho, \rho^*, \theta_0) = \frac{\rho_f^{\lambda_1-1} K_{1\rho} \{4 \cos[(1 + \lambda_1)\theta_0] \varphi_1 + \cos[(1 - \lambda_1)\theta_0] (1 + \lambda_1 + \tilde{\psi}_{12}(\theta_0) \tilde{\chi}_{11}(\theta_0) + \tilde{\psi}_{11}(\theta_0))\}}{\sqrt{2\pi} (1 + \lambda_1 + \varphi_1)} \tag{12}$$

$$\lim_{\rho^* \rightarrow 0} di2(\rho, \rho^*, \theta_0) = \frac{\rho_f^{\lambda_2-1} K_{2\rho} \{4 \sin[(1 + \lambda_2)\theta_0] \varphi_2 + \sin[(1 - \lambda_2)\theta_0] (1 + \lambda_2 - \tilde{\psi}_{22}(\theta_0) \tilde{\chi}_{22}(\theta_0) - \tilde{\psi}_{21}(\theta_0))\}}{\sqrt{2\pi} (+\lambda_2 - \varphi_2 - 1)} \tag{13}$$

The equation  $\bar{\sigma}(\rho, \rho^*) = \lim_{\rho^* \rightarrow 0} (\bar{\sigma})$ , according to the procedure given by Neuber [1,3], results in the following equation with di1( $\rho, \rho^*, \theta_0$ ) and di2( $\rho, \rho^*, \theta_0$ ) according to Eqs. (10-11):

$$di1(\rho, \rho^*, \theta_0) - di2(\rho, \rho^*, \theta_0) = \frac{\rho_f^{\lambda_1-1} K_{1\rho} \{4 \cos[(1 + \lambda_1)\theta_0] \varphi_1 + \cos[(1 - \lambda_1)\theta_0] (1 + \lambda_1 + \tilde{\psi}_{12}(\theta_0) \tilde{\chi}_{11}(\theta_0) + \tilde{\psi}_{11}(\theta_0))\}}{\sqrt{2\pi} (1 + \lambda_1 + \varphi_1)} - \frac{\rho_f^{\lambda_2-1} K_{2\rho} \{4 \sin[(1 + \lambda_2)\theta_0] \varphi_2 + \sin[(1 - \lambda_2)\theta_0] (1 + \lambda_2 - \tilde{\psi}_{22}(\theta_0) \tilde{\chi}_{22}(\theta_0) - \tilde{\psi}_{21}(\theta_0))\}}{\sqrt{2\pi} (+\lambda_2 - \varphi_2 - 1)} \tag{14}$$



Solving Eq. (14), with  $K_{1\rho}=K_1$  and  $K_{2\rho}=K_2$ , it is possible to determine the fictitious radius  $\rho_f$  and therefrom the support factor  $s$  by using the following expression derived from Eq. (1):

$$s = \frac{\ell_f - \rho}{\rho^*} \quad (15)$$

The factor  $s$  under mixed mode loading results as a function of the mode 1 and mode 2 stress intensity factors and of the crack propagation angle  $\theta_0$ . Eq. (14) can be solved in general terms for any ratio  $K_{2\rho}/K_{1\rho}$  of the notch stress intensity factors, the crack propagation angle  $\theta_0$  being dependent on this ratio. The mode ratio  $M$  is defined as follows based on the ratio  $\chi$  of the notch stress intensity factors  $K_{2\rho}$  and  $K_{1\rho}$ :

$$M = \frac{2}{\pi} \arctan \chi \quad (16)$$

$$\chi = \frac{K_{2\rho}}{K_{1\rho}} \ell_f^{\lambda_2 - \lambda_1} \quad (17)$$

Pure mode 1 loading results in  $M = 0$  and pure mode 2 loading in  $M = 1$ . The dimension of the notch stress intensity factors being dependent on the relevant eigenvalues  $\lambda_2$  or  $\lambda_1$ , which are not identical in general, a length parameter  $\ell_f$  is introduced in Eq. (17) with the condition  $\ell_f=1$ .

In the case of pointed notches,  $\rho = 0$ , Eq. (14) can be applied by replacing  $K_{1\rho}$  by  $K_1$  and  $K_{2\rho}$  by  $K_2$  while  $\theta_0$  remains the angle of crack propagation. Pointed notches are relevant in worst case considerations of strength assessments. The following expression is valid for  $\chi$ , updating Eq. (17):

$$\chi = \frac{K_2}{K_1} \ell_f^{\lambda_2 - \lambda_1} \quad (18)$$

The simplified hypothesis,  $K_1=K_{1\rho}$  and  $K_2=K_{2\rho}$ , which has been inherently assumed for the analytical calculations, avoids an iterative procedure for the determination of  $s$  and then of  $\rho_f$ . This is the hypothesis implicitly used by Neuber without the explicit introduction of the stress intensity factors [1], but is approximately true only when the fictitiously enlarged notch root radius is sufficiently small. In the reality, for large  $\rho_f$ ,  $K_{1\rho}$  and  $K_{2\rho}$  do not match  $K_1$  and  $K_2$  [20,34].

## EVALUATION OF THE SUPPORT FACTOR UNDER MIXED MODE LOADING CONDITIONS

By considering Eq. (14) and solving it for different values of the mode ratio  $M$ , it is possible to obtain the support factor  $s$  as a function of the mode ratio  $M$ . To evaluate the values of  $s$ , the condition  $\rho \approx 0$  approximating pointed notches has been used. This choice was made for two specific reasons. The first reason is that pointed notches are the most critical ones in strength assessments. In many practical cases, the real notch radius is not equal to zero, but it is very small and it can be approximated in the safe direction by the worst case condition  $\rho = 0$ . The second reason is that by considering pointed notches, it is possible to define the mode ratio  $M$  uniquely.

The mode ratio  $M$  is evaluated here by using  $K_{1\rho} = K_1$  and  $K_{2\rho} = K_2$ . Eq. (8) has been employed here to determine the crack propagation angle  $\theta_0$  by applying the MTS criterion to pointed V-notches.

Due to the fact that Eq. (14) has been expressed in terms of the real notch radius  $\rho$ , this radius has to remain finite.

In this contribution,  $\rho = 0.001$  mm has been introduced in order to model quasi-pointed V-notches. The values of  $s$  have been evaluated by solving Eq. (14) numerically for different values of the mode ratio  $M$  and keeping constant  $\rho^*=0.1$  mm together with  $\rho=0.001$  mm. The data are plotted in Fig. 2. The support factor  $s$  rises with the mode ratio  $M$  varying from  $M = 0$  (pure mode 1) to  $M = 1$  (pure mode 2). Whereas the rise is rather weak for the keyhole ( $2\alpha = 0$ ), it is rather strong



for larger notch opening angles ( $2\alpha = 60^\circ$ ). It has to be noted that large values of  $s$  mean large fictitious notch radii  $\rho_f$ , i.e. uncritical strength conditions.

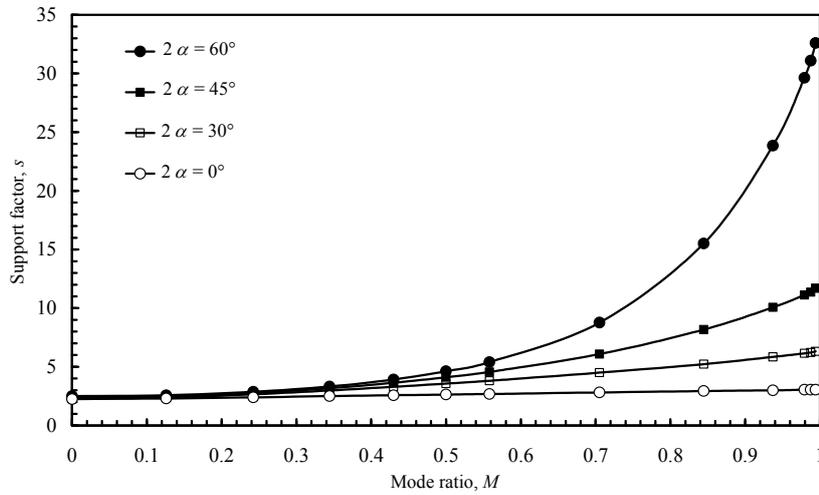


Figure 2: Support factor  $s$  as a function of the mode ratio  $M$  for different values of the notch opening angle  $2\alpha$ .

### VALIDATION OF THE PROPOSED FNR APPROACH FOR MIXED MODE LOADING CONDITIONS

The FNR approach applied to in-plane mixed mode loading is validated by comparisons based on the FE method considering inclined V-notches with end holes (notch radius  $\rho_f = s\rho^*$ ) characterized by different notch opening angles  $2\alpha$ . Geometry and dimensions of the double-V-notched plate are shown in Fig. 3 together with the applied remote boundary conditions. The values of the notch stress intensity factors  $K_1$  and  $K_2$  found by FE analysis for the corresponding pointed V-notches ( $\rho=0$ ) are presented in Tab. 1 for different notch opening angles  $2\alpha$  and notch inclination angles  $\beta$ . Tab. 2 also summarises the mode ratio  $M$ , the crack propagation angle  $\theta_0$  and the support factor  $s$  determined by solving Eq. (14) for the specific parameters.

The following parameters are considered in the numerical investigation by FE analysis:

- the notch opening angle  $2\alpha = 0, 30, 45$  and  $60^\circ$ ,
- the notch inclination angle  $\beta = 15, 30, 45$  and  $60^\circ$  corresponding to a value of  $M$  varying between 0.16 and 0.73,
- the microstructural support length  $\rho^*$  varying between 0.05 and 0.3 mm,
- the notch depth  $2a = 10\sqrt{2}$  mm combined with the plate width  $w = 100$  mm.

The averaged stress has been obtained by numerical integration over the length  $\rho^*$  of the hoop stress component ( $\sigma_{th} = \sigma_{\theta\theta}$ ) along the direction  $\theta_0$  evaluated by using Eq. (8).

$$\bar{\sigma} = \frac{1}{\rho^*} \int_0^{\rho^*} \sigma_{th}(r, \theta_0) dr \quad (19)$$

The stress concentration factor characterising the averaged notch stress over  $\rho^*$  in pointed V-notches is defined in Eq. (20) with reference to the nominal stress  $\sigma_n$ .

$$\bar{K}_t = \frac{\bar{\sigma}}{\sigma_n} = \frac{1}{\rho^* \sigma_n} \int_0^{\rho^*} \sigma_{th} dr \quad (20)$$

The stress concentration factor  $K_t(\rho_f)$  of the fictitiously rounded notch is defined as follows:



$$K_t(\rho_f) = \frac{\sigma_{\max}(\rho^*, s)}{\sigma_n} \quad (21)$$

The relative deviation can be defined as follows:

$$\Delta = \frac{K_t(\rho_f) - \bar{K}_t}{K_t(\rho_f)} \quad (22)$$

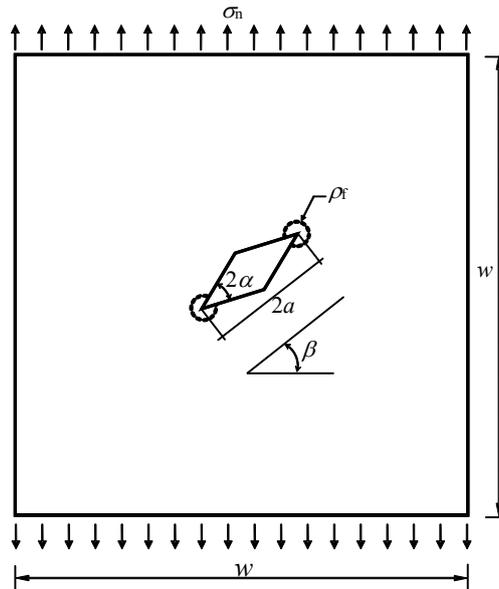


Figure 3: Geometry and dimensions of the double-V-notched quadratic plate specimen considered in the FE analyses; remote loading by prescribed nominal stress  $\sigma_n$ ; dimensions  $w = 100$  mm and  $2a = 14.14$  mm; real pointed versus fictitiously rounded (root hole) V-notch.

$2\alpha$ ( $^\circ$ )	$\beta$ ( $^\circ$ )	$K_1$ (MPa mm $^{1-\lambda_1}$ )	$K_2$ (MPa mm $^{1-\lambda_2}$ )	$\lambda_1$	$\lambda_2$	$\theta_0$ ( $^\circ$ )	$\chi$	M	s
45	15	472	128	0.550	0.660	-13.49	0.272	0.169	2.56
	30	376	222	0.550	0.660	-25.73	0.592	0.340	3.16
	45	244	257	0.550	0.660	-36.52	1.053	0.516	4.24
	60	112	223	0.550	0.660	-46.70	1.993	0.704	6.08

Table 1: Notch stress intensity factors  $K_1$  and  $K_2$  and crack propagation angle  $\theta_0$  for different mode ratios M resulting in support factors s.

### COMPARISON WITH FE RESULTS

The maximum stress  $\sigma_{\max}(\rho^*, s)$  has been determined in this section directly by means of finite element analyses modeling the plate shown in Fig. 3 with  $\rho_f = s\rho^*$ , while  $\bar{\sigma}$  values are those evaluated analytically. The finite element analyses (FEA) were carried out by using Ansys (release 13.0). In total 2000 models have been analysed, each model characterized by the specific values of  $2\alpha$ ,  $\beta$  and  $\rho_f$ . The results from some models exhibiting relative deviations less than or equal to 10% are listed in Tab. 2 as example. The errors are due not only to the adopted simplified hypothesis,  $K_1 = K_{1p}$  and  $K_2 = K_{2p}$ , but also to the fact that the maximum stress component  $\sigma_{\max}$  takes place outside the notch bisector,



creating an additional stress concentration effect linked to the nominal stress component parallel to the notch inclination angle  $\beta$ .

Finally it is worth mentioning that another consistent approach for mixed-mode loading is based on the strain energy density (SED) averaged on a defined control volume [35]. This approach has been successfully used to assess the static behaviour of notched plates made of brittle material as well as the high-cycle fatigue strength of welded joints. In the presence of pointed notches subjected to mixed mode 1–2 loading conditions, the control volume (a semicircular sector under plane strain or plane stress conditions) is introduced as constant, at first for mode 1 loading conditions. In the case of blunt U- or V-notches, the crescent shape control volume is rigidly rotated with respect to the notch bisector line and centred in the point of maximum tangential stress (or maximum strain energy density) on the notch edge resulting in an ‘equivalent’ mode 1 loading condition. One of the main advantages of the average SED approach is that it can be applied using coarse meshes.

$2\alpha$ (°)	$\beta$ (°)	$\chi$	M	$\theta_0$ (°)	s	$\rho^*$ (mm)	$\rho_f$ (mm)	$\bar{K}_t$ Eq.(28)	$K_t(\rho_f)$ FEM	$\Delta$ (%)
45	15	0.272	0.169	–13.50	2.56	0.05	0.128	16.90	16.93	0.17
						0.1	0.256	12.06	12.57	4.07
						0.2	0.511	8.61	9.24	6.86
						0.3	0.767	7.07	7.81	9.47
	30	0.592	0.340	–25.73	3.16	0.05	0.158	14.62	14.73	0.75
						0.1	0.316	10.58	10.92	3.11
						0.2	0.631	7.67	8.18	6.28
						0.3	0.947	6.36	6.97	8.79
	45	1.053	0.516	–36.52	4.24	0.05	0.212	11.08	11.19	0.99
						0.1	0.424	8.18	8.56	4.48
						0.2	0.848	6.05	6.65	9.03

Table 2: FNR results for mixed mode loading conditions; normal stress failure criterion combined with the MTS criterion for  $\theta_0$ ; different values of the microstructural support length  $\rho^*$  and the mode ratio M.

## CONCLUSIONS

Based on FNR concept used in combination with the normal stress criterion for the averaged notch stress and the maximum tangential stress criterion for the crack propagation angle, the support factor has been analytically and numerically determined for V-notches with root hole subjected to in-plane mixed mode loading. A suitable definition and quantification of the mode ratio for pointed V-notches has been found. Taking advantage of a recently conceived analytical frame for V-notches with root hole, the original Neuber procedure for determining the fictitious notch radius and the support factor has been applied to out-of-bisector crack propagation, the propagation direction being determined by the maximum tangential stress criterion as a function of the mode ratio and the notch opening angle. Different values of this length, of the notch depth and of the notch opening angle have been considered as well as different mode ratios. The obtained values of the support factor are well suited for engineering usage in structural strength assessments. The relative deviations have been found variable from case to case. A large set of cases have been found to be characterized by relative deviations less than 10%, directly using the original Neuber’s procedure.

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