A multiaxial incremental fatigue damage formulation using nested damage surfaces

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ABSTRACT. Multiaxial fatigue damage calculations under non-proportional variable amplitude loadings still remains a quite challenging task in practical applications, in part because most fatigue models require cycle identification and counting to single out individual load events before quantifying the damage induced by them. Moreover, to account for the non-proportionality of the load path of each event, semi-empirical methods are required to calculate path-equivalent ranges, e.g. using a convex enclosure or the MOI (Moment Of Inertia) method. In this work, a novel Incremental Fatigue Damage methodology is introduced to continuously account for the accumulation of multiaxial fatigue damage under service loads, without requiring rainflow counters or path-equivalent range estimators. The proposed approach is not based on questionable Continuum Damage Mechanics concepts or on the integration of elastoplastic work. Instead, fatigue damage itself is continuously integrated, based on damage parameters adopted by traditional fatigue models well tested in engineering practice. A framework of nested damage surfaces is introduced, allowing the calculation of fatigue damage even for general 6D multiaxial load histories. The proposed approach is validated by non-proportional tension-torsion experiments on tubular 316L stainless steel specimens.

KEYWORDS. Multiaxial fatigue; Variable amplitude loads; Non-proportional multiaxial loads; Nested fatigue damage surfaces; Incremental damage calculation.

INTRODUCTION

Most fatigue crack initiation models need to properly identify load events before computing the damage induced by them. Hence their fatigue damage calculation routines need to include cycle counting algorithms like the well-known rainflow methodology for uniaxial loads. Cycle counting is necessary because traditional fatigue models are discrete in nature, since they only can accumulate damage after a load event (e.g. a half-cycle) is properly identified, detected e.g. from a load reversal or from a hysteresis loop that closes. However, the detection and counting of loading events can be a quite challenging task under multiaxial non-proportional (NP) histories. The existing multiaxial rainflow algorithms [1] are not trivial to apply. In fact, they are not even robust, since they can output very different half-

cycles depending on the choice of the initial counting point of a periodic load history [2, 3]. Furthermore, multiaxial fatigue damage evaluation requires the semi-empirical calculation of path-equivalent stress or strain ranges from the rainflow-counted paths, increasing even more its computational burden [4].
On the other hand, a completely different fatigue calculation approach assumes damage as a continuous variable, whose increments can be computed as the loading proceeds. Most works based on this idea use Continuum Damage Mechanics concepts [5], which need to be supplemented by purely phenomenological damage evolution equations that are difficult to calibrate, to say the least. In fact, despite their academic appeal, such models remain controversial and have not found a wide acceptance in the fatigue design community.
Other continuous damage approaches are based on an integration of elastoplastic work. However, the accumulated total work required to initiate a microcrack by fatigue certainly is not a material property. Moreover, the elastoplastic work still depends on the number of cycles, thus it is impossible to calculate without previous load cycle and/or reversal detection. Therefore, even if it could be assumed that fatigue damage can be quantified by this parameter, its calculation routine still would need to include a rainflow or other similar load event counter.
Alternatively, instead of integrating dubious strain energy or energy-based damage parameters, a more reasonable path is to continuously quantify fatigue damage itself, using some well-proven model that can properly describe multiaxial fatigue damage in the material in question. The so-called Incremental Fatigue Damage (IFD) approach integrates the chosen parameter until reaching *1.0* or any other suitable critical-damage value using traditional accumulation concepts, as originally performed a long time ago for the uniaxial case by Wetzel under Topper's guidance in 1971 [6], and again by Chu in 2000 [7]. It is important to emphasize that in such calculations fatigue damage is continuously computed after each infinitesimal stress or strain increment, so its quantification does not require the prior identification of load cycles. In this work, the IFD approach is revisited and extended to general multiaxial fatigue problems, including no-proportional ones, based on a direct analogy with non-linear incremental plasticity concepts, however calculating damage instead of plastic strains at each load increment.

THE INCREMENTAL FATIGUE DAMAGE APPROACH

The Incremental Fatigue Damage approach was proposed for uniaxial load histories in Wetzel and Topper's rheological model [6, 8]. It makes use of the derivative of the normal stress σ with respect to damage D, called here *generalized damage modulus* D_σ, thus

$$D_\sigma \equiv d\sigma / dD \quad \Rightarrow \quad D = \int dD = \int (1/D_\sigma) \cdot d\sigma \tag{1}$$

Consider, for instance, a uniaxial constant amplitude loading history with stress amplitude σ_a. During a loading half-cycle, the excursion of the stress σ from $-\sigma_a$ to $+\sigma_a$ could be integrated according to Eq. (1) to find the associated fatigue damage $D = 1/2N$, however without explicitly calculating the fatigue life N. Assuming the material is initially virgin, the damage D from the first half-cycle is initially zero in the initial valley when $\sigma = -\sigma_a$ and thus $\Delta\sigma = \sigma-(-\sigma_a) = 0$, and continuously grows toward $D = 1/2N$ until σ reaches the peak $+\sigma_a$, when $\Delta\sigma = \sigma-(-\sigma_a) = 2\sigma_a$.
For simplicity, Wöhler's stress-based fatigue damage model is adopted below (but strain-based models will be considered later). A simplified relation between the current stress state σ and the continuous damage D from the half-cycle excursion $-\sigma_a \rightarrow +\sigma_a$ can then be obtained from Wöhler's curve e.g. written in Basquin's notation:

$$\sigma_a = \sigma_c \cdot (2N)^b \quad \Rightarrow \quad \Delta\sigma/2 = [\sigma - (-\sigma_a)]/2 = \sigma_c / D^b \quad \Rightarrow \quad D = [(\sigma + \sigma_a)/2\sigma_c]^{-1/b} \tag{2}$$

The generalized damage modulus D_σ during this half-cycle is thus such that

$$1/D_\sigma = dD/d\sigma = -[(\sigma + \sigma_a)/2\sigma_c]^{-1/b} / [b(\sigma + \sigma_a)] \tag{3}$$

from which the fatigue damage $D = 1/2N$ can be calculated using the integral

$$D = \int_{-\sigma_a}^{+\sigma_a} -\frac{1}{b(\sigma + \sigma_a)} \left(\frac{\sigma + \sigma_a}{2\sigma_c} \right)^{-1/b} \cdot d\sigma = \left(\frac{\sigma + \sigma_a}{2\sigma_c} \right)^{-1/b} \Bigg|_{-\sigma_a}^{+\sigma_a} = \left(\frac{\sigma_a}{\sigma_c} \right)^{-1/b} = \frac{1}{2N} \tag{4}$$

If this conceptually simple procedure could be generalized to multiaxial NP variable amplitude loading (VAL) histories, integrating damage along a general multiaxial load path, then cycle identification, multiaxial rainflow counting, and stress (or strain) range calculations would not be required to obtain the fatigue damage D. However, this bold statement is easier said than done, since D_σ depends not only on the current stress state (σ in this uniaxial case), but also on the previous loading history (the value $-\sigma_a$ from the last reversal), see Eq. (3). So, Incremental Fatigue Damage models need to allow D_σ to vary as a function of the stress level and of the existing state of damage [9].

The history dependence of D_σ, often neglected or overly simplified in the few IFD models proposed in the literature, is analogous to the load-order dependence of elastoplastic hysteresis loops. Chu [7] outlined the generalization of Wetzel's rheological model to multiaxial loadings, indirectly detecting cycles using two simple rules. However, damage memory is not properly stored in that simple model for general NP VAL histories, where often no hysteresis loop actually closes and thus any virtual loop closure detection makes no sense. The main purpose of this work is to propose the improvements needed to properly extend the interesting IFD idea to general multiaxial loads.

MULTIAXIAL INCREMENTAL FATIGUE DAMAGE APPROACH

Stress-based Incremental Fatigue Damage Formulation

In this work, instead of using rheological models, a direct analogy between IFD and incremental plasticity is adopted instead to store fatigue damage memory, using internal material variables. In incremental plasticity, a 5D deviatoric stress increment $d\vec{s}'$ can be used to calculate the associated 5D plastic strain increment $d\vec{e}'_{pl}$ from the current generalized plastic modulus P, using a plastic flow rule [10-11].

In particular, it is well known that in the non-linear kinematic (NLK) incremental plasticity formulation, *plastic memory* is stored by the current arrangement among the hardening surfaces defined by their backstresses $\vec{\beta}'_i$, from which the *surface translation directions* $\vec{v}'_i$ are calculated (according to some translation rule) and combined with material coefficients p_i to calculate the current plastic modulus P [10-11]. Therefore, no *plastic straining* occurs if the stress increment $d\vec{s}'$ happens inside the *yield surface*, whose radius should be equal or smaller than the *cyclic yield strength* S_{Yc}. The *accumulated plastic strain* p is then proportional to the integral of the scalar norm $|d\vec{e}'_{pl}|$ of the deviatoric plastic strain increments.

Let's now rephrase the previous paragraph for the desired IFD model, based on the proposed direct analogy between plasticity and fatigue damage. In the IFD model presented here, a 5D deviatoric stress increment $d\vec{s}'$ can be used to calculate the associated 5D *damage* increment $d\vec{D}'$ from the current *generalized damage modulus* D_σ, using a *damage evolution rule*. In the IFD formulation, *damage memory* is stored by the current arrangement among *damage surfaces* defined by their *damage backstresses* $\vec{\beta}'_{\sigma i}$, from which the *damage surface translation directions* $\vec{v}'_{\sigma i}$ are calculated (according to some translation rule) and combined with material coefficients $d_{\sigma i}$ to calculate the current damage modulus D_σ. No *damage* occurs if the deviatoric stress increment $d\vec{s}'$ happens inside the *fatigue limit surface*, whose radius should be equal or smaller than the *fatigue limit* of the material S_L. The *accumulated damage* D is then equal to the integral of the scalar norm $|d\vec{D}'|$ of the 5D damage increments.

The damage backstress vector $\vec{\beta}'_\sigma$ locates the center of the current fatigue limit surface, which can be decomposed as the sum of M damage backstresses $\vec{\beta}'_{\sigma 1}$, $\vec{\beta}'_{\sigma 2}$, …, $\vec{\beta}'_{\sigma M}$ that describe the relative positions between centers of consecutive damage surfaces, as illustrated in Fig. 1 for a 2D case.

Notice in Fig. 1 that each damage surface has a *constant* radius $r_{\sigma i}$, while the radius differences between consecutive surfaces are $\Delta r_{\sigma i} \equiv r_{\sigma i+1} - r_{\sigma i}$. The fatigue limit and failure surfaces are defined, respectively, for $i = 1$ and $i = M + 1$, while the remaining $i = 2, 3, \ldots, M$ are the damage surfaces. The damage backstress lengths are always between $|\vec{\beta}'_{\sigma i}| = 0$, if consecutive centers coincide, and $|\vec{\beta}'_i| = \Delta r_{\sigma i}$, if they are mutually tangent.

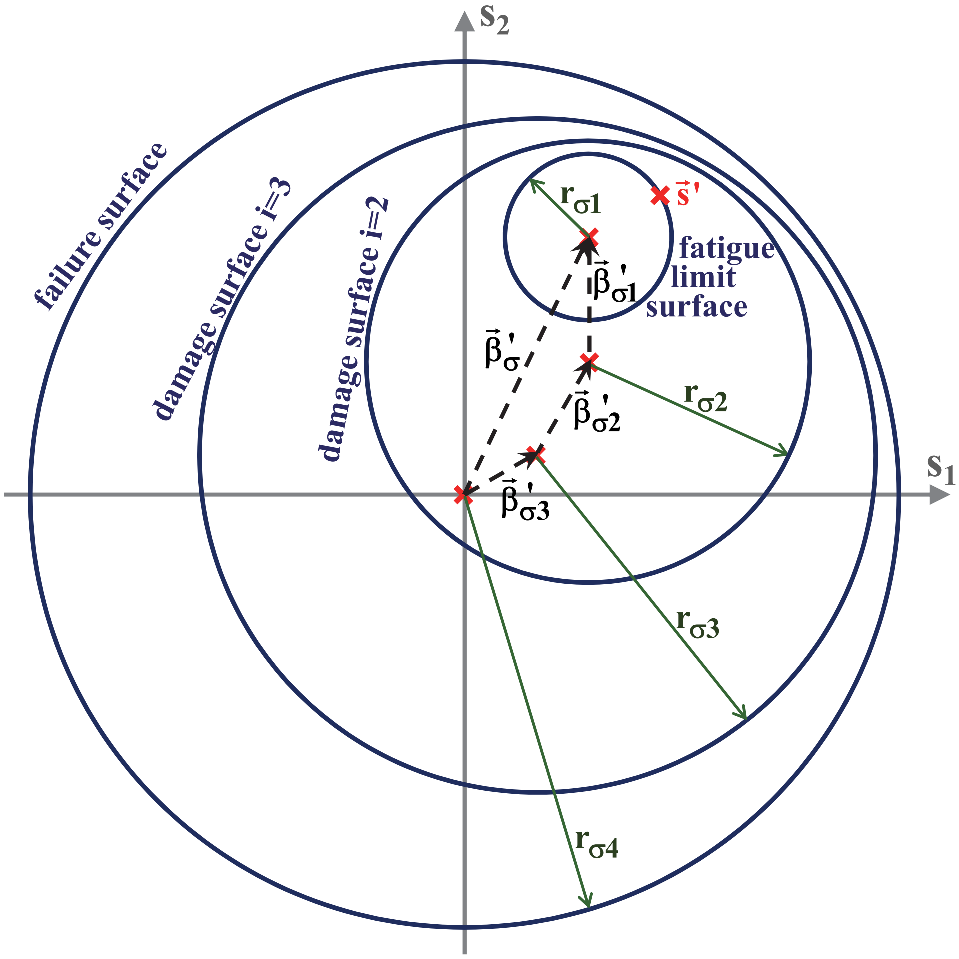

Figure 1: Fatigue limit, damage, and failure surfaces in a 2D deviatoric stress space for three moving nested surfaces, showing the damage backstress vector that defines the location of the fatigue limit surface center, and its three components that describe the relative positions between the centers of consecutive surfaces at each load event.

The proposed multiaxial IFD model uses a 5D damage vector $\vec{D}' \equiv [D_1\ D_2\ D_3\ D_4\ D_5]^T$ that acts as an internal variable that stores the current multiaxial fatigue damage state (to account for the damage memory). The scalars D_1 through D_5 are signed damage quantities associated with each one of the directions of the 5D deviatoric stress vector $\vec{s}'$, defined in [11]. In this way, the total accumulated damage D (which thus works for multiaxial fatigue problems analogously to the accumulated plastic strain p for multiaxial plasticity problems) is obtained from the length of the path described by the 5D damage vector $\vec{D}'$, calculated in either continuous or discrete formulations from

$$D = \int dD = \int |d\vec{D}'| \cong \sum \Delta D = \sum |\Delta \vec{D}'| \tag{5}$$

If a given stress state $\vec{s}'$ is on the fatigue limit surface with a normal unit vector $\vec{n}'_\sigma$, and if its infinitesimal increment $d\vec{s}'$ is in the outward direction, then $d\vec{s}'^T \cdot \vec{n}'_\sigma > 0$ and a fatigue damage increment is obtained from a *damage evolution rule* (inspired on the analogous Prandtl-Reuss flow rule [10-11]):

$$d\vec{D}' = (1/D_\sigma) \cdot (d\vec{s}'^T \cdot \vec{n}'_\sigma) \cdot \vec{n}'_\sigma \cdot f_{MS}(\vec{\sigma}) \cdot f_{NP}(\vec{\beta}'_\sigma, \vec{n}'_\sigma) \tag{6}$$

where $f_{MS}(\vec{\sigma})$ is a scalar *mean stress function* of the current 6D stress $\vec{\sigma}$ to account for mean/maximum-stress effects, which can be defined e.g. from Goodman's or Gerber's $\sigma_a\sigma_m$ relations when applicable; and $f_{NP}(\vec{\beta}'_{\sigma}, \vec{n}'_{\sigma})$ is a *NP function* to account for the additional effects introduced by the non-proportionality of the load path. For materials that fail due to distributed damage in all directions, the mean stress function $f_{MS}(\vec{\sigma})$ could be based on the current hydrostatic stress σ_h from $\vec{\sigma}$. On the other hand, for materials that fail due to a single dominant crack, like most metallic alloys (whose multiaxial fatigue damage parameters tend to be better described by the critical-plane approach), then $f_{MS}(\vec{\sigma})$ could be based on the normal stress $\sigma_{\perp}$ perpendicular to the considered candidate plane.

Except for the failure surface (which never translates), during this damage process the fatigue limit and all damage surfaces suffer translations

$$d\vec{\beta}'_{\sigma i} = d_{\sigma i} \cdot \vec{v}'_{\sigma i} \cdot dD, \quad \text{if } |\vec{\beta}'_{\sigma i}| < \Delta r_{\sigma i} \quad \text{or} \quad d\vec{\beta}'_{\sigma i} = 0, \quad \text{if } |\vec{\beta}'_{\sigma i}| = \Delta r_{\sigma i} \tag{7}$$

where $d_{\sigma i}$ are coefficients calibrated for each surface, and $\vec{v}'_{\sigma i}$ are the *damage surface translation directions* adapted e.g. from the general translation rule from [11].

The current generalized damage modulus D_{σ} is then obtained from the consistency condition, which guarantees that the current stress state is never outside the fatigue limit surface, taken from an analogy to the NLK hardening formulation for plasticity problems

$$D_{\sigma} = \left(\sum_{i=1}^{M} d_{\sigma i} \cdot \vec{v}'^{T}_{\sigma i} \right) \cdot \vec{n}'_{\sigma} \tag{8}$$

allowing the calculation of the evolution of the damage vector $\vec{D}'$ using Eq. (6).

The (scalar) accumulated damage D is then obtained from Eq. (5). This formulation can deal with any multiaxial stress history, proportional or NP, and eliminates the need to count cycles and find equivalent ranges, or even to define them. Indeed, for instance, Fig. 2 shows continuous IFD damage predictions for a material whose elastic Coffin-Manson's parameters are $\sigma_c = 772.5MPa$ and $b = -0.09$, under the uniaxial loading history $\sigma_x = \{0 \rightarrow 300 \rightarrow -300 \rightarrow 300\}MPa$. Jiang-Sehitoglu's translation rule was adopted with $M = 16$ surfaces, calibrated between logarithmically spaced damage levels 10^{-8} and 0.01.

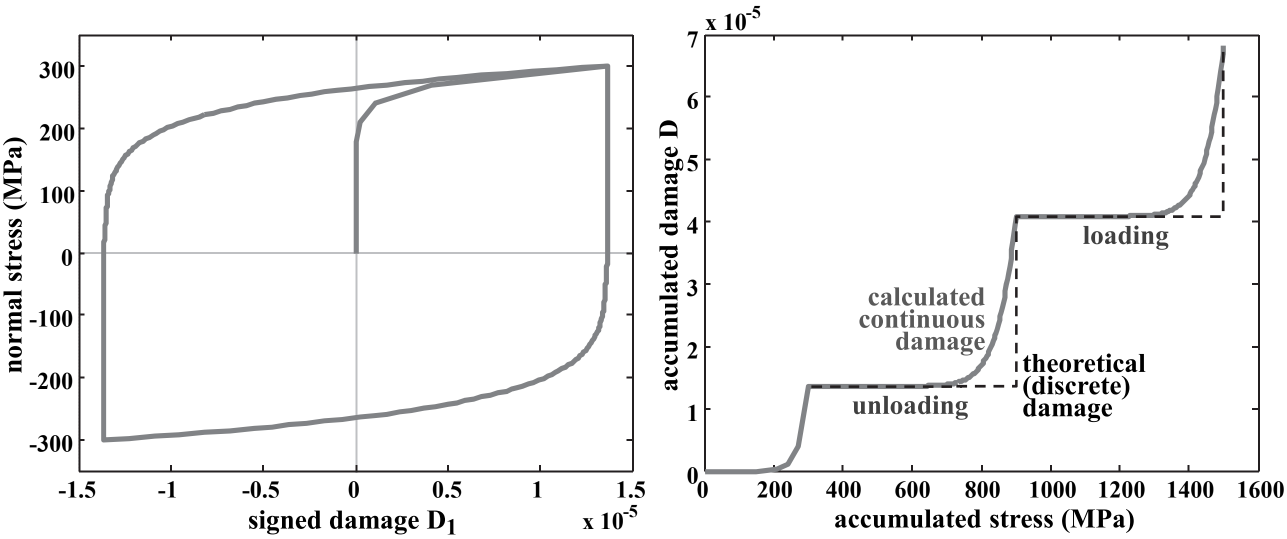

Figure 2: Hysteresis loops relating applied stress and a signed damage state (left) and resulting accumulated damage (right) for a uniaxial constant amplitude loading history.

Strain-based Incremental Fatigue Damage Formulation

All the formulations and the example presented above assumed nominally linear elastic loading histories, whose damage can calculated from SN models such as Wöhler-Basquin's and Goodman, but this is not a limitation for this methodology.

Indeed, the proposed IFD approach can be as well extended for elastoplastic loading histories, whose fatigue damage must be quantified by εN models. However, instead of using fatigue limit and damage surfaces defined in stress spaces, strain spaces should be used in the continuous damage calculations in such cases. A generalized damage modulus D_ε (instead of D_σ) is thus defined, which for uniaxial loading histories becomes the derivative of the normal strain ε with respect to damage D, thus $D_\varepsilon \equiv d\varepsilon/dD$.

In the strain-based version of the proposed IF approach, a 5D deviatoric strain increment $d\vec{e}'$, defined in [10], is used to calculate the associated 5D *damage* increment $d\vec{D}'$ from the current D_ε, using a suitable *damage evolution rule*. To do so, *damage memory* is stored by the current arrangement among *damage surfaces* defined by their *damage backstrains* $\vec{\beta}'_{\varepsilon i}$, from which the *damage surface translation directions* $\vec{v}'_{\varepsilon i}$ are calculated according to some translation rule and combined with material coefficients $d_{\varepsilon i}$ to calculate the current D_ε. The *accumulated damage* D is then equal to the integral of the scalar norm $|d\vec{D}'|$ of the damage increments. The same equations from the stress-based version can be used in the strain-based one, as long as the M damage surface backstrains $\vec{\beta}'_{\varepsilon 1}$, $\vec{\beta}'_{\varepsilon 2}$, …, $\vec{\beta}'_{\varepsilon M}$, radii $r_{\varepsilon i}$, and radius differences $\Delta r_{\varepsilon i} \equiv r_{\varepsilon i+1} - r_{\varepsilon i}$ between consecutive damage surfaces are all defined as strain (instead of stress) quantities.

Experimental Results

The proposed IFD formulation is experimentally evaluated using complex 2D tension-torsion stress histories, applied on annealed tubular 316L stainless steel specimens in a multiaxial servo-hydraulic testing machine. The Coffin-Manson curve for this material is $\Delta\varepsilon/2 = 0.0119\cdot(2N)^{-0.277} + 0.758\cdot(2N)^{-0.582}$, obtained from uniaxial εN tests.

The experiments consist of strain-controlled tension-torsion cycles applied to eight tubular specimens, each of them following one of the eight periodic $\varepsilon_x \times \gamma_{xy}/\sqrt{3}$ histories from Fig. 3. Tab. 1 compares the predicted and observed fatigue lives in number of blocks, where each block consists of a full load period. All predictions were performed using the strain-based version of the proposed incremental plasticity formulation, assuming for simplicity $f_{MS}(\vec{\sigma}) \equiv 1$ and $f_{NP}(\vec{\beta}'_\sigma, \vec{n}'_\sigma) \equiv 1$ in Eq. (6).

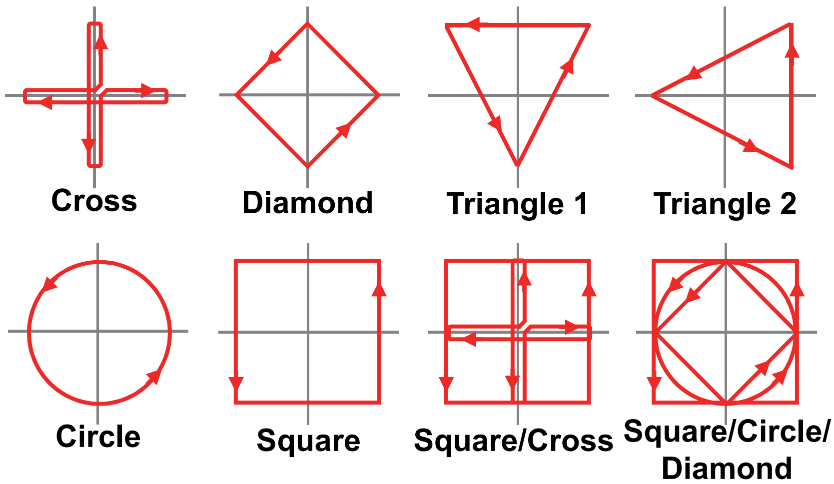

Figure 3: Applied periodic $\varepsilon_x \times \gamma_{xy}/\sqrt{3}$ strain paths on eight tension-torsion tubular specimens, all of them with normal and effective shear amplitudes *0.6%*.

As shown in Tab. 1, albeit the proposed IFD method does not use any cycle detection or counting algorithm, all fatigue lives are predicted with relatively small errors, well within the usual scatter found in all fatigue life measurements. It also automatically applies Miner's rule under VAL, as it can be seen in the loading path consisting of blocks of consecutive square and cross paths, since the predicted number of blocks *482* is such that $1/482 \cong 1/751 + 1/1314$. Similarly, the predicted *327* blocks of consecutive square, circle and diamond paths is such that $1/327 \cong 1/751 + 1/996 + 1/1436$. Miner's rule was also confirmed within the observed experimental results, since e.g. in this latter case it would predict a life of $1/(1/772 + 1/837 + 1/976) = 285$ blocks, almost the same value as the measured *288* blocks. It is important to note that all the predictions listed in Tab. 1 were based only on uniaxial Coffin-Manson data, without any posterior curve fitting procedure.

Tension-Torsion path:	predicted	observed	error
Cross	1314	1535	−14%
Diamond	1436	976	+47%
Triangle 1	1135	842	+35%
Triangle 2	1180	840	+40%
Circle	996	837	+19%
Square	751	772	−3%
Square + Cross	482	342	+41%
Square + Circle + Diamond	327	288	+14%

Table 1: Predicted and observed lives, in number of blocks, for each applied path.

CONCLUSIONS

In this work, a continuous multiaxial Incremental Fatigue Damage formulation that does not needs cycle counting or path-equivalent estimations is proposed, based on a direct analogy with incremental plasticity models. Both proposed stress and strain-based approaches can be formulated using traditional stress, strain, or even energy-based SN and εN damage models, such as Wöhler-Basquin, Coffin-Manson, Smith-Watson-Topper, or Fatemi-Socie, making it an attractive and practical tool for engineering use. In particular, the proposed IFD models do not require additional fitting parameters, or complex calibration routines, as opposed to equally continuous models that are based on traditional Continuum Damage Mechanics approaches. The results show that the proposed method is able to predict quite well multiaxial fatigue lives under complex tension-torsion histories, even though it does not require any cycle detection, multiaxial rainflow counting, or path-equivalent range computations.

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