Effect of the Number of Gaussian Points and Their Distribution on Image Quality

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Abstract

This research involves studying the influence of increasing the number of Gaussian points and the style of their distribution, on a circular exit pupil, on the numerical calculations accuracy of the point spread function for an ideal optical system and another system having focus error of $(0.25\lambda$ and $0.5\lambda)$

It was shown that the accuracy of the results depends on the type of distributing points on the exit pupil. Also, the accuracy increases with the increase of the number of points (N) and the increase of aberrations which requires on increas (N).

Introduction

The calculation of image assessment criteria (e.g. the strehl ratio, the point spread function (PSF) and the optical transfer function (OTF)) involves the evaluation of an integral of the general form (1):

$$F(u^{-}) = \int_{a}^{b} e^{iq(x,y)} dx \qquad \text{...[1]}$$

Where a,b are any arbitrary limit of integration and u' is a conaconal coordinate (1) which is related with the image that coordinates z' with the relation $z'=2\pi u'$. $q'(x,y)$ which is a general function of x,y .

This kind of integration contains an oscillatory function (sinc function) Therefore no analytical solution was found except for an aberration–free system. To solve equation [1], a complex numerical calculation is needed especially when the optical system has a high order aberration or one needs to calculate the intensity on the image plane far from the origin of the image (i.e $Z^{'}>0$).

Three questions must be answered, first; on which kind of numerical method should be applied to solve equation [1]) first, to a given accuracy; second, on how many rays (no. of points) should be taken, and third, on the type of rays (point) distribution among the exit pupil which shall give the accurate results.

Many approaches to these problems were done, First, by using simple quadrate, like Simpson rule which has been used for this purpose, but it was found the need for a large amount of computations even to obtain a reasonable degree of accuracy. Hopkins (2) used different technique, to evaluate the above integral .In this method he needed to calculate the values of the 1st order derivative $q^{-}(x,y)$ of the function q (x,y) of the points X_n (n) (n = 1,2,3,....,N) where N is the number of points. The use of these derivatives in Hopkins formula leads to significant errors in the results especially for ,a small value of N. Qusay K. Ahmed (3) used Filon method to calculate the edge spread function (ESF) and he found that Filon method gave an accurate method for the (ESF) on the image plane far from the origin. So he suggested to use Gauss quadrate so as to calculate the (ESF) at the origin of the image and Filon method to calculate (ESF) outside the origin ($z^{-}>0$).

A new method that obviates the need of the knowledge of the derivatives q(x,y) in solving equation [1] and also obviates the poor accuracy in Filon method in the origin of the image and the answers for the three questions mentioned above are presented in this paper.

Mathematical Formulation

The normalized complex amplitude $F(u^{'},v^{'})$ due to a point on the image plane is.

$$F(u',v') = NF \iint_{A^{-}} f(x,y)\exp[i2\pi(u'x+v'y)]dxdy \quad ..[2]$$

where A^{-} denotes the region of the exit pupil, NF is the normalization factor,$(u^{'},v^{'})$ are Hopkins canonical coordinates(1), and f(x,y) is the complex pupil function, which consists of two term, the first is the amplitude transmittance function τ (x,y) and the second is wave aberration polynomial W (x, y). It is given by :

$$f(x,y) = \tau(x,y)e^{ikW(x,y)} \quad ...[3]$$

where $k=2\pi/\lambda$

$$\tau(x,y) \text{ is taken } \left.\begin{array}{l} = 1 \text{ for the region } x^2 + y^2 \leq 1 \\ = 0 \text{ for the region } x^2 + y^2 > 1 \end{array}\right] \quad ...[4]$$

Strehll ratio $I_N(0)$ is defined as

$$I_N(0) = \left|\frac{F(0)}{F_O(0)}\right|^2 = |F_N(0)|^2 \quad ...[5]$$

where F(0) and F_o (0) is the central amplitude for the optical system and perfect optical system (i.e w (x, y) = 0 and τ(x, y) = 1) respectively.

The limit of integration is taken over a circular exit pupil of normalized area equal to π. So the limits of integration are from –1 to +1 and $-\sqrt{1-y^2}$ to $\sqrt{1-y^2}$ for y and x respectively.

Since the intensity distribution over the image plane is symmetrical, therefore one coordinates is sufficient and v $^{-}$ may be set equal to zero. So eq.[2] becomes :

$$F(z') = \frac{1}{\pi}\int_{-1}^{+1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} e^{ik(W(x,y)+\bar{z}x)}dxdy \quad ...[6]$$

where $z' = 2\pi u'$

$$PSF(Z') = |F(Z')|^2 \quad ...[7]$$

Eq. [7] has not generally an analytical solution except for perfect optical system (aberration free system) which is first order Bessel function (J_1)

$$PSF = \left| \frac{2J_1(2\pi P)}{2\pi P} \right|^2 \qquad ...[8]$$

Where $P = \sqrt{(u'^2 + v'^2)}$ and for optical system having focus error W_{20} which represent sinc function of the form

$$PSF(0,0) = \left| \sin c(\frac{\pi W_{20}}{\lambda}) \right|^2 \qquad ..[9]$$

The most cases approximate numerical methods have to be used to compute this diffraction pattern.

A new method (Gaussian quadrate) was used in this study. By this method a subdivision of the pupil into small area elements was made and an approximate integration was carried out for each case, Then, the irradiant of the image point was obtained by adding the results.

As shown in Fig. (1) the exit pupil was divided into a finite number of subintervals by using a very fine mesh. By using Gaussian quadrature (4), suppose the formula of the general form is

$$\int_a^b f(s)ds = \frac{b-a}{2} \sum_{i=1}^{N} H_i f(x_i) \qquad ...[10]$$

Where s and x are related by,

$$s = \frac{b+a}{2} + \frac{b-a}{2} x$$

and N is the number of Gauss points which is taken from Legendre 2N-1 polynomial [4] and H_i are the corresponding weigth factors. Fortunantly H_i and x_i for the integral (-1 to +1) have been tabulated by AAbramowitz(5) up to N=96.

$$\int_a^b f(s) = \frac{b-a}{2} \sum H_i f(\frac{b+a}{2}) + \frac{b-a}{2} x)$$

By using equation [10],(Gaussian numerical method), and since the intensity is real, the imagenary part of the above equation must be zero, then equation [6] become,

$$F(z^{-}) = \frac{1}{\pi}\sum_{i} H_i \sum_{j} H_j(j)(1-y^2)^{0.5}\cos(k(W(x,y)+z^{-}x) \quad ..[11]$$

where i represents the Gaussian points along the y-axis and j represents the point along x-axis ,Hi and Hj are the weights for these points.
To compute equation [7], a program has been written which gives the value of F (z^{-}) at any point of the image plane .The number of points and the shape of its distribution among the exit pupil is an input for this program.

Results and Discussion

The influence of the distribution of the points on the accuracy of the results has been investigated. Two kinds of distributions were studied; the first one is symmetric (represented by Fig. (1)). The second one (modified distribution) is unsymmetrical depending on the increase of the points in areas where aberrations increase, which is normally near the edges, at the exit pupil (represented by Fig. (2)). We have used less points in the areas with short distance from the center and more points when coming closer to it.

The compression between the numerical results (equation [11]) after squaring it with that of the analytical solution for perfect optical system (equation [8]) and the results for system with focus error W_{20}= 0.25 λ, 0.5 λ) which is shown in tables (1 and 2) that the accuracy of the second distribution (modified distribution) is better than the first one for different values of (N).
The influence of the number of the mesh (No. of Gauss points) on the accuracy of the PSF evaluations for an optical system with focus error (W_{20} = 0 ,0.25λ,0.5λ) is shown in tables (1 ,2,3,and 4).
Bellow is the summary of figure(3):
-For an aberration free system without focus error (W_{20} = 0), the number of points which gives accurate results is (10) for the PSF on the image plane outside the origin less than p20 (i.e. z^{-}<20). As

z increases N must be increased but as shown in Fig. (3; first column) 10 points are of enough accuracy.

-For a system with focus error ($W_{20} = 0.25$) 10 points gives accurate results for $z^- < 20$ and N = 15 gives good results for $z^- < 25$. As z increases N must be increased as shown in Fig. (3;second column)

-For a system having focus error ($W_{20} = 0.5$) N > 18 must be taken and N must be increased as z is increased (Fig. (3;third column)).

- From tables 2 and 4 , which represent the error percentage between numerical calculation of equation [6] using Gauss method and the analytical solution results of equation [8] for perfect system and analytical solution results of equation [9] for a system with focus error W_{20}= 0.0, 0.25λ ,0.5 λ . It is shown that the second distribution (modified distribution) give an accurate result (error percentage =0.1248 for 20 Gauss points than the first distribution (symmetrical distribution) (error percentage =0.0904 for 20 Gauss points whose number is suggested to be used in calculations).

Conclusion

From the results it is shown that ,

a. More accuracy requires increasing N points.

b. High aberrations demand increasing in N points.

c. Increasing z^- (away from the center of the image) needs large numbers of N.

d. Because of the difficulty of changing N (for it is an input data) and to avoid changing it every time when calculating different types of aberrations, we suggest the use of (20) points because z^- in our calculations requires not more than (25).

At last, from the result above, we suggest that the modified distribution to be used instead of the symmetrical one.

References

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Table (1) Values of Strehl ratio for symmetrical distribution with different values of (N)

W(x,y) N	W_{20}=0.0	W20=0.25	W20 = 0.5	W20 = 1
4	1	.8177297	.3987392	5.7321E-2
6	1	.813491	.4039382	6.1573E-2
8	1	.8118558	.403987	1.9139E-2
10	1	.8112482	.4045198	5.2593E-3
12	1	.8109726	.4048262	1.7656E-3
14	1	.8089282	.4013312	7.5397E-4
16	1	.8092661	.402043	8.7977E-4
18	1	.810049	.40383	2.6255E-4
20	1	.809940	.4034438	3.3604E-4
22	1	.8103027	.4043387	7.8231E-4
24	1	.810148	.4039729	9.7398E-4
26	1	.8102333	.404222	9.5635E-4
28	1	.8103099	.404538	8.2822E-4
30	1	.8103759	.4046561	6.6595E-4
32	1	.8104324	.4048299	5.1113E-4
34	1	.8104376	.4049658	3.7942E-4
36	1	.8105128	.4050783	2.7629E-4
38	1	.8100313	.4039826	3.4750E-4
40	1	.8100841	.4041075	3.43109E-4

Table (2) Absolut error percentage between analytical and numerical calculations PSF for different values of (N). (Symmetrical distribution)

N	W20 = 0.25	W20=0.5	W20= 1
4	.8704	1.6721	2294.06
6	0.3475	0.2887	2471.6
8	0.1458	0.2767	699.3
10	0.0709	0.1451	119.6
12	0.0369	0.0695	26.258
14	0.2152	0.9322	1341.48
16	0.1735	0.7565	136.74
18	0.0769	0.3154	110.96
20	0.0904	0.4108	85.96
22	0.0457	0.1913	67.32
24	0.0647	0.2801	59.32
26	0.0542	0.2187	60.05
28	0.0448	0.1614	65.40
30	0.0366	0.1115	72.18
32	0.0297	0.0686	78.65
34	0.0243	0.0351	84.15
36	0.0197	0.0073	88.46
38	0.0791	0.2778	115.64
40	0.0726	0.2469	114.32

Table (3) Strehl ratio for modified distribution with different values of (N)

N	0	$W_{20} = 0.25$	$W_{20} = 0.5$	$W_{20} = 1$
4	1	.8177297	.3987392	5.732E-2
6	1	.8137603	.4070483	4.85659E-2
8	1	.8119709	.4051068	2.13307E-2
10	1	.8113036	.4049488	1.12777E-2
12	1	.8110061	.4050174	6.71111E-3
14	1	.8108425	.4050853	4.30710E-3
16	1	.8043577	.3891387	1.97127E-3
18	1	.8112419	.4065035	2.54300E-3
20	1	.8105688	.4049512	1.54712E-3
22	1	.8092161	.4010942	4.86850E-3
24	1	.8101271	.4039764	9.01047E-4
26	1	.8106369	.4052821	8.42296E-4
28	1	.8078637	.3984478	4.48503E-4
30	1	.8094763	.4027038	1.28868E-4
32	1	.8071343	.3970445	3.22749E-4
34	1	.8091857	.4017493	6.56485E-4
36	1	.8098481	.4035054	1.51313E-4
38	1	.8103285	.4046341	4.88732E-4
40	1	.8089477	.4013932	3.21509E-6

Table (4) Absolut error percentage between analytical and numerical calculations for different values of (N). (Modified distribution)

N	W20 = 0.25	W20 = 0.5	W20 = 1
6	0.3812	0.4789	1932.2
8	0.16047	0.002962	790.89
10	0.0781	0.03929	371.01
12	0.04145	0.02236	180.298
14	0.02127	0.005603	79.8897
16	0.77865	3.94198	0.3858
18	0.0705	0.3444	6.2105
20	0.01248	0.03870	35.384
22	0.17934	0.99079	103.337
24	0.06696	0.27933	62.369
26	0.40810	0.04297	64.821
28	0.34617	1.64405	81.2680
30	0.14724	0.593471	105.382
32	0.4361	1.99045	13.479
34	0.18309	0.8290	72.5665
36	0.10138	0.39559	632.97
38	0.042125	0.11698	79.587
40	0.21245	0.916999	99.8657

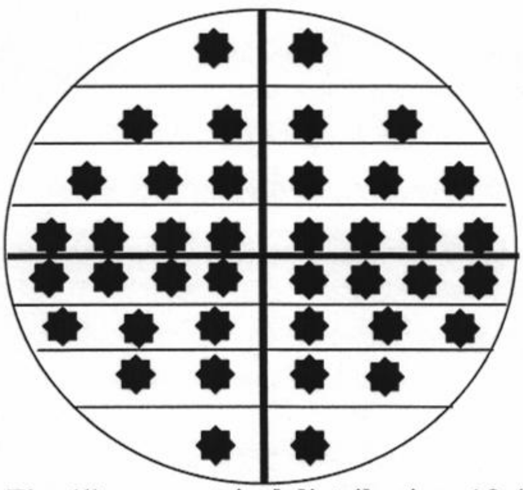

Fig. (1) symmetrical distribution (2,4,6,8,8,6,4,2)

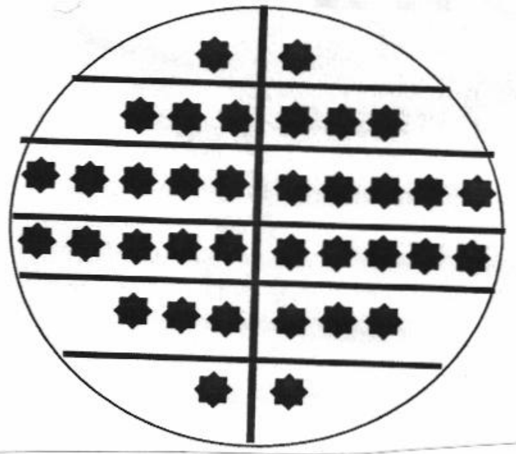

Fig.(2) non-symmetrical distribution (Modified distribution) (2,6,10,10,6,2)

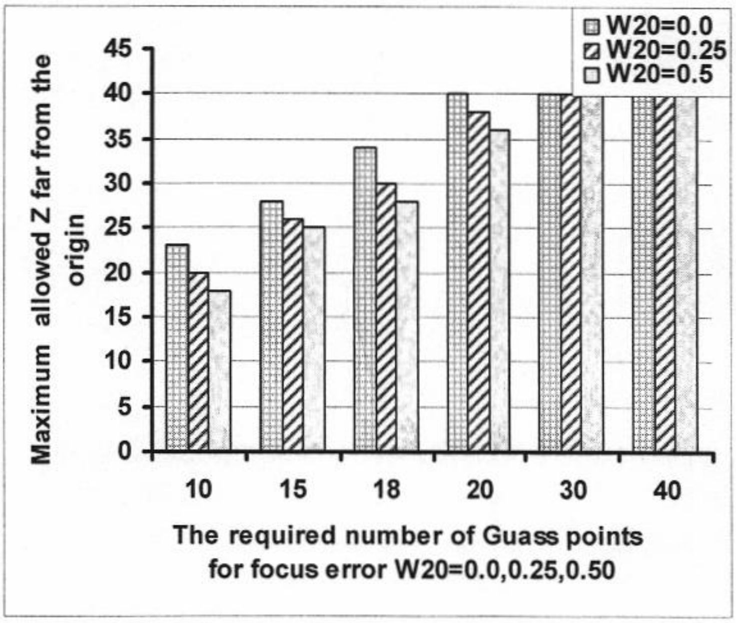

Fig. (3) Maximum allowed Z and Gauss points for accurate PSF for system with focus error W20=0.0, 0.25 and 0.50

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تأثير عدد وشكل توزيع نقاط كاوس في تقييم الصورة

علي هادي الحمداني
الجامعة التكنولوجية/ قسم هندسة الليزر والبصريات ألالكترونية

الخلاصة

يتناول هذا البحث دراسة تأثير الزيادة في عدد نقاط كاوس (Gauss) وأسلوب توزيع هذه النقاط على فتحة خروج (Exit pupil) دائرية الشكل على دقـة الحـسابات العددية (Numerical Calculations) لدالة (PSF) لمنظومة بصرية مثالية ومنظومة أخـرى تحتـوي على الزيغ البصري (Focus Error) بمقـدار (0.25 و0.5) . تبين أن دقة النتائج تعتمد على طريقة التوزيع للنقاط على فتحة الخروج، وكـذلك تـزداد الدقة مع زيادة عدد النقاط. وكلما ازدادت العيوب ظهرت الحاجة لزيـادة عـدد النقـاط.