Comparison of the Suggested loss Function with Generalized Loss Function for One Parameter Inverse Rayleigh Distribution

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Abstract

The experiences in the life are considered important for many fields, such as industry, medical and others. In literature, researchers are focused on flexible lifetime distribution.
In this paper, some Bayesian estimators for the unknown scale parameter θ of Inverse Rayleigh Distribution have been obtained, of different two loss functions, represented by Suggested and Generalized loss function based on Non-Informative prior using Jeffery's and informative prior represented by Exponential distribution. The performance of θ estimators is compared empirically with Maximum Likelihood estimator, Using Monte Carlo Simulation depending on the Mean Square Error (MSE). Generally, the preference of Bayesian method of Suggested loss function with Exponential informative prior are the best estimator compared to others.

Key words: Inverse Rayleigh Distribution, Bayes estimator, Suggested loss function(SLF), Generalized Loss Function(GLF), Maximum likelihood (MLE), Jeffery prior; Exponential informative prior MSE.

Introduction

In the term of reliability studies many applications used the Distribution of Inverse Rayleigh. It was introduced in literary (Trayer 1964) of reliability with survival studies, life distribution which characterized via a monotonic failure rate. In 1972 Voda has explained that is lifetimes distribution that related with served types of experimental unite can approximated by the Inverse Rayleigh distribution [1] in this regard let consider $x_1, x_2, \dots, x_n$ to be a randomize sample of independent observation from a one parameter Inverse Rayleigh distribution with probability of density (p.d.f) to scale parameter (θ) as shown in[2].

$$f(x,\theta) = \frac{2\theta}{x^3} . e^{\frac{-\theta}{x^2}} \;\; ; \;\; x > 0 \;\; , \;\; \theta > 0 \qquad (1)$$

The Corresponding Cumulative of distribution Function (CDF) is:

$$F(x;\theta) = e^{\frac{-\theta}{x^2}} \;\; ; \qquad x > 0 \;\; , \;\; \theta > 0 \qquad (2)$$

AL-Shareefi. E. F. (2015). Suggested loss Function in estimating the Scale parameter for Laplace distribution [3].

$$L(\hat{\theta},\theta) = \frac{\left(\sum_{i=0}^{k} a_j \theta^j\right)\left(\hat{\theta}-\theta\right)^2}{\theta^c} \;\; , \;\; j = 0,1,2,3,\dots,k \;\; , \qquad c\,,a : are\ constants$$

Maximum Likelihood Estimator (MLE).

Maximum Likelihood can be obtained for the scale parameterθ, as following:
let $x_1, x_2, \dots, x_n$ Suppose to be random sample with density function(1). The likelihood function illustrated by [4].

$$L(\, x_1, x_2, \dots, x_n;\, \theta) = 2^n \theta^n \prod_{i=1}^{n} \frac{1}{x_i^3} exp\left[-\theta \sum_{i=1}^{n} \frac{1}{x_i^n}\right] \qquad (3)$$

R. A. Fisher (1920) proposed The Maximum Likelihood method [5], since then used extensively. This method consider the most popular algorithm to estimate the unknown parameter θ to specify the probability function $f(\,x;\,\theta)$, based on the observation $(\,x_1, x_2, \dots, x_n)$ which were independently sample from the Inverse Rayleigh distribution.

In Equation (3). By using the logarithm of the likelihood function and differentiation with respect to (θ), will get:

$$\frac{\partial ln(x_{i,}\,\theta)}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \frac{1}{x_i^2}$$

$$let\ \frac{\partial ln(x_{i,}\,\theta)}{\partial\theta} = 0$$

Hence:

$$\hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i^2}} = \frac{n}{T} \qquad ,Wher\ T = \sum_{i=1}^{n} \frac{1}{x_i^2} \qquad (4)$$

Some Bayes Estimators

1- Bayes Estimator by using Jeffreys prior Information [6].

We assumed that θ has non-information prior density, which is defined as:

$$g \propto \sqrt{I(\theta)}$$

Where $I(\theta)$ that considered Fisher information which is defined as following:

$$I(\theta) = -nE\left[\frac{\partial^2 \ln f(x_i;\,\theta)}{\partial\theta^2}\right] \;\; , \quad Hence:$$

$$g_1(\theta) = b\sqrt{-nE\left(\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right)} \qquad (5)$$

where b is a constant.

$$\ln f(x;\theta) = ln(2) + ln(\theta) - 3ln(x) - \frac{\theta}{x^3}$$

$$\frac{\partial \ln f(x;\theta)}{\partial\theta} = \frac{1}{\theta} - \frac{1}{x^3}$$

$$\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2} = -\frac{1}{\theta^2}$$

Hence, we get:

$$E\left[\frac{\partial^2 \ln f(x;\theta)}{\partial\theta^2}\right] = -\frac{1}{\theta^2} \qquad (6)$$

by substitution (6) with (5), will get:

$$g_1(\theta) = \frac{1}{\theta}\sqrt{n} \quad , \quad \theta > 0$$

The posterior density function is:

$$h_1(\theta|x_1, x_2, \dots x_n) = \frac{g_1(\theta).L(\theta;\ x_1, x_2, \dots, x_n)}{\int_0^\infty g_1(\theta).L(\theta;\ x_1, x_2, \dots, x_n)d\theta}$$

$$h_1(\theta|x_1, x_2, \dots x_n) = \frac{\theta^{n-1}.e^{-\theta T}}{\int_0^\infty \theta^{n-1}.e^{-\theta T}d\theta}$$

Hence, the posterior density functions of (θ) with Jefferys prior is:

$$h_1(\theta|x_1, x_2, \dots x_n) = \frac{T^n.\ \theta^{n-1}.e^{-\theta T}}{\Gamma n} \qquad (7)$$

The posterior density function of $(\theta)$is recognized as the density of Gamma distribution with parameters n and T

i.e: $\theta \sim Gamma\ (n, T)$

Hence:

$$E(\theta) = \frac{n}{T} \quad , \quad Var\ (\theta) = \frac{n}{T^2} \qquad Wher\ T = \sum_{i=1}^{n} \frac{1}{x_i^2}$$

2- Bayes Estimator by using Exponential prior Information [4].

Hence, (θ) reflect the information of prior exponential distribution:

$$g_2(\theta) = \frac{1}{b}.e^{\frac{-\theta}{b}} \quad ; \quad \theta > 0\ , \quad b > 0 \qquad (8)$$

$$h_2(\theta|x_1, x_2, \dots x_n) = \frac{g_2(\theta).L(\theta;\ x_1, x_2, \dots, x_n)}{\int_0^\infty g_2(\theta).L(\theta;\ x_1, x_2, \dots, x_n)d\theta} = \frac{\theta^n e^{-\theta\left(T+\frac{1}{b}\right)}}{\int_0^\infty \theta^n e^{-\theta\left(T+\frac{1}{b}\right)}d\theta}$$

$$h_2(\theta|x_1, x_2, \dots x_n) = \frac{\left(T+\frac{1}{b}\right)^{n+1}\theta^n e^{-\theta\left(T+\frac{1}{b}\right)}}{\Gamma(n+1)} \qquad (9)$$

Notice that:

$$\theta \sim \text{Gamma}(n+1\,, P) \quad , P = \sum_{I=1}^{N}\frac{1}{x_i^2} + \frac{1}{b} = T + \frac{1}{b}$$

3- Bayesian Estimators under Suggested Loss function

A new loss function suggested here which is called Modified Generalized loss function defined as following [4]:

$$L_{MGS}(\hat{\theta}, \theta) = \frac{\left(\sum_{i=0}^{k} a_j\theta^j\right)(\hat{\theta}-\theta)^2}{\theta^c} \quad , \quad \text{C=0, 1, 2, …, n is constant}$$

Where $L_{MGS}(\hat{\theta}, \theta)$ is modified by Al-Sherefi

Then, the Risk function under the Suggested loss function denoted by $R_{MGS}(\hat{\theta}, \theta)$ will be:

$$R_{MGS}(\hat{\theta}, \theta) = E[L_{MGS} = (\hat{\theta}, \theta)] = \int_0^\infty \frac{1}{\theta^C}\left(\sum_{j==0}^{k} a_j\,\theta^j\right)(\hat{\theta}-\theta)^2 h(\theta|\underline{x})d\theta$$

$$= a_0\hat{\theta}^2 E\left(\frac{1}{\theta^C}|\underline{x}\right) - 2a_0\hat{\theta}E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) + a_0 E\left(\frac{1}{\theta^{C-2}}|\underline{x}\right) + \; a_1\hat{\theta}^2 E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) - 2a_1\hat{\theta}E\left(\frac{1}{\theta^{C-2}}|\underline{x}\right) + a_1 E\left(\frac{1}{\theta^{C-3}}|\underline{x}\right) + \cdots + \; a_K\hat{\theta}^2 E\left(\frac{1}{\theta^{C-K}}|\underline{x}\right) - 2a_K\hat{\theta}E\left(\frac{1}{\theta^{C-(K+1)}}|\underline{x}\right) + a_K E\left(\frac{1}{\theta^{C-(K+2)}}|\underline{x}\right)$$

By taking partial derivative of $R_{MGS}(\hat{\theta},\theta)$ with respect to $\hat{\theta}$ and making it equal to zero yields:

$$\frac{R_{MGS}(\hat{\theta},\theta)}{\partial\hat{\theta}} = 2a_0\hat{\theta}\, E\left(\frac{1}{\theta^C}|\underline{x}\right) - 2a_0\, E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) + 2a_1\hat{\theta}E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) - 2a_1 E\left(\frac{1}{\theta^{C-2}}|\underline{x}\right) + \cdots + 2a_K\hat{\theta}E\left(\frac{1}{\theta^{C-K}}|\underline{x}\right) - 2a_K E\left(\frac{1}{\theta^{C-(K+1)}}|\underline{x}\right) = 0$$

$$2a_0\hat{\theta}\, E\left(\frac{1}{\theta^C}|\underline{x}\right) + 2a_1\hat{\theta}E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) + \cdots + 2a_K\hat{\theta}E\left(\frac{1}{\theta^{C-K}}|\underline{x}\right) = 2a_0\, E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) + 2a_1 E\left(\frac{1}{\theta^{C-2}}|\underline{x}\right) + \cdots + 2a_K E\left(\frac{1}{\theta^{C-(K+1)}}|\underline{x}\right)$$

$$\hat{\theta}_{MGS} = \frac{a_0 E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) + a_1 E\left(\frac{1}{\theta^{C-2}}|\underline{x}\right) + \cdots + a_K E\left(\frac{1}{\theta^{C-(K+1)}}|\underline{x}\right)}{a_0 E\left(\frac{1}{\theta^C}|\underline{x}\right) + a_1 E\left(\frac{1}{\theta^{C-1}}|\underline{x}\right) + \cdots + a_K E\left(\frac{1}{\theta^{C-K}}|\underline{x}\right)} \qquad (10)$$

- **With Jeffrey's prior information**

According to the posterior density function $h_1(\theta|\underline{x})$, we derived $E(\theta^m|\underline{x})$, $E\left(\frac{1}{\theta^m}|\underline{x}\right)$.

Since $\theta \sim Gamma\ (n, T)$, $then$

$$E(\theta^m) = \int_{\forall\theta} \theta^m\, h_1(\theta|\underline{x})d\theta = \frac{\Gamma(n+m)}{\Gamma(n)T^m} \quad , \qquad m = 0, 1, 2, \ldots$$

$$E\left(\frac{1}{\theta^m}\right) = \int_{\forall\theta} \frac{1}{\theta^m}\, h_1(\theta|\underline{x})d\theta = \frac{T^m\Gamma(n-m)}{\Gamma(n)}$$

Which can be substituted to (10) to obtain Bayes estimator Jeffrey's prior information:

1- When k =1 and c=1

$$\hat{\theta}_{MJ1} = \frac{a_0\, T(n-1) + a_1(n)}{a_0\, T^2(n-2) + a_1 T(n-1)} \qquad (11)$$

2- When k =2 and c=1

$$\hat{\theta}_{MJ2} = \frac{a_0\, T^2(n-1) + a_1 T(n) + a_2(n+1)}{a_0\, T^3(n-2) + a_1 T^2 + a_2 T(n)} \qquad (12)$$

3- When k =1 and c=2

$$\hat{\theta}_{MJ3} = \frac{a_0\, T(n) + a_1(n-1)}{a_0\, T^2 + a_1 T(n)} \qquad (13)$$

4- When k =2 and c=2

$$\hat{\theta}_{MJ4} = \frac{a_0\, T^4(n) + a_1 T^3(n-1) + a_2(n+2)}{a_0\, T^5(n-1) + a_1 T^4(n) + a_2 T^3(n-1)} \qquad (14)$$

5- When k =1 and c=3

$$\hat{\theta}_{MJ5} = \frac{a_0\, T^2(n-1) + a_1 T(n)}{a_0\, T^2(n-2) + a_1 T^2(n-1)} \qquad (15)$$

6- When k =2 and c=3

$$\hat{\theta}_{MJ6} = \frac{a_0\, T^2(n-1) + a_1 T(n) + a_2(n-1)}{a_0\, T^3(n-2) + a_1 T^2 + a_2 T(n)} \qquad (16)$$

- **With Exponential prior information**

According to the posterior density function $h_2(\theta|\underline{x})$, can derived $E(\theta^m|\underline{x})$, $E\left(\frac{1}{\theta^m}|\underline{x}\right)$ and get some estimators for θ based on Exponential prior as follows:

$since\ \ \theta \sim Gamma\ (n+1, P)$, $\quad P = T + \frac{1}{b}$

then $E(\theta^m) = \int_{\forall\theta} \theta^m h_2(\theta|\underline{x})\, d\theta = \frac{P^m\Gamma(n+1-m)}{\Gamma(n+1)}$

$$E\left(\frac{1}{\theta^m}\right) = \int_{\forall\theta} \frac{1}{\theta^m} h_2(\theta|\underline{x})\, d\theta = \frac{\Gamma(n+m+1)}{\Gamma(n+1)P^m}$$

Putting k=1 and c=1 we get:

$$\hat{\theta}_{ME1} = \frac{a_0P(n)+a_1(n+1)}{a_0P^2(n-1)+Pa_1(n)} \quad (17)$$

Putting k=2 and c=1 we get:

$$\hat{\theta}_{ME2} = \frac{a_0P(n)+a_1P(n+1)+a_2(n+2)}{a_0P^3(n-1)+P^2a_1(n)+Pa_2(n+1)} \quad (18)$$

Putting k=1 and c=2 we get:

$$\hat{\theta}_{ME3} = \frac{a_0P(n-1)+a_1(n)}{a_0P^2(n)+Pa_1(n-1)} \quad (19)$$

Putting k=2 and c=2 we get:

$$\hat{\theta}_{ME4} = \frac{a_0P^2(n-1)+Pa_1+a_2(n+1)}{a_0P^3(n)+P^2a_1(n+1)+Pa_2} \quad (20)$$

Putting k=1 and c=3 we get:

$$\hat{\theta}_{ME5} = \frac{a_0P^2(n)+a_1(n-1)}{a_0P^3(n-1)+P^2a_1(n)} \quad (21)$$

Putting k=2 and c=3 we get:

$$\hat{\theta}_{ME4} = \frac{a_0P^2(n)+Pa_1+a_2(n-1)}{a_0P^3(n-1)+P^2a_1(n)+a_2P(n-1)} \quad (22)$$

3.4 Bayesian Estimator under Generalized Loss function.

The Generalized loss function **[7]** can be written as:

$\boldsymbol{L(\hat{\theta}, \theta) = \left(\sum_{j=0}^{k} a_j\theta^j\right)(\hat{\theta}-\theta)^2}$ Where a_j, j=0, 1,2 , …,k are constant

So, Risk function of Generalized loss function, denoted by $R_{GS}(\hat{\boldsymbol{\theta}}, \boldsymbol{\theta})$is:

$$R_G(\hat{\theta}, \theta) = E\left[L_{GS}(\hat{\theta}, \theta)\right]$$

$$= \int_0^\infty L_{GS}(\hat{\theta}, \theta)\; h(\theta|\underline{x})d\theta$$

$$= \int_0^\infty (a_0 + a_1\theta + \cdots + a_k\theta^k)(\hat{\theta}^2 - 2\hat{\theta}\theta + \theta^2)h(\theta|\underline{x})d\theta$$

$$= a_0\hat{\theta}^2 - 2a_0\hat{\theta}E(\theta|\underline{x}) + a_0E(\theta^2|\underline{x}) + a_1\hat{\theta}^2E(\theta|\underline{x})$$
$$-2a_1\hat{\theta}E(\theta^2|\underline{x}) + a_1E(\theta^3|\underline{x}) + \cdots + a_k\hat{\theta}^2\, E(\theta^k|\underline{x})$$
$$-2a_k\hat{\theta}E(\theta^{k+1}|\underline{x}) + a_kE(\theta^{k+2}|\underline{x})$$

By taking the partial derivative for $R_{GS}(\hat{\theta}, \theta)$ with respect to $\hat{\theta}$ and make it equal to zero yields:

$$\hat{\theta}_G = \frac{a_0E(\theta|\underline{x})+a_1E(\theta^2|\underline{x})+\cdots+a_kE(\theta^{k+1}|\underline{x})}{a_0+a_1E(\theta|\underline{x})+a_2E(\theta^2|\underline{x})+\cdots+a_kE(\theta^k|\underline{x})} \quad (23)$$

- **With Jeffrey's prior information**

From the posterior density function $h_1(\theta|\underline{x})$**,** can show the Bayes estimator θ of One parameter Inverse Rayleigh distribution under Generalized loss function , $\hat{\theta}_{GJ}$ can be

$$\hat{\theta}_{GJ} = \frac{a_0\frac{n}{T}+a_1\frac{(n+1)n}{T^2}+\cdots+a_k\frac{(n+k)(n+k-1)\ldots(n+1)n}{T^{K+1}}}{a_0+a_1\frac{n}{T}+\cdots+a_k\frac{(n+k-1)(n+k-2)\ldots(n+1)n}{T^K}} \quad (24)$$

In this paper, the first and second polynomials are used as follows:

$$\hat{\theta}_{GJ1} = \frac{a_0\frac{n}{T}+a_1\frac{(n+1)n}{T^2}}{a_0+a_1\frac{n}{T}} \quad (25)$$

$$\hat{\theta}_{GJ2}=\frac{a_0\frac{n}{T}+a_1\frac{(n+1)n}{T^2}+a_2\frac{(n+2)(n+1)n}{T^3}}{a_0+a_1\frac{n}{T}+a_2\frac{(n+1)n}{T^2}} \qquad (26)$$

- **With Exponential prior information**

When consider posterior density function $h_2(\theta|\underline{x})$, the Bayes estimator of One Parameter to Inverse Rayleigh distribution when generalized loss function $\hat{\theta}_{GE}$ can be obtained as follows:

$$\hat{\theta}_{GE}=\frac{a_0\frac{(n+1)}{P}+a_1\frac{(n+12)(n+1)}{p^2}+\cdots+a_k\frac{(n+k)(n+k+1)\ldots(n+1)}{P^{K+1}}}{a_0+a_1\frac{(n+1)}{p}+\cdots+a_k\frac{(n+k)\ldots(n+1)}{P^K}}$$

In this paper, the first and second polynomials are used as follows:

$$\hat{\theta}_{GE1}=\frac{a_0\frac{(n+1)}{P}+a_1\frac{(n+12)(n+1)}{p^2}}{a_0+a_1\frac{(n+1)}{p}} \qquad (27)$$

$$\hat{\theta}_{GE2}=\frac{a_0\frac{(n+1)}{P}+a_1\frac{(n+12)(n+1)}{p^2}+a_2\frac{(n+3)(n+2)(n+1)}{P^3}}{a_0+a_1\frac{(n+1)}{p}+a_2\frac{(n+2)(n+1)}{P^2}} \qquad (28)$$

Simulation Results

In this research Q basic program is used to simulate the results and the tables considered in Monte Carlo simulation study is explained for comparing 17 estimators of the scale parameter θ with One Parameter Inverse Rayleigh distribution, using Mean Square Error (MSE) of an estimator which is defined as follows:

$$MSE(\hat{\theta})=\frac{\sum_{i=1}^{R}(\hat{\theta}_i-\theta)}{R} \qquad ; \qquad i=1,2,3,\ldots,R$$

Where, R is the number of replications. we generated R=5000 samples each with size (n=5, 10, 30, 50, 100) respectively, from the Inverse Rayleigh distribution. With the scale parameter $(\theta=1,3).(a_0=5000,\ a_1=5,\ a_2=0.5)$ and one values of the hyper-parameter of Exponential Prior (b=0.8).

Discussion

The experimental results of simulation study for estimating the scale parameter θ of Inverse Rayleigh distribution are summarized and tabulated in (1 and 2) involved both expected values and MSE's .

The result can be summarized as the following important points.

1- In general, Bayesian methods with proposed loss function achieved better performance when compared with generalized one.

2- In term of estimation Bayes method performed good with two different loss functions when consider exponential prior, noted better performance of corresponding estimator when consider Jeffrey's non- informative prior.

3- The values of MSE's for Bayes estimators using Exponential informative prior, are decreasing with using the value of the parameter of Exponential Prior (b=0.8)

4- For all estimates, (MSE's) of the scale parameter is getting high with the increase of the scale in parameter value, with all cases.

5- Tables (1) shows the Bayes estimator performance under suggested loss function with Exponential prior information (ME3) is better estimator, when comparing with other estimator this includes all sample sizes and values of the scale parameter, and followed by the (ME4).

6- In the tables (2), Bayes estimator performance under suggested loss function with Exponential of prior information (ME1) also become best estimator, when comparing with other estimator including all size samples and all values of the scale parameter, and followed by the (ME2).

Conclusions

The results in table (1) show that, Bayes estimators under suggested loss function (c = 2, k = 1) with Exponential prior (ME3 and ME4) are the best estimators when comparing with other estimators, for all sample sizes.
The results in table (2) show that, Bayes estimators under suggested loss function (c = 1, k = 1) with Exponential prior (ME1 and ME2) are the best estimators comparing to other estimators, for all sample sizes.

Recommendations

1- Preferably use a suggested loss function with Exponential prior information of Inverse Rayleigh distribution, for all sample size.
2- The researchers in literature of life research field use of Inverse Rayleigh distribution with a Suggested loss function with Exponential prior information (ME3), to get best estimator.

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Table (1) :Estimated Value and MSE of Different Estimates of $\theta$when $\theta = 1$ and $b = 0.8$

Estimator	n / Criteria	5	10	30	50	100
MLE	EXP	1.003300	1.00212	1.000135	.999999	1.00086
	MSE	0.196414	0.99520	0.0332317	0.199787	0.0099792
MJ1	EXP	1.003663	1.002243	1.000171	1.000015	1.00097
	MSE	0.1966515	0.99569	0.0323591	0.0199804	0.0099796
MJ2	EXP	1.003804	1.00228	1.000179	1.00001	1.00098
	MSE	0.1968111	0.0959146	0.0332375	0.019980	0.0099797
MJ3	EXP	0.8354679	0.9101985	0.9669396	0.97942	0.99007
	MSE	0.1629812	0.089998	0.0320929	0.019550	0.0098143
MJ4	EXP	0.836459	0.9112321	0.9680131	0.980509	0.991166
	MSE	0.1634007	0.090224	0.0321699	0.019590	0.0098661
MJ5	EXP	0.7173007	0.835866	0.938536	0.962476	0.982313
	MSE	0.180493	0.0963126	0.0309993	0.019952	0.0099418
MJ6	EXP	0.7162212	0.834426	0.9367566	0.96061	0.980381
	MSE	0.180455	0.096291	0.0330970	0.019951	0.009938
ME1	EXP	0.969682	0.9838793	0.9937132	0.996092	0.99898
	MSE	0.1374343	0.082540	0.03116556	0.019219	0.007830
ME2	EXP	0.9697628	0.983879	0.993721	0.99609	0.99898
	MSE	0.1374981	0.8255471	0.0031166	0.0192202	0.009783
ME3	EXP	0.8311052	0.9018459	0.9626591	0.976935	0.989192
	MSE	0.1287969	0.0787670	0.0306053	0.0190048	0.009707
ME4	EXP	0.8311435	0.901866	0.9626662	0.976939	0.989150
	MSE	0.1288123	0.078772	0.0360598	0.019005	0.009708
ME5	EXP	0.7271882	0.832643	0.933485	0.958503	0.97958
	MSE	0.1511843	0.0869719	0.03189133	0.0195043	0.0098224
ME6	EXP	0.727209	0.83247	0.9334905	0.958506	0.979590
	MSE	0.1511865	0.0869731	0.0318915	0.0195044	0.0098225
GJ1	EXP	1.2548250	1.113630	1.0346620	1.0204240	1.011080
	MSE	0.372426	0.135847	0.0367694	0.0212210	0.0103042
GJ2	EXP	1.255265	1.11369	1.034671	1.020429	1.01109
	MSE	0.3732667	0.1358988	0.0332375	0.0212220	0.010304
GE1	EXP	1.163750	1.082262	1.0268420	1.016013	1.008970
	MSE	0.2234833	0.1358988	0.0339564	0.020236	0.010059
GE2	EXP	1.163942	1.082306	1.026851	1.016019	1.008980
	MSE	0.2237444	0.1063669	0.0339586	0.020237	0.010050
Best Estimator		ME3	ME3	ME3	ME3	ME3

Table (2): Estimated Value and MSE of Different Estimates of θ when $\theta = 3$ and $b = 0.8$

Estimator	n / Criteria	5	10	30	50	100
MLE	EXP	1.767754	3.006362	3.000406	2.99998	3.002902
	MSE	3.010088	0.8956877	0.2990804	0.1798087	0.08982460
MJ1	EXP	3.012783	3.007453	3.000724	3.000167	3.002992
	MSE	1.774066	0.8970026	0.299208	0.1798533	0.08982461
MJ2	EXP	3.016506	3.00844	3.00095	3.000289	3.003651
	MSE	3.368372	0.8987561	0.299339	0.1798967	0.08983491
MJ3	EXP	2.50141	2.725148	2.89503	2.932422	2.464293
	MSE	3.436136	0.8071163	0.2878014	0.1753457	0.0882645
MJ4	EXP	2.514436	2.737391	2.406849	2.944174	2.976032
	MSE	1.479307	0.8157457	0.2905892	0.1769841	0.08911095
MJ5	EXP	2.1562	2.512616	2.821238	2.843148	2.452834
	MSE	1.624319	0.8668747	0.2979653	0.1796418	0.0895777
MJ6	EXP	2.145678	2.499107	2.85000	2.876301	2.935389
	MSE	1.624806	0.8668865	0.2979754	0.179650	0.0844888
ME1	EXP	2.643373	2.806646	2.929722	2.957023	2.973261
	MSE	1.357742	0.7793146	0.2851485	0.1747153	0.0887607
ME2	EXP	2.645119	2.807335	2.929918	2.957134	2.43319
	MSE	1.361288	0.7793164	0.2852321	0.174745	0.0887675
ME3	EXP	2.265344	2.572621	2.838151	2.900155	2.944105
	MSE	1.443162	0.8052614	0.2841583	0.1762513	0.08945159
ME4	EXP	2.266204	2.573094	2.838321	2.900258	2.944159
	MSE	1.443919	0.8055394	0.289198	0.176267	0.089493
ME5	EXP	1.981964	2.374336	2.75213	2.845418	2.915525
	MSE	1.727804	0.9215045	0.3087012	0.1839584	0.0917951
ME6	EXP	1.982431	2.373971	2.75228	2.845521	2.915571
	MSE	1.727811	0.921531	0.308707	0.1839612	0.091765
GJ1	EXP	3.768214	3.34193	3.104227	3.061404	3.033327
	MSE	3.368372	1.224658	0.3310709	0.1910416	0.09275016
GJ2	EXP	3.774822	3.343488	3.104478	3.061542	3.033385
	MSE	3.436136	1.228741	0.331278	0.1911055	0.09276488
GE1	EXP	3.173027	3.087507	3.027406	3.016173	3.002992
	MSE	1.804121	0.9046382	0.29996	0.181149	0.0898246
GE2	EXP	3.177203	3.088561	3.027628	3.016297	3.003051
	MSE	1.819349	0.9066377	0.3001041	0.1801626	0.08983491
Best Estimator		ME1	ME1	ME1	ME1	ME1