Direct Estimation for Approximation by Bernstein Polynomial by Using Ditzian-Totik and Average in $L_p[a,b], 1 \le p < \infty$ Modulus of Smoothness

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Abstract

The purpose of the paper is to find the degree of the approximation of a functions f be bounded , measurable and defined in interval $[a,b]$ by Bernstein polynomial in L_p space $1 \le p < \infty$ by using Ditzian-Totik modulus of smoothness and k^{th} average modulus of smoothness.

Introduction

Let f be abounded and measurable function in $[0,1]$ for which

$$\Big(\int_{[a,b]} |f|^p dx \Big)^{\frac{1}{p}} < \infty \qquad 1 \le p < \infty$$

Bernstein polynomial of f is defined by

$$B_n(f,x) = \sum_{k=0}^{n} f(\frac{k}{n}) P_{n,k}(x);$$

$$P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad , n \ge 2 \qquad [1.1]$$

For $1 \le p < \infty$, let us define

$$\Delta x = \frac{\sqrt{\varphi(x)}}{\sqrt{n} + \frac{1}{n}}$$

Where [1.2]

$$\varphi(x) = \sqrt{x}(1-x)$$

The Ditzian-Totik modulus of smoothness which is defined for such f will be follows

$$\omega_r^{\varphi}(f,\delta,[a,b])_p = \sup_{0 \le h \le \delta} \left\| \Delta_{h\varphi(.)}^r (f,.) \right\|_{L_p[a,b]} \qquad [1.3]$$

where

$$\Delta_{h\varphi}^r f(x) = \begin{cases} \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x+i\varphi h) & x + ih\varphi \in [a,b] \\ 0 & x + ih\varphi \notin [a,b] \end{cases} \qquad [1.4]$$

The K^{th} average modulus of smoothness for $f \in L_p[a,b], 1 \le p < \infty$ is defined by

$$\tau_2(f,\delta,[a,b]) = \left\| \omega_k(f,\delta,[a,b]) \right\|_p \qquad [1.5]$$

Let $W_p^r[a,b]$ be the set of all functions f on $[a,b] \ni f^{(k-1)}$ which are absolutely continous and $f^{(k)} \in L_p$, (1).

Consider now

$$S_u(x) = \begin{cases} u(x-1) & 0 \le u \le x \le 1 \\ x(u-1) & 0 \le x \le u \le 1 \end{cases}$$

At a number of places in this paper use some standard inequalities concerning sums and integrals which(2),(3) are:

1-If $f, g \in L_p, 1 \le p < \infty$ then $\|f+g\|_p \le c(\|f\|_p + \|g\|_p)$, c is constant [1.6]

2- $f, g \in L_p, 1 \le p < \infty$ then

$$\left(\int_x |f.g|^p dx\right)^{\frac{1}{p}} \le \left(\int_x |f|^p dx\right)^{\frac{1}{p}} . \left(\int_x |g|^q dx\right)^{\frac{1}{q}} \ni \frac{1}{p} + \frac{1}{q} = 1 \qquad [1.7]$$

Lemma 1 (4)

S_u has he following properities

$$\begin{cases} (i)\ (B_n(S_u, x) - S_u(x)) \ge 0 \\ (ii)\ \int_0^1 (B_n(S_u, x) - S_u(x))dx \le n^{-1}\varphi(u), \qquad \text{for } 0 \le u \le 1 \end{cases}$$

Lemma 2

Let $n \in N$ and g, g' are absolutely continuous functions on $[0,1]$ then

$$B_n(g,x) - g(x) = \int_0^1 (B_n(S_u, X) - S_u(x))g''du \ , 0 \le x \le 1 .$$

Proof:-

For x=0 or x=1, we obviously have the result , so let 0<x<1 then by using definition of $S_u(x)$ we get:

$$\int_0^1 S_u(x)g''(u)du = \int_0^x u(x-1)g''(u)du + \int_x^1 x(u-1)g''(u)du$$

$$= (x-1)\int_0^x ug''(u)du + x\int_x^1 (u-1)g''(u)du \qquad [1]$$

By integrating by parts, we get that

$$\int_0^x ug''(u)du = xg'(x) - g(x) + g(0) \qquad [2]$$

And

$$\int_x^1 (u-1)g''(u)du = -(x-1)g'(x) - g(1) + g(x) \qquad [3]$$

Then by substituting [2]and(3) in (1),we obtain

$$\int_0^1 S_u(x)g''(u)du = g(x) + g(0)(x-1) - g(1)x$$

Now , for $\int_0^1 B_n(S_u, x)g''(u)du$, we have

$$\int_0^1 B_n(S_u, x)g''(u)du = \int_0^1 \sum_{k=0}^n S_u(\frac{k}{n})P_{n,k}(x)g''(u)du$$

$$= \sum_{k=0}^n P_{n,k}(x)\int_0^1 S_u(\frac{k}{n})g''(u)du$$

Evaluate $\int_0^1 S_u(\frac{k}{n})g''(u)du$, we have:

$$\int_0^1 S_u(\frac{k}{n})g''(u)du = \int_0^{k/n} u(\frac{k}{n}-1)g''(u)du + \int_{k/n}^1 \frac{k}{n}(u-1)g''(u)du$$

By the same lines used for $\int_0^1 S_u(x)g''(u)du$ we obtain

$$\int_0^1 S_u(\frac{k}{n})g''(u)du = g(\frac{k}{n}) + g(0)(\frac{k}{n}-1) - g(1)\frac{k}{n}$$

Then

$$\int_0^1 B_n(S_u, x)g''(u)du = \sum_{k=0}^n P_{n,k}(x)(g(\frac{k}{n}) + g(0)(\frac{k}{n}-1) - g(1)\frac{k}{n})$$

$$= B_n(g;x) + B_n(g(0)(x-1);x) - B_n(g(1)x;x)$$

Now since Bernstein polynomial is linear operator and preserves linear function , we obtain

$$\int_0^1 B_n(S_u, x)g''(u)du = B_n(g;x) + g(0)(x-1) - g(1)x$$

Hence

$$\int_0^1 (B_n(S_u, x) - S_u(x))g''(u)du = B_n(g;x) + g(0)(x-1) - g(1)x - g(x) - g(0)(x-1) + g(1)x$$

$$= B_n(g, x) - g(x).$$

The main result

We shall prove a direct theorem in $L_p[a,b], 1 \le p < \infty$ for Bernstein operator $B_n(f)$ interms of Ditzian modulus of smoothness and average modulus of smoothness .

The direct theorem

If f is a measurable and bounded function on $[a,b]$ then $1 \le p < \infty$
We have

$$\|f(x) - B_n(f)\| \le c\omega_r^{\varphi}(f,\delta,[a,b])_p + \tau_2(f,\delta,[a,b]).$$

Where δ may be a function of x .

Now we also need the following lemmas to prove our theorem:

Lemma 3

For $n \in N$ and $g \in \omega_p^r[a,b], 1 \le p < \infty$ we have

$$\|B_n(g) - g\|_p \le cn^{-p}\|\varphi.g''\|_p \text{ where } c \text{ is constant.}$$

Proof

Consider the linear operator[3]

$$T(f) = \int_0^1 (B_n(S_u, X) - S_u(X))\varphi^{-1}(u) f(u) du$$

Then

$$\|T(f)\|_p = \left\| \int_0^1 (B_n(S_u, X) - S_u(X))\varphi^{-1}(u) f(u) du \right\|_p$$

$$= \left[\int_0^1 \left| \int_0^1 (B_n(S_u, X) - S_u(X))\varphi^{-1}(u) f(u) du \right|^p dx \right]^{\frac{1}{p}}$$

$$= \left[\int_0^1 \left| n^{-1}\varphi(u)\varphi^{-1}(u) f(u) du \right|^p dx \right]^{\frac{1}{p}}$$

by (1.6) we get

$$\|T(f)\|_p \le \frac{c}{n}\left(\int_0^1 |f(u)|^p du \right)^{\frac{1}{p}}$$

$$\|T(f)\|_p \le \frac{c}{n}\|f\|_p$$

then let $f = \varphi.g''$ and by lemma (2)we obtain

$$\|T(\varphi.g'')\|_p = \left\|\int_0^1 (B_n(S_u, X) - S_u(X))\varphi^{-1}(u)\varphi(u)g''(u)du\right\|_p$$

$$= \|B_n(g) - g\|_p$$

So that

$$\|B_n(g) - g\|_p = \|T(\varphi.g'')\|_p$$

$$\le \frac{c}{n^p}\|\varphi.g''\|_p .$$

lemma 4

Let $f \in W_p^r[a,b]$ then $1 \le p < \infty$ we have

$$\|\Delta^2 F''\|_p \le \frac{c}{n^p}\omega_r^p(f,\delta,[a,b])$$

Proof:

Setting

$$F(x) = \frac{1}{n^{1+p}}\sum_{i=1,2}(-1)^{3-i}\binom{2}{i}\left(\frac{2}{\varphi ih}\right)^2 \int_0^{\frac{\varphi ih}{2}}\int_0^{\frac{\varphi ih}{2}} \varphi^{-1}(x)f(x+u_1+u_2)du_1du_2, h = \frac{1}{n}$$

$$F''(x) = \frac{\phi^{-1}(x)}{n^{1+p}}\sum_{i=1,2}(-1)^{3-i}\binom{2}{i}\left(\frac{2}{\phi ih}\right)^2 \Delta^2_{\frac{\phi ih}{2}} f(x)$$

Now clear from (1.2) we have

$$\|\Delta^2 F''\|_p = \left\|\left(\frac{\sqrt{\phi(x)}}{\sqrt{n}+\frac{1}{2n}}\right)^2 .F''\right\|_p$$

$$\leq \left\| \frac{\phi(x)}{n} \cdot \frac{\phi^{-1}(x)}{n^{1+p}} \sum_{i=1,2} (-1)^{3-i} \binom{2}{i} \left(\frac{2}{\phi i h} \right)^2 \Delta^2_{\frac{\phi i h}{2}} f(x) \right\|_p$$

$$\left\| \Delta^2 F'' \right\|_p \leq \frac{1}{n^{2+p}} \left\| 8\Delta^2_{\frac{\phi h}{2}} f(x) - \Delta^2_{\phi h} f(x) \right\|_p$$

$$\leq \frac{c}{n^{2+p}} \left(\left\| \Delta^2_{\frac{\phi h}{2}} f(x) \right\|_p + \left\| \Delta^2_{\phi h} f(x) \right\|_p \right)$$

$$\leq \frac{c}{n^{2+p}} \left(\sup_{h \leq \delta} \left\| \Delta^2_{\frac{\phi h}{2}} f(x) \right\|_p + \sup_{h \leq \delta} \left\| \Delta^2_{\phi h} f(x) \right\|_p \right)$$

$$\leq \frac{c}{n^{2+p}} \omega_2^{\phi}(f, \delta, [a,b])_p$$

$$\leq \frac{c}{n^p} \omega_2^{\phi}(f, \delta, [a,b])_p .$$

Lemma 5

If $f \in L_p[a,b]$, then $(1 \leq p < \infty)$ we have

$$|F(x) - f(x)| \leq 2\omega_2^{\phi}(f, x+h, h)$$

Proof:-

$$|F(x) - f(x)| = \left| \frac{\phi^{-1}(x)}{n^{1+p}} \sum_{r=1,2} (-1)^{3-r} \binom{2}{r} \left(\frac{2}{rh\phi} \right)^2 \int_0^{rh\phi/2} \int_0^{rh\phi/2} f(x + u_1 + u_2) du_1 du_2 - f(x) \right|, h = \frac{1}{n}$$

$$\leq \left| \sum_{r=1,2} (-1)^{2-r} \binom{2}{r} (\phi h)^{-2} \int_0^{\phi h} \int_0^{\phi h} f(x + r(\frac{t_1 + t_2}{2}) dt_1 dt_2 + f(x) \right|$$

$$\leq \left| (\phi h)^{-2} \int_0^{\phi h} \int_0^{\phi h} \sum_{r=0}^{2} (-1)^{2-r} \binom{2}{r} f(x + r(\frac{t_1 + t_2}{2}) dt_1 dt_2 \right|$$

$$\leq (\phi h)^{-2} \int_0^{\phi h} \int_0^{\phi h} \left| \Delta^2_{\frac{t_1 + t_2}{2}} f(x) \right| dt_1 dt_2$$

Let $t_1 = 2u_1 - u_2$ and $t_2 = u_2$ we obtain

$$|F(x)-f(x)| \le 2(\phi h\ )^{-2}\int_0^{\phi h}\int_0^{\phi h}|\Delta_{u_1}^2 f(x)|du_1 du_2$$

$$\le 2(\phi h)^{-1}\int_0^{\phi h}|\Delta_{u_1}^2 f(x)|du$$

$$\le 2(\phi h)^{-1}\int_0^{\phi h}\sup_{0\le u\le\phi h}|\Delta_{u_1}^2 f(x)|du$$

$$\le 2\omega_2^{\phi}(f,x+h,h).$$

Lemma 6

If $f\in L_p[a,b]$, then $(1\le p<\infty), n\ge 2$ we have

$$\|B_n(f)-B_n(F)\|\le \tau_2(f,\delta,[a,b])_p.$$

Proof

By (1.1)and (lemma 5) we have

$$\|B_n(f)-B_n(F)\|_p = \int_0^1\left|\sum_{k=0}^{n}(f(\frac{k}{n})-F(\frac{k}{n}))P_{n,k}(x)\ \right|^p dx$$

$$\le c\int_0^1\left|\sum_{k=0}^{n}\omega_2^{\phi}\left(f,\frac{\phi(k+1)}{n},[a,b]\right)P_{n,k}(x)\right|$$

By (1.6)we get

$$\|B_n(f)-B_n(F)\|_p \le c_p\left(\sum_{k=0}^{n}\int_0^1\left|\omega_2^{\phi}\left(f,\frac{\phi(k+1)}{n},[a,b]\right)\right|^p\right)^{\frac{1}{p}}dx.\left(\int_0^1(P_{n,k}(x)^q\right)^{\frac{1}{q}}dx \ni \frac{1}{p}+\frac{1}{q}=1$$

$$= M_1.M_2 \quad , \quad M_2=c \quad ,\text{c is constant}$$

$$M_1\le c_p\left(\sum_{k=0}^{n}\int_{k/n}^{k+1/n}\left|\omega_2^{\phi}\left(f,\frac{\phi(k+1)}{n},[a,b]\right)\right|^p\right)^{\frac{1}{p}}dx, h=\frac{1}{n}$$

$$\le c_p\int_{\phi h}^{1+\phi h}\omega_2^{\phi}(f,x+\phi h,[a,b])^p$$

$$=\tau_2(f,x,[a,b])_p.$$

Lemma 7

For $n \in N$ and $F \in W_p^r[a,b], 1 \le p < \infty$ we have

$$\|F - B_n(F)\| \le c\omega_2^\phi(f,\delta,[a,b])_p.$$

Proof

By using (lemma 3) and (lemma 4) we get

$$\|F - B_n(F)\|_p \le \frac{c}{n^p}\|\phi F''\|_p$$
$$\le cn^{-3-p}\|\Delta^2 F''\|_p$$
$$\le c\omega_2^\phi(f,\delta,[a,b])_p.$$

Proof of the direct theorem

$$\|f - B_n(F)\|_p = \|f - F + F - B_n(F) + B_n(F) + B_n(f)\|_p$$
$$\le \|f - F\|_p + \|F - B_n(F)\|_p + \|B_n(F) - B_n(f)\|_p$$

Using (lemma 5), (lemma 6), (lemma 7)

$$\|f - B_n(F)\|_p \le 2\omega_2^\phi(f, x+h, h)_p + c_1\varpi_2^\phi(f,\delta,[a,b])_p + \tau_2(f,x,[a,b])_p$$

$$\le c_2\varpi_2^\phi(f,\delta,[a,b])_p + \tau_2(f,x,[a,b])_p \quad .$$

References

1 . Hu,Y.; Kopotun, K. Yu, X.M. (1997) . Canada.J.of math.,49(1);74-99.

2 . Ygmund, A. Z (1958): Trigonometric Series . Voi.I.II comberidge ,p.28.

3 . Siddiq ,A.H. (1986): functional analysis with applications , Professor of Mathematics Aligarh Muslim university , Algarh ,p.292.

4 . Ivanov ,K.G. (1984).Bulg, Acad of Sci . 48 :421-429.

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التقدير المباشر للتقريب باستعمال متعددة حدود برنشتاين في الفضاءات $L_p[a,b], 1 \le p < \infty$ باستخدام مقياس نعومة ديتيزين ومتوسط المقياس

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المستخلص

الغرض من هذا البحث ايجاد درجة اقتراب الدوال المقيدة القابلة للقياس و المعرفة على المدة $1 \le p < \infty$ باستعمال متعددة حدود برنشتاين بدلالة مقياس نعومة ديتيزين ومتوسط المقياس .. .