https://doi.org/ 10.30526/2017.IHSCICONF.1806

Q uasi-inner product spaces of quasi-Sobolev spaces and their completeness

Jawad Kadhim Khalaf Al-Delfi
jawadaldelfi@uomustansiriyah.edu.iq
Dept. of Mathematics / College of Science / Al-Mustansiriyah University

Abstract

Sequences spaces ℓ_{p}^{m}, m $\in \mathbb{R}$, p $\in \mathbb{R}_{+}$ that have called quasi-Sobolev spaces were introduced by Jawad . K. Al-Delfi in 2013 [[1. In this paper, we deal with notion of quasi-inner product space by using concept of quasi-normed space which is generalized to normed space and given a relationship between pre-Hilbert space and a quasi-inner product space with important results and examples. Completeness properties in quasi-inner product space gives us concept of quasi-Hilbert space. We show that , not all quasi-Sobolev spaces ℓ_{p}^{m} , are quasi-Hilbert spaces. The best examples which are quasi-Hilbert spaces and Hilbert spaces are ℓ_{2}^{m}, where m $\in \mathbb{R}$. Finally, propositions, theorems an examples are our own unless otherwise referred.

Keywords: quasi-Sobolev space, quasi-Banach space, G$\hat{a}$teaux derivative , quasi-inner product space, quasi-Hilbert space. smooth quasi-Hilbert space.

1. Introduction

The family of sequence spaces ℓ_p , $1 < p < \infty$ are normed space where, ℓ_2 is the only inner product space in this family. Completeness of these spaces can be proved with respect to appropriate norms [2, 3]. Since the triangle inequality fails in the family of sequence spaces ℓ_p , $0 < p < 1$ where, there is no norm for this range, then imply that it is not Banach space . For a sequence space ℓ_p, where $0 < p < 1$ and others , many concepts were introduced . One of these concepts is a quasi- Banach space which is based on the definition of a quasi- norm [4]. A quasi- Banach space is a topological linear space [5].

In [1] , we were constructed a set of all sequence spaces of power real number m, m $\in \mathbb{R}$. The new spaces have called quasi-Sobolev spaces and have denoted by ℓ_p^m . We were proved that these spaces are quasi-Banach spaces in case $0 < p < \infty$ and they are Banach spaces for $1 < p < \infty$. In our work, we need study these spaces with other concepts such as a pre-Hilbert space and a quasi- inner product space (q. i .p) and their completeness.

In normed spaces, mathematicians have used Gâteaux derivatives to introduce notion of quasi- inner product space and have investigated properties of this concept such as completeness, smoothness and others [6,7, 8] . This paper is devoted transference above ideology on quasi-normed space to given (q. i .p) and is studied the relationship between this notion and others, in order to study quasi-inner product spaces for ℓ_p^m and their completeness.

The paper consists of two sections. Section one includes definitions of quasi- normed space and quasi-Banach space with some useful results which are needed in the section two. One of important theorems which is presented in this section is Jordan-van Neumann theorem. This theorem gives necessary and sufficient conditions to be generated by an inner product space. The second two presents a Gâteaux derivative that has big role to define many concepts, such as quasi- inner product space with completeness property of it. Also, this section shows that this functional is an inner product function in pre-Hilbert spaces. A space ℓ_p^m, for every $m \in \mathbb{R}$ and $p \in \mathbb{R}_+$ is a quasi-Hilbert space if it is a quasi-inner product space. Hence,
with ℓ_p^m, we find spaces which are quasi-Hilbert spaces and are not Hilbert spaces , spaces neither quasi-Hilbert spaces nor Hilbert spaces and spaces are quasi-Hilbert spaces and Hilbert space.

2. Quasi-normed spaces of sequence spaces.

This section contains notions such as quasi-normed space, a pre-Hilbert space and others with the relationship between them. Also, theorems and equations which are useful in section two are introduced.

Definition 1.1. [4]:
A quasi-norm ${}_q\|\,.\,\|$ on vector space V over the field of real numbers $\mathbb{R}$ is a function ${}_q\|\,.\,\|\colon V \longrightarrow [0,+\infty)$ with the properties:

(1) ${}_q\| v \| \geq 0, \quad \forall v \in V, \; {}_q\| v \| = 0 \leftrightarrow$ v= 0.
(2) ${}_q\| \alpha v \| = | \alpha |\, {}_q\| v \|, \; \forall v \in V, \forall \alpha \in \mathbb{R}$.
(3) ${}_q\| v + w \| \leq C \left({}_q\| v \| + {}_q\| w \| \right) \quad \forall\, v, w, \in V$, where $C \geq 1$ is a constant independent of v, w.

A quasi-normed space is denoted by $\left(V, {}_q\|.\|\right)$ or simply V.

A function ${}_q\| . \|$ be a norm if C = 1, thus it is generalization of norm . Every norm function is quasi-norm. The converse does not hold, in general.

Since every quasi-normed space V is a metric space by d(v, w) = ${}_q\| v - w \|$, then it is atopological linear space and the concepts of fundamental sequences and completeness in quasi-normed spaces are given [5]. A quasi- Banach space is a complete quasi-normed space.

Definition 1.2.
A symmetric linear functional on V^2 is a functional L such that:
(1) $L(\beta\, v + \mu\, w, u) = \beta\, L(v, u) + \mu L(w, u)$;
(2) $L(v, w) = L(w, v), \quad \forall\, \beta, \mu \in \mathbb{R}, \forall\, v, w, u \in V$.

Remark 1.3.
It is obvious, any inner product function satisfies definition 1.2 and generates a quasi - norm which is ${}_q\| v \| = (< v, v >)^{1/2}$, $\forall\, v \in V$

Lemma 1.4.
In a pre-Hilbert space V , one has the equality:

$${}_q\| v + w \|^4 - {}_q\| v - w \|^4 = 8\left({}_q\| v \|^2 + {}_q\| w \|^2\right) □□v, w □□□□ \; \forall\, v, w, \in V \qquad (1)$$

Proof:
Using remark 1.3, we get ${}_q\| v + w \|^2 = < v + w, v + w > = {}_q\| v \|^2 + 2 < v, w > + {}_q\| w \|^2 \Rightarrow \left({}_q\| v + w \|^2\right)^2 = \left({}_q\| v \|^2 + {}_q\| w \|^2\right)^2 + 4$ □□v, w □ $\left({}_q\| v \|^2 + {}_q\| w \|^2\right) + 4(< v, w >)^2$.

Also, ${}_q\| v - w \|^2 = {}_q\| v \|^2 - 2 < v, w > + {}_q\| w \|^2 \Rightarrow$

$${}_q\| v - w \|^4 = \left({}_q\| v \|^2 + {}_q\| w \|^2\right)^2 - 4 □□v, w □ \left({}_q\| v \|^2 + {}_q\| w \|^2\right) + 4(< v, w >)^2.$$

Thus, ${}_q\| v + w \|^4 - {}_q\| v - w \|^4 = 8\left({}_q\| v \|^2 + {}_q\| w \|^2\right)$ □□v, w □□□ and this is the desired result.

Definition 1.5. [1]:
Let $\{\lambda_k\} \subset \mathbb{R}_+$ is monotonically increasing sequence such that $\lim_{K \to \infty} \lambda_k = +\infty$, quasi-Sobolev spaces are sequence spaces ℓ_p^m, where $0 < p < \infty$ and $m \in \mathbb{R}$ which are defined as :

$$\ell_p^m = \left\{ v = \{v_k\} : \sum_{k=1}^{\infty} \lambda_k^{\frac{mp}{2}} |v_k|^p < +\infty \right. .$$

When $m = 0$ then $\ell_p^0 = \ell_p$, $0 < p < \infty$.

Theorem 1.6. [1]:

For every $m \in \mathbb{R}$ and $p \in \mathbb{R}_+$ a space ℓ_p^m, is a quasi-Banach space with the function :

$$_q\|v\| = \left(\sum_{k=1}^{\infty} \lambda_k^{\frac{mp}{2}} |v_k|^p\right)^{1/p}.$$

We note that the constant $C = 2^{1/p}$ for $p \in (0, 1)$, and $C = 1$ for $p \in [1, +\infty)$.

Theorem 1.7. (parallelogram equality)

Let V be a pre-Hilbert space. Then $\forall\, v, w \in V$,

$$_q\|v+w\|^2 + {}_q\|v-w\|^2 = 2\,{}_q\|v\|^2 + 2\,{}_q\|w\|^2 \qquad (2)$$

Proof:

Since V be a pre-Hilbert space and $< v, w > = \left(\frac{1}{4}\,{}_q\|v+w\|^2 - \frac{1}{4}\,{}_q\|v-w\|^2\right)$ from remark 1.3 and proof of lemma 1.4 , then putting this function in equation (1) we obtain the desired result.

Now , we introduce Jordan-van Neumann theorem in quasi- normed spaces.

Theorem 1.8. (Jordan – van Neumann **)**

A quasi-normed space V is a pre-Hilbert space iff equality (2) is satisfied by the quasi-norm of V.

Proof:

The proof of this theorem is very technical and proceeds in a way similar to its version in normed space (see [3]).

The next example shows the importance of the parallelogram equality mentioned in the previous theorem.

Example 1.9:

Let v and w belong to the quasi-normed space $\ell_{1/2}^{-1}$, where $v = \{v_k\} = \{0.1, 0, 0, 0, \ldots\}$, $w = \{w_k\} = \{0, 0.2, 0, 0, \ldots\}$ and take $\{\boldsymbol{\lambda}_k\} = \{k\}$, $k \in \mathbb{N}$. Then we have:

$$_{½}\|v+w\|^2 = \left(\sum_{k=1}^{\infty} \lambda_k^{\frac{-1}{4}} |x_k + y_k|^{1/2}\right) = 0.4792627792275938 = {}_{½}\|v-w\|^2, \quad \text{so}$$

$_{½}\|v+w\|^2 + {}_{½}\|v-w\|^2 = 0.9585255584551875$, and, $2\,{}_{½}\|v\|^2 + 2\,{}_{½}\|w\|^2 = 0.482842712474619$. It is clear that two sides of the equation (2) do not hold. Thus , $\ell_{1/2}^{-1}$ is not pre-Hilbert space.

3.Quasi-inner product spaces of sequence spaces

A Gâteaux derivative is used to define many concepts, such as quasi- inner product function, and smooth quasi-Hilbert space with some important results and examples.

Definition 2.1.

Let V be a vector space over the field $\mathbb{R}$ equipped with ${}_q\|.\|$. A Gâteaux derivative of ${}_q\| v \|$ is a functional $\delta\,(v\,,w)$ at $v \in V$ in the direction $w \in V$ which is defined as:
$\delta(v\,,w) = (\,\delta_1(v,w) + \delta_2(v,w)\,)$ such that:

$\delta_1(v,w) = \lim_{h\to+0} h^{-1}\,({}_q\| v + hw \| - {}_q\| v \|)$, and $\delta_2(x,y) = \lim_{h\to-0} h^{-1}\,({}_q\| v + hw \| - {}_q\| v \|)$, where $h \in \mathbb{R} \setminus \{0\}$. In similar way, we define $\delta\,(w\,,v)$.

Gâteaux derivatives $\delta(v,w)$ and $\delta(w,v)$ inspires the functionals $\tau(v,w) = \frac{{}_q\| v \|}{2}\,\delta(v\,,w)$ and $\tau\,(w\,,v) = \frac{{}_q\| w \|}{2}\,\delta\,(w\,,v)$ sequentially .

Definition 2.2

A Gâteaux derivative $\tau\,(v\,,w)$ is said to be quasi-inner product function if $\tau\,(w\,,v)$ exists and the next equality is satisfied:

$${}_q\| v + w \|^4 - {}_q\| v - w \|^4 = 8\,(\,{}_q\| v \|^2\ \tau\,(v\,,w) + {}_q\| w \|^2\ \tau\,(w,v)\,),\ \forall\ v,\ w \in V \quad (3)$$

Similarly, $\tau\,(w\,,v)$. A space V is said to be a quasi-inner product if both $\tau\,(v\,,w)$ and $\tau\,(w\,,v)$ are quasi-inner product functions .

Lemma 2.3

For every positive integer $p \geq 1$ and $m \in \mathbb{R}$, the functional $\tau\,(v\,,w)$ in quasi-Sobolev spaces ℓ_p^m exists and is defined as :

$$\tau\,(v\,,w) = {}_q\| v \|^{2-p} \sum_k \lambda_k^{\frac{mp}{2}} |v_k|^{p-1} (\text{sng}\, v_k\,) w_k,\ \ \forall\, v \in \ell_p^m \ \text{ s.t. }\ {}_q\| v \| \in E\,,$$

where, $E = \left\{ {}_q\| v \| : \begin{array}{ll} {}_q\| v \| \geq 0\,, & P = 1 \\ {}_q\| v \| > 0\,, & P \geq 2 \end{array} \right\}$ and

$$\text{sng}\ v_k = \begin{cases} 1, & v_k > 0 \\ 0, & v_k = 0 \\ -1, & v_k < 0 \end{cases}. \qquad (4)$$

Similarly, we define $\tau\,(w\,,v)$.

Proof:

In definition 2.1, we use properties of limits of functions and applying definition of a quasi-norm function of ℓ_p^m which is in theorem 1.6 with help of the binomial theorem, which is for every positive integer p, $(v + w)^p = \sum_{k=0}^{p} \binom{p}{k} v^k w^{p-k}$, we get Eq. (4).

Proposition 2.4.

 https://doi.org/ 10.30526/2017.IHSCICONF.1806

The existence of the limit in definition of Gâteaux functions is necessary condition , not sufficient, in order that any quasi-normed space be a quasi-inner product space.

Proof

Suppose V is a quasi-normed space. From definition 2.1, we observe that existence of $\delta_1(v,w)$ and $\delta_2(v,w)$ are connected by the limit on behavior of the quasi-norm as $h \to \pm 0$. hence, $\tau(v,w)$ is exist if this limit is exist. Also , with $\tau(w,v)$ similarly.

To explains above condition is not sufficiently, we take the example:

Example 2.5:

Suppose $v, w \in \ell_3^1$, where $v = \{v_k\} = \{1,0,0,0,\ldots\}$, $w = \{w_k\} = \{1,1,0,0,\ldots\}$ and take $\{\lambda_k\} = \{\sqrt{k}\}$, $k \in \mathbb{N}$. Then, using lemma 2.3,we get $\tau(v,w) = 1$, $\tau(w,v) = 0.372884880824589$. Thus, $\tau(v,w)$ and $\tau(w,v)$ are exist . However, equation (3) is not satisfied. Therefore, the space ℓ_3^1 is not quasi-inner product space.

Remark 2.6.

If cases the values of p differ from those values considered in lemma 2.3, we have quasi-Sobolev spaces ℓ_{p}^m which are not quasi- inner product. For instance, in case $\mathrm{p} \in (0,1)$, *as* it is shown in the example 1.9. Indeed,
with the space $\ell_{1/2}^{-1}$, $\delta_1(v,w)$ and $\delta_2(w,v)$ do not exist, since there is no limit as $h \to \pm 0$ from definition 2.1. Then right hand in Eq. (3) is not finite, while left hand equal zero.

Definition 2.7

A quasi-normed space V is smooth if $\delta_1(v,w)$ and $\delta_2(v,w)$ have one value.
When V is smooth quasi-normed space , then $\tau(v,w) = {}_q\| v \| \lim_{h\to 0} h^{-1} \left({}_q\| v + hw \| - {}_q\| v \| \right)$. Similarly, $\tau(w,v)$.

Proposition 2.8.

Every pre-Hilbert space.is a quasi-inner product space.

Proof:

Let V *is* a pre-Hilbert space. According to lemma 1.4, an inner product function gives eq. (1) . Also, By remark 1.3 and definition 2.1 , we obtain $\tau(v,w) = \langle v, w \rangle$ and $\tau(w,v) = \langle w, v \rangle$. Hence, we have equation (3), and the definition 2.2 is hold. Thus, V is an quasi-inner product space.

The converse of proposition does not hold**, consider** the following example:

Example 2.9:

Take example 2.5 with replace space ℓ_3^1 by ℓ_4^1 . Since Eq. (3) is satisfied with quasi-normed space ℓ_4^1 , where the left and right hand of Eq. (3) are equal to 16, so it is quasi-inner product space. But the left and right hand of Eq. (2) are not equal, hence this space is not a pre-Hilbert space.

Definition 2.10.

A complete quasi- inner product space is called a quasi-Hilbert space.
If a quasi-Hilbert space is smooth, then it is called a smooth quasi-Hilbert space.
We recall that completeness property is coming from this property of quasi-normed space.

Theorem 2.11.

 https://doi.org/ 10.30526/2017.IHSCICONF.1806

For every $m \in \mathbb{R}$, ℓ_2^m is a smooth quasi-Hilbert space and Hilbert space.

Proof:

According to lemma 2.3, we get $\tau(v,w) = \sum_k \lambda_k{}^m |v_k|(sng\, v_k)\ w_k$,and $\tau(w,v) = \sum_k \lambda_k{}^m |w_k|\ (sng\, w_k) v_k$ which are linear by definition 1.2, with definition of $\tau(v,w)$ and $\tau(w,v)$ as above, then they are symmetric, that is, $\tau(v,w) = \tau(w,v)$, and $\tau(v,v) = {}_q\|v\|^2 \geq 0$, with equality iff $v = 0$. Hence, ℓ_2^m is a pre-Hilbert space. By proposition 2.8, it is a quasi-inner product space, where $8\sum_k \lambda_k{}^{2m}|v_k|^3(\text{sng}\, \text{v}_{\text{k}})\ \text{w}_\text{k} + 8 \sum_k \lambda_k{}^{2m}|w_k|^3(sng\, w_k)v_k$ is value to both sides of equation (3). If we apply quasi-norm function of ℓ_2^m in definition 2.1, we obtain $\delta_1(v,w) = \delta_2(v,w)$ since the limit in $\delta_1(v,w)$ itself one $\delta_2(v,w)$. Then ℓ_2^m is smooth.

Now, since ℓ_2^m is a quasi-Banach space for every $m \in \mathbb{R}$ by theorem 1.6, then it is complete under ${}_q\|v\| = (\tau(v,v))^{1/2}$, i.e. every fundamental sequence $\{v_k\}$, $k \in \mathbb{N}$ is convergent in it. Therefore, Theorem is proved.

Remark 2.12.

Since a space ℓ_p^m, for every $\text{m} \in \mathbb{R}$ and $\text{p} \in \mathbb{R}_+$ is a quasi-Banach space, then ℓ_p^m is a quasi-Hilbert space if it is a quasi-inner product space.

References

[1] J.K. Al-Delfi., Quasi-Sobolev Spaces ℓ_p^m ., Bulletin of South Ural State University , Series of "*Mathematics.Mechanics* . Physics", 5, 1. , 107–109. (In Russian). 2013.

[2] W. Rudin, Functional Analysis., McGraw-Hill, Inc., New York, 1991.

[3] A. H. Siddiqi, Functional Analysis With Applications., Tata McGraw-Hill Publishing Company, Ltd, New Delhi, India, 1986.

[4] N. Kalton, Quasi-Banach Spaces., Handbook of the Geometry of Banach Spaces, Vol. Edit. by. Johnson W.B and. Lindenstrauss. J – Amsterdam etc.: Elsevier, 1099–1130. 2003

[5] J. Bergh ; J. Löfström, Interpolation Spaces. An Introduction., Berlin–Heidelberg–New York, Springer-Verlag, 1976.

[6] P.M. Milicic , On the g-orthogonal projection and the best approximation of vector in a quasi- inner product spaces., Scientiae Mathematicae Japonicae, 4, 3, 941-944. 2001.

[7] R .A . Tapia., A characterization of inner product spaces., Proc. Amer. Math. Soc., 41, 569-574. 1973

[8] A. Sahovi ; F.Vajzovi and Peco . S., Continuity conditions for the Hilbert transform on quasi-Hilbert spaces., Sarajevo journal of mathematics, (2014), Vol.10, No. 22, pp- 111–120.