



Investigations in static response and free vibration of a functionally graded beam resting on elastic foundations

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ABSTRACT. In this article, an analytical study was done to predict the behavior of the beam vis-à-vis bending, buckling, and dynamic responses of isotropic homogeneous beams based on an elastic foundation. The material properties of the FG-beams vary across the thickness using the power law. In this work, the sinusoidal shear deformation beams theory is used to investigate the static and dynamic behavior of FG beams. The present theory fulfills the condition of nullity of edge stresses and does not require the use of a shear correction factor. Hamilton's principle is used to deduce equations of motion, and analytical solutions for simply supported beams were obtained using the Navier resolution method. Nondimensional displacements, eigenfrequencies and critical-buckling loads of isotropic homogeneous beams were obtained for various values of the foundation parameters. The numerical results obtained by the present technique have been compared with the results of literature and are in excellent agreement with them. It can be concluded that the current HSDBT is simple and accurate in solving the bending, eigenfrequency and critical-buckling load problems for FGM beams.

KEYWORDS. Undetermined integral terms; Free vibration; isotropic-homogeneous beams; Navier's solution; Elasticity.



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INTRODUCTION

The functionally graded materials (FGM) may be defined as materials having a progressive variation of material properties. This material is produced by mixing two or more materials in a certain percentage of volume (ceramic and metal). The mixing ratio of the constituents varies regularly and the material properties change without any interruption throughout the thickness. There are a large number of works have been done on the dynamics, flexion and buckling behavior of FGM structures.

The Conventional composite structures suffer from a discontinuity in the properties of materials at the interface of layers and constituents. Therefore, constraint fields in intersection areas create interface problems and thermal stress concentrations in high-temperature environments. Many authors have studied the dynamic behavior of FGM beams, mostly,

by means of both the classical beam theory (CBT), FSDBT and HSDBT Wang et al. [1] given a solution to solve the free vibration, buckling and bending problems of the Timoshenko and Euler-Bernoulli beams based on different models of elastic foundations.

There are many areas of application for composite materials (Chikh et al. [2]; Akbaş et al. [3]; Chikh et al. [4]; Fahsi et al. [5]) same the aircraft and aerospace industry. Omidi et al [6] studied the dynamic stability of simple supported FG beams reposing on a linear elastic foundation; with piezoelectric-layers under a periodic axial compression load. Zhong et al. [7] provided an analytical solution for console beams subjected to various types of mechanical loads. Thai et al. [8] studied the free vibration and bending of FG beams by the use of different higher-order beams theories. Zhu, H. [9] established three-dimensional finite element model using finite element software to simulate and compare the stress performance of the strengthening beams with different numbers of CFRP plates. Bouchikhi, A. S et al. [10] investigated the 2D simulation used to calculate the J-integral of the main crack behavior emanating from a semicircular notch and double semicircular notch and its interaction with another crack which may occur in various positions in (TiB/Ti) FGM plate under mode I. Yassine Khalfi et al. [11] developed a refined and simple shear deformation theory for mechanical buckling of composite plate resting on two-parameter Pasternak’s foundations. Meftah Kamel [12] presented a finite element method for analyzing the elasto-plastic plate bending problems. Saidi Hayat [13] presented a new shear deformation theory for free vibration analysis of simply supported rectangular functionally graded plate embedded in an elastic medium.

In this paper, a higher-order shear deformation beams theory for bending; buckling and free vibration of FG beams are developed. The present theory differs from other higher-order theories because, in present theory the displacement field which includes undetermined integral terms, which is not considered by the other researchers. The results of the present model are compared with the known data in the literature.

VARIATIONAL FORMULATION AND CINEMATICS

Consider an FG beam with length L , width b , and thickness h made of Al/Al_2O_3 as represents in Fig. 1. The lower part of the FG-beam was totally ceramic and the upper surface was completely made of metal. The beam $0 \leq x \leq L$; $-b/2 \leq y \leq b/2$; $-h/2 \leq z \leq h/2$ in the Cartesian coordinate systems. assumed to be positive in the proposed direction, and the beam is deformed in the x - z plane solely. The x -axis coinciding with the beam inert axis. The beam is supported by Winkler–Pasternak foundations.

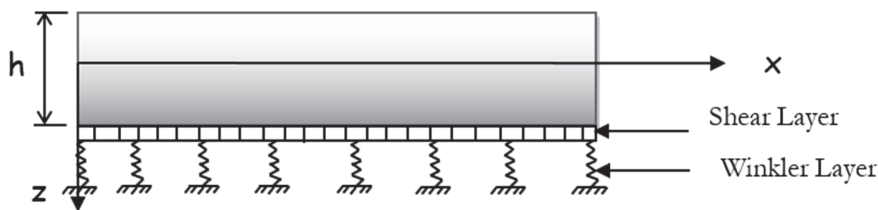


Figure 1: FGM beam supported by Winkler–Pasternak type elastic foundation.

KINEMATICS AND CONSTITUTIVE EQUATIONS

On the fundamental of the assumptions expressed in the previous section, the displacement field of present theory can be obtained by:

$$u(x, y, z, t) = u_0(x, y, t) - z \frac{\partial w_0(x, y, t)}{\partial x} + k_1 f(z) \int \theta(x, y, t) dx \quad (1)$$

$$w(x, y, z, t) = w_0(x, y, t)$$

where:

$$f(z) = \frac{b}{\pi} \sin\left(\frac{\pi z}{b}\right), \quad g(z) = \frac{\partial f(z)}{\partial z}$$



The deformations related to the displacement-field in Eq. (1) contains only three unknowns (u_0, w_0, θ) . The linear strains corresponding with the displacement field in Eq. (1) are:

$$\varepsilon_x = \varepsilon_x^0 + z \varepsilon_x^1 + f(z) \varepsilon_x^2, \quad \gamma_{xz} = g(z) \gamma_{xz}^0$$

where

$$\varepsilon_x^0 = \frac{\partial u_0(x, y, t)}{\partial x}, \quad \varepsilon_x^1 = -\frac{\partial^2 w_0(x, y, t)}{\partial x^2}, \quad \varepsilon_x^2 = k_1 A' \theta(x, y, t), \quad \gamma_{xz}^0 = k_1 \int \theta(x, y, t) dx \quad (2)$$

The integral appearing in the above expressions shall be resolved by a Navier type solution and can be represented as:

$$\int \theta dx = A' \frac{\partial \theta}{\partial x} \quad (3)$$

where the coefficient "A'" is depending on the type of solution chosen, in this case via Navier. Therefore, "A'" and k_1 is expressed as follows:

$$A' = -\frac{1}{\alpha^2}, \quad k_1 = \alpha^2 \quad (4)$$

According to the polynomial material law, the effective Young's modulus $E(z)$

$$E(z) = E_m + (E_c - E_m)(0.5 + z/h)^p \quad (5)$$

The constitutive relations of an FG plate can be written as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} \quad (6)$$

where C_{ij} are, the three-dimensional elastic constants given by:

$$C_{11} = \frac{E(z)}{(1-\nu^2)}, \quad C_{55} = \frac{E(z)}{2(1+\nu)} \quad (7)$$

The equilibrium equations can be obtained using the Hamilton principle, in the present case yields:

$$\int_{t_1}^{t_2} (\delta U - \delta V + \delta K) dt = 0 \quad (8)$$

$$\begin{aligned} \delta U &= \int_{-b/2}^{b/2} \int_{\Omega} [\sigma_x \delta \varepsilon_x + \tau_{xz} \delta \gamma_{xz}] d\Omega dz, \\ \delta V &= \int_{\Omega} (q - f_e) \delta w d\Omega, \\ \delta K &= \int_0^L \int_{-b/2}^{b/2} \rho(z) (\ddot{u} \delta u + \ddot{w} \delta w) dz dy dx \end{aligned} \quad (9)$$



where Ω is the top surface, and f_e is the density of reaction force of foundation. For the Pasternak foundation model:

$$f_e = k_m w(x, y) - k_p \frac{\partial^2 w(x, y)}{\partial x^2} \quad (10)$$

The equilibrium equations can be acquired using the Hamilton principle.

$$\begin{aligned} \delta u_0 : \frac{\partial N_x}{\partial x} - I_1 \frac{\partial^2 u_0}{\partial t^2} + I_2 \frac{\partial^3 w_0}{\partial t^2 \partial x} - I_3 k_1 A^2 \frac{\partial^3 \theta}{\partial t^2 \partial x} &= 0 \\ \delta w_0 : \frac{\partial^2 M_x^b}{\partial x^2} + q(x, t) + N_0 \frac{\partial^2 w_0}{\partial x^2} - f_e - I_1 \frac{\partial^2 w_0}{\partial t^2} - I_2 \frac{\partial^3 u_0}{\partial t^2 \partial x} + I_4 \frac{\partial^4 w_0}{\partial t^2 \partial x^2} + I_5 \frac{\partial^4 \theta}{\partial t^2 \partial x^2} &= 0 \\ \delta \theta : k_1 A \frac{\partial^2 M_x^s}{\partial x^2} + k_1 A^2 \frac{\partial Q_{xz}}{\partial x} + I_3 k_1 A^2 \frac{\partial^2 u_0}{\partial t^2 \partial x} - I_5 k_1 A \frac{\partial^4 \theta}{\partial t^2 \partial x^2} + I_6 k_1^2 A^4 \frac{\partial^4 \theta}{\partial t^2 \partial x^2} &= 0 \end{aligned} \quad (11)$$

where (N_x) denote the resulting force in-plane, (M_x^b) , (M_x^s) denote the total moment resultants and (Q_{xz}) are transverse shear stress resultants and they are defined as

$$\begin{aligned} N_x &= \int_{-b/2}^{b/2} \sigma_x d\zeta, \quad M_x^b = \int_{-b/2}^{b/2} \sigma_x \zeta d\zeta, \\ M_x^s &= \int_{-b/2}^{b/2} \sigma_{xz} f(\zeta) d\zeta, \quad Q_{xz} = \int_{-b/2}^{b/2} \sigma_{xz} g(\zeta) d\zeta \end{aligned} \quad (12)$$

Following the Navier solution process, we assume the following solution form for (u_0, w_0, θ) and that check the boundary conditions,

$$\begin{Bmatrix} u_0 \\ w_0 \\ \theta \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U \cos(\lambda x) \\ W \sin(\lambda x) \\ \theta \sin(\lambda x) \end{Bmatrix} e^{i\omega t} \quad (13)$$

where U, W , and θ are arbitrary parameters to be determined, ω is the natural frequency, and $\lambda = \frac{m\pi}{L}$. The transverse load $q(x)$ is also expanded in Fourier series as:

$$q(x) = \sum_{m=1}^{\infty} Q_m \sin(\lambda x) \quad (14)$$

where

$$Q_m = \frac{2}{L} \int_0^L q(x) \sin(\lambda x) dx \quad (15)$$

In the case where a sinusoidally distributed load, we have

$$m = 1, Q_1 = q_0 \quad (16)$$

In the case where uniform distributed the load, we have



$$m = 1, Q_m = \frac{4q_0}{m\pi}, (m = 1, 3, 5, \dots) \tag{17}$$

In the case where static problems, we have the following equation:

$$[K]\{\Delta\} = \{F\} \tag{18}$$

where $\{\Delta\} = \{U, W, \theta\}^t$ and $[K]$ is the symmetric matrix given by

$$[K] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} \tag{19}$$

In the case of free vibration problem problems, the analytical solutions can be obtained by:

$$([K] - \omega^2[M])\{\Delta\} = 0 \tag{20}$$

where $[M]$ is the symmetric matrix given by

$$[M] = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} \tag{21}$$

For buckling problems, can be expressed as

$$([K] - \bar{N})\{\Delta\} = 0 \tag{22}$$

in which:

$$\begin{aligned} S_{11} &= -A_{11}\lambda^2, S_{12} = B_{11}\lambda^3, S_{13} = k_1 A' \lambda D_{11}, \\ S_{22} &= \lambda^4 E_{11} k_1 - k_p \lambda^2 - N_0 \lambda^2 - k_w \\ S_{23} &= -k_1 A' \lambda^2 F_{11}, S_{33} = -k_1^2 A'^3 A_{,55} \lambda^2 - k_1^2 A'^2 G_{11}, \end{aligned} \tag{23}$$

$$\begin{aligned} m_{11} &= I_1, m_{12} = -I_2 \lambda, m_{13} = k_1 A' I_3 \lambda, m_{22} = I_1 + I_4 \lambda \\ m_{23} &= -I_5 \lambda^2, m_{33} = k_1^2 A'^4 \lambda^2 I_6, \end{aligned}$$

where

$$(A_{11}, B_{11}, D_{11}, E_{11}, F_{11}, G_{11}) = \int_{-h/2}^{h/2} C_{11} (1, \xi, f(\xi), \xi^2, \xi f(\xi), f(\xi)^2) d\xi, \tag{24}$$

$$A_{55}^i = \int_{-h/2}^{h/2} C_{55} g(\xi)^2 d\xi$$

$$(I_1, I_2, I_3, I_4, I_5, I_6) = \int_{-h/2}^{h/2} \rho(\xi) (1, \xi, f(\xi), \xi^2, \xi f(\xi), f(\xi)^2) d\xi \tag{25}$$



RESULTS AND DISCUSSION

In this study, bending; buckling and free vibration investigation on SS FG beam by the present theory is suggested for investigation. The FG beams are made of Aluminum (Al; $E_m = 70$ GPa, $\rho_m = 2702$ kg/m³, $\nu_m = 0.3$) and alumina (Al₂O₃; $E_c = 380$ GPa, $\rho_c = 3960$ kg/m³, $\nu_c = 0.3$) and their properties vary in the direction of the thickness of the beam according to power-law. The lower part of the FG-beam is rich in aluminum, while the upper part of the FG-beam is alumina rich. For convenience, the following dimensionless parameters are used:

$$\bar{\omega} = \sqrt[4]{\frac{\rho_c AL^4 \omega^2}{EI}}, \bar{w}(L/2) = \frac{w(L/2)100E_c I}{qL^4}, \bar{K}_w = \frac{k_w L^4}{EI}, \bar{K}_p = \frac{k_p L^2}{EI}, \bar{N} = \frac{N_0 L^2}{EI} \tag{26}$$

The buckling answer of an FG beam under axial force (N_0) has been studied. A dimensionless; critical-buckling load is shown in Tab. 2. The critical-buckling load was obtained for various values regarding the foundation parameters \bar{K}_w and \bar{K}_p . The results were contrasted with those delivered by Rao et al. [16]. Tab. 2 reveals that this study's results agreed with those available in the literature.

Tab. 3 present the comparisons of the dimensionless natural frequency obtained by the present beam theory with other beams theories results of Chen et al. [14] and Ying et al. [15] for three divers values of the thickness-to-length ratio, and for divers values of foundation parameters \bar{K}_w and \bar{K}_p . As can be seen, the new results are in excellent concordat with previous ones.

Foundation Parameters		L/h = 120			L/h = 15			L/h = 5		
\bar{K}_w	\bar{K}_p	Chen et al. [14]	Ying et al. [15]	Present	Chen et al. [14]	Ying et al. [15]	Present	Chen et al. [14]	Ying et al. [15]	Present
	0	1.30229	1.30229	1.30416	1.31528	1.31527	1.30416	1.42026	1.42024	1.30416
0	10	0.64483	0.64483	0.64527	0.64835	0.64830	0.64527	0.67820	0.67451	0.64527
	25	0.36611	0.36611	0.36624	0.36742	0.36735	0.36624	0.38170	0.37667	0.36624
	0	1.18057	1.18057	1.18210	1.19140	1.19134	1.18210	1.28260	1.27731	1.18210
10	10	0.61333	0.61333	0.61372	0.61656	0.61649	0.61372	0.64639	0.64025	0.61372
	25	0.35567	0.35567	0.35579	0.35692	0.35684	0.35579	0.37206	0.36568	0.35579
	0	0.64007	0.64007	0.64051	0.64377	0.64343	0.64051	0.69610	0.66848	0.64051
10 ²	10	0.42558	0.42558	0.42576	0.42741	0.42716	0.42576	0.45927	0.43881	0.42576
	25	0.28285	0.28285	0.28291	0.28380	0.28360	0.28291	0.30516	0.28944	0.28291

Table 1 Comparisons of the mid-span deflection $\bar{w}(L/2) = \frac{w(L/2)100E_c I}{qL^4}$ of an isotropic-homogeneous beam on elastic foundations due to a uniform pressure.



\bar{K}_w	Theories	\bar{K}_p / π^2			
		0	0.5	1	2
0	Rao et al. [16]	9.8696	14.8040	19.7390	34.5440
	Present	9.8538	14.7886	19.7234	29.5930
1	Rao et al. [16]	9.9709	14.9070	19.8410	34.6450
	Present	9.9551	14.8899	19.8247	29.6943
10^2	Rao et al. [16]	20.0020	24.9370	29.8710	44.6760
	Present	19.9859	24.9207	29.8555	39.7251
10^4	Rao et al. [16]	1023.1000	1028.0000	1032.9000	1047.7000
	Present	1023.0656	1028.0004	1032.9352	1042.8048

Table 2 Comparisons of buckling load parameter \bar{N} of an isotropic-homogeneous beam on elastic foundations $L/h = 20$

Foundation Parameters		$L/h = 120$			$L/h = 15$			$L/h = 5$		
\bar{K}_w	$\frac{\bar{K}_p}{\pi^2}$	Chen et al. [14]	Ying et al. [15]	Present	Chen et al. [14]	Ying et al. [15]	Present	Chen et al. [14]	Ying et al. [15]	Present
0	0	3.14143	3.14145	3.14028	3.13025	3.13227	3.13730	3.04799	3.06373	3.11161
	1	3.73588	3.73587	3.73520	3.72657	3.72775	3.73165	3.65802	3.66645	3.70107
	2.5	4.29687	4.29689	4.29646	4.28809	4.28886	4.29237	4.21834	4.22319	4.25717
10^2	0	3.74823	3.74823	3.74757	3.73895	3.74012	3.74400	3.67050	3.67882	3.71333
	1	4.14356	4.14357	4.14309	4.13472	4.13558	4.13915	4.06636	4.07200	4.10521
	2.5	4.58227	4.58227	4.58192	4.57347	4.57410	4.57757	4.49914	4.50278	4.53999
10^4	0	10.02403	10.02403	10.02407	9.99582	9.99583	10.01451	7.34081	7.34081	7.84931
	1	10.04813	10.04812	10.04816	10.01970	10.01971	10.03857	7.34095	7.34095	7.84931
	2.5	10.08394	10.08393	10.08398	10.05519	10.05520	10.07435	7.34116	7.34116	7.84931

Table 3. Comparisons of the fundamental frequency parameter $\bar{\omega} = \sqrt[3]{\frac{\rho_c AL^4 \omega^2}{EI}}$ of an isotropic-homogeneous beam on to elastic foundations using diverse beam theories

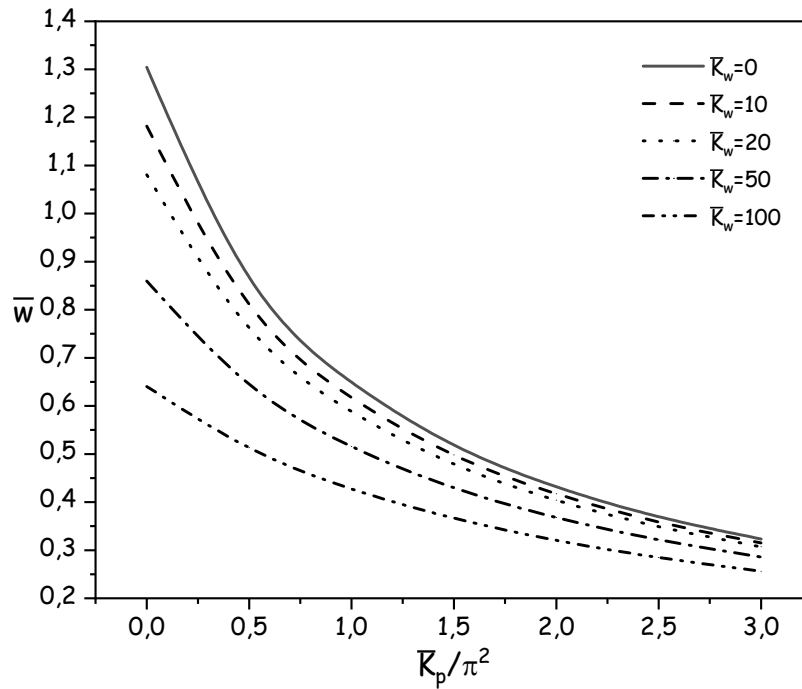


Figure 2: Variation of the non-dimensional transverse displacement $\bar{w}(L/2) = \frac{w(L/2)100E_c I}{qL^4}$ of an isotropic-homogeneous beam with Pasternak parameter \bar{K}_p and Winkler parameter \bar{K}_w .

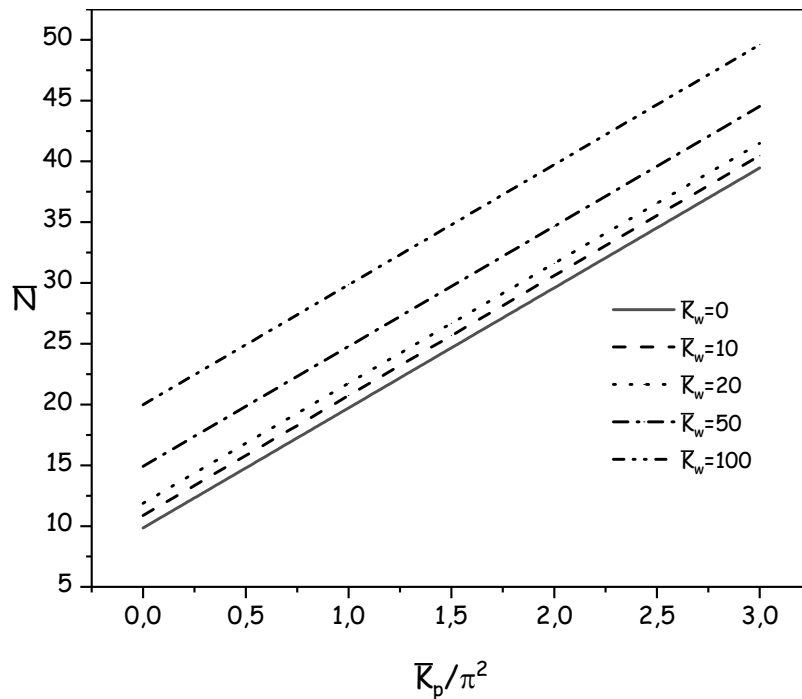


Figure 3: Variation of the non-dimensional buckling load parameter \bar{N} of an isotropic-homogeneous beam with Pasternak parameter \bar{K}_p and Winkler parameter \bar{K}_w .

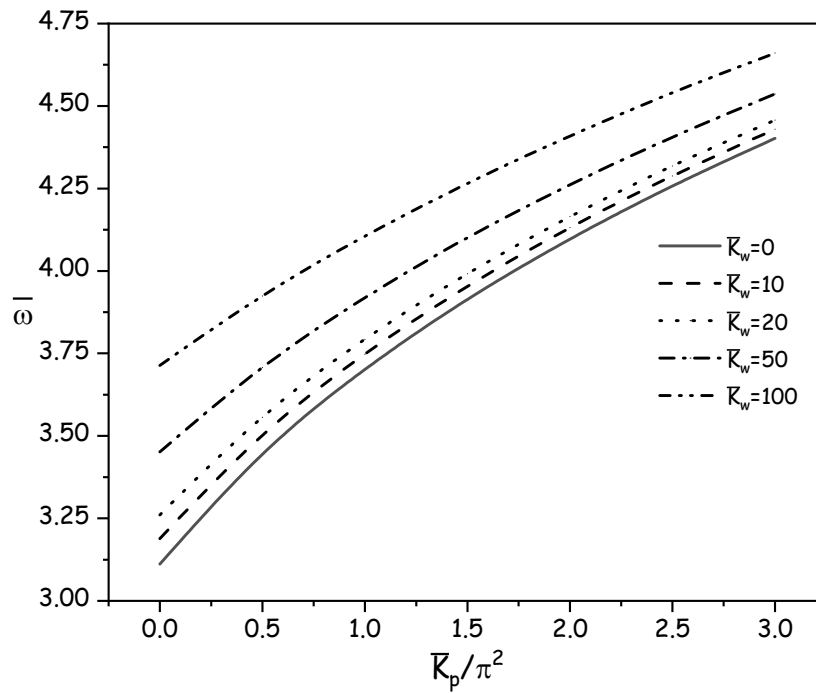


Figure 4: Variation of the nondimensional fundamental frequency $\bar{\omega} = \sqrt[4]{\frac{\rho_c AL^4 \omega^2}{EI}}$ of isotropic homogeneous beam with Pasternak parameter \bar{K}_p and Winkler parameter \bar{K}_w .

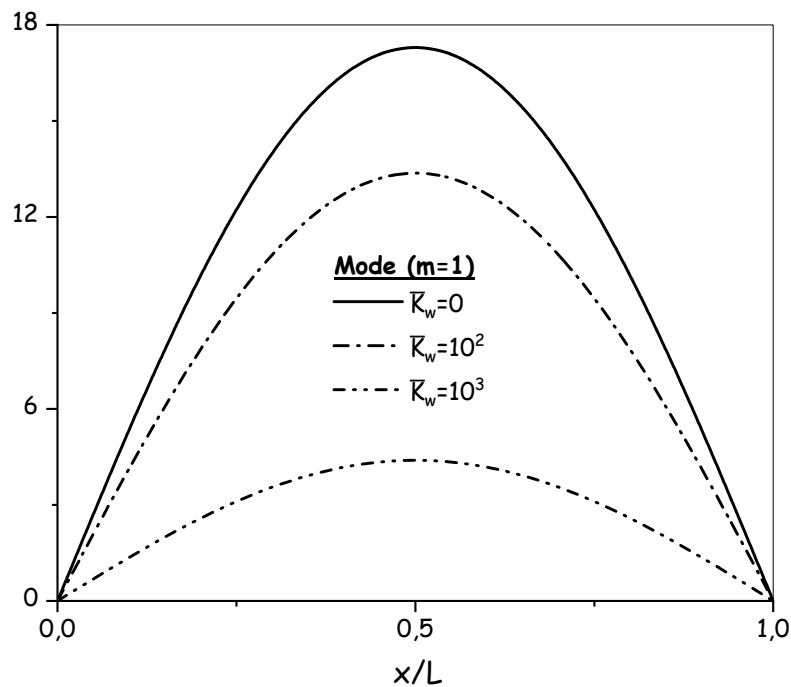


Figure 5: Mode shape of \bar{w} at the base surface of the beam with various aspect ratios for the first mode frequency ($m=1, L/h=20, \bar{K}_p/\pi^2=2.5$)

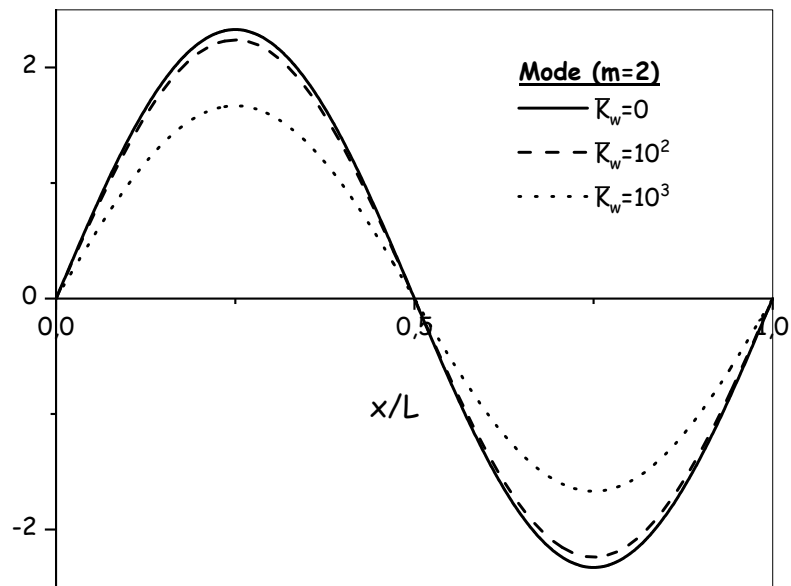


Figure 6: Mode shape of \bar{w} at the base surface of the beam with various aspect ratios for the second mode frequency ($m = 2, L/h = 20, \bar{K}_p / \pi^2 = 2.5$)

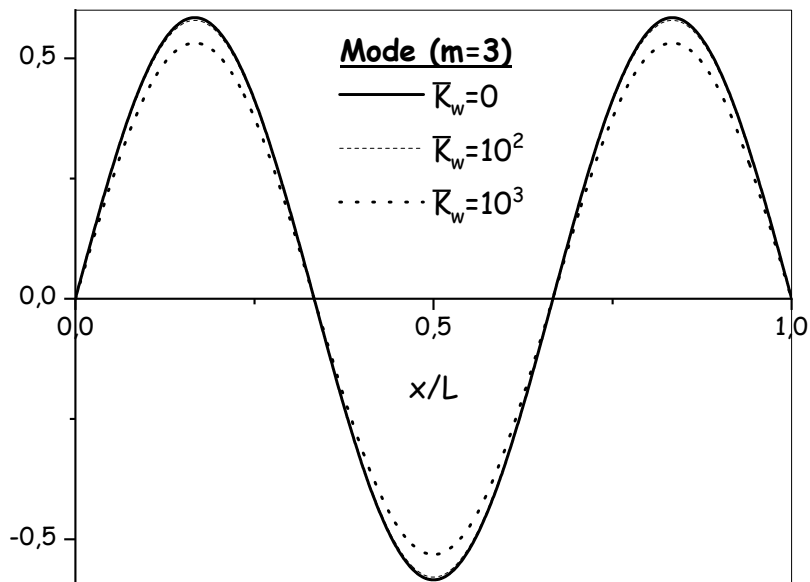


Figure 7: Mode shape of \bar{w} at the base surface of the beam with various aspect ratios for the third mode frequency ($m = 3, L/h = 20, \bar{K}_p / \pi^2 = 2.5$).

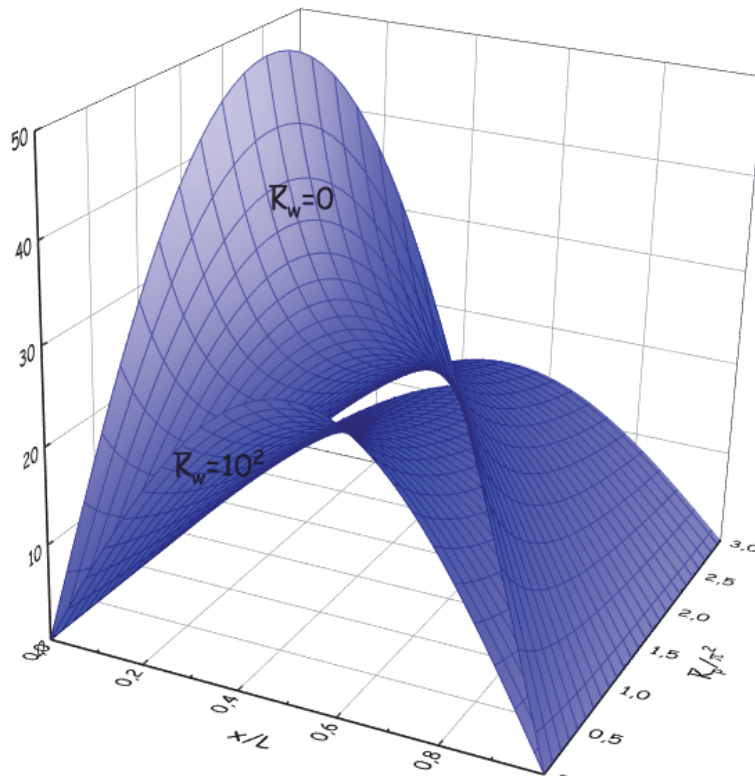


Figure 8: Effect of shear deformation, Pasternak parameter \bar{K}_p , and Winkler parameter \bar{K}_w on the deflection of isotropic homogeneous beams under uniform load.

In Fig. 2, the non-dimensional transverse displacement is plotted against the Pasternak parameter and several values of the Winkler parameter. It can be drawn from this curve that the higher the Pasternak's foundation parameter, the lower the transverse displacement and the same thing for the Winkler parameter.

Fig. 3 presents the variation of the dimensionless critical-buckling load as a function of the Pasternak parameter and for various values of the Winkler parameter. It can be drawn from this curve that the dimensionless critical-buckling load increases linearly with the Pasternak parameter.

Fig. 4 presents the variation of the non-dimensional fundamental frequency in function of the Pasternak parameter and for various values of the Winkler parameter. It can be drawn from this curve that the higher the Pasternak's foundation parameter is, the higher the vibration frequency.

Figs. 5, 6 and 7 are respectively the first, second and third-order of mode shapes of the displacement w at the lower surface of the isotropic homogeneous beam on an elastic foundation.

The impact of shear deformation on the deflection of FG beams is shown in Fig. 8 for various values of Pasternak parameter and tow values of Winker parameter ($\bar{K}_w = 0, \bar{K}_w = 10^2$).

CONCLUSION

In this paper; an efficient theory is presented for bending; free vibration and analysis of the dimensionless critical - buckling load for functionally graded simply-supported beams reposed on two elastic parameters. This theory incorporates both shear deformation.

The governing equations and the boundary conditions are calculated using Hamilton's principle. The closed-form solutions are obtained by using Navier solution.

Numerical comparisons are made to illustrate the mastery of the current theory. The present theory satisfies the stress-free boundary conditions on the conditions on the upper and lower surfaces of the beam, and do not need a shear correction factor.



Detailed mathematical formulations are given and numerical results are established, while the emphasis is set on examining the effect of the several parameters. The results of the actual theory are almost identical to each other and conform well with the existing solutions.

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